A discrete probability distribution expressed by Racah polynomial arising from Schur-Weyl duality

Shintarou Yanagida (Graduate School of Mathematics, Nagoya University)

2022/06/16 MINI-SYMPOSIA of OPSFA 2022

Contents

Based on the collaboration [HHY]:

Masahito Hayashi (SUSTech/Nagoya), Akihito Hora (Hokkaido), S.Y., "Asymmetry of tensor product of asymmetric and invariant vectors arising from Schur-Weyl duality based on hypergeometric orthogonal polynomial", arXiv:2104.12635, 71pp.

I will focus on the mathematical part of this paper.

1. Conclusion and setting (9 pages)
   Based on §2 of our paper [HHY].
   1.1. Preliminary: Racah polynomial
   1.2. Conclusion: The discrete probability distribution $P_{n,m,k,l}$.
   1.3. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$.
2. How to prove Main Theorem
3. Asymptotic behavior of $P_{n,m,k,l}$
4. Concluding remarks

A. q-analogue

1.1. Racah polynomial

The (generalized) hypergeometric series

$$ {}_{r+1}F_r\left[\begin{matrix}\alpha_1, & \alpha_2, & \dots, & \alpha_{r+1} \\ \beta_1, & \beta_2, & \dots, & \beta_r\end{matrix}; z\right] := \sum_{i=0}^{\infty} \frac{(\alpha_1)_i(\alpha_2)_i \cdots (\alpha_{r+1})_i}{(\beta_1)_i(\beta_2)_i \cdots (\beta_r)_i(1)_i} z^i $$

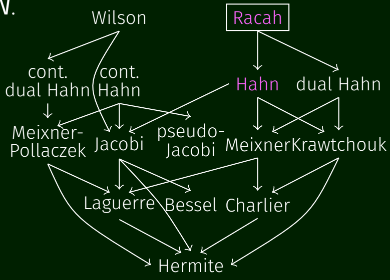
with $(a)_i := a(a+1)\cdots(a+i-1)$ the rising factorial.

Racah polynomial $R_n(z)$ of variable z and degree $n = 0, 1, \dots, N$ for $N \in \mathbb{Z}_{\geq 0}$:

$$ R_n(z; \alpha, \beta, \gamma, \delta) := {}_4F_3\left[\begin{matrix}-n, & n+\alpha+\beta+1, & -z, & z+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & \gamma+1\end{matrix}; 1\right] $$

with $\alpha + 1 = -N$ or $\beta + \gamma + 1 = -N$ or $\delta + 1 = -N$.

- The family $\{R_n(z) \mid n = 0, 1, \dots, N\}$ is orthogonal with respect to some discrete weight function w: $\sum_{i=0}^{N} R_m(i)R_n(i)w(i) = \delta_{m,n}$.
- It sits in the top line of Askey scheme of hypergeometric orthogonal polynomials.



1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$ (1/4)

Racah polynomial of degree $n = 0, 1, \dots, N$:

$$R_n(z; \alpha, \beta, \gamma, \delta) := {}_4F_3\left[\begin{matrix} -n,\ n+\alpha+\beta+1,\ -z,\ z+\gamma+\delta+1 \\ \alpha+1,\ \beta+\gamma+1,\ \delta+1 \end{matrix}; 1\right]$$

with $\alpha + 1 = -N$ or $\beta + \delta + 1 = -N$ or $\gamma + 1 = -N$.

Theorem 1

For $n, m, k, l \in \mathbb{Z}$ satisfying $0 \le 2m, k, l \le n$, $m - l \ge 0$ and $n - m - k + l \ge 0$,

$$p(x) := \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} R_x(m-l; -m-1, -n+m-1, -(n-k)-1, 0)$$

gives a discrete probability distribution $P_{n,m,k,l}$ for $x \in \{0, 1, \dots, n\}$.

$\binom{a}{k} := \frac{1}{k!} a(a-1)\cdots(a-k+1) \in \mathbb{Q}[a]$ for $k \in \mathbb{Z}_{\ge 0}$.

- Theorem 1 says that the Racah part $= \sum_{i=0}^{M \wedge N} (-1)^i \frac{\binom{x}{i}\binom{n+1-x}{i}\binom{M}{i}\binom{N}{i}}{\binom{m}{i}\binom{n-m}{i}\binom{M+N}{i}} \ge 0$ for $0 \le x \le n$.
- Theorem 1 also says that the total sum $\sum_{x=0}^{n} p(x) = 1$.
  It is generalized to a nontrivial summation formula in the next page.

1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$ (2/4)

The probability distribution function (pdf) again:

$$P_{n,m,k,l}[X = x] = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ -m, & m-n, & -M-N \end{matrix}; 1\right].$$

$(n, m, k, l \in \mathbb{Z},\ 0 \le 2m, k, l \le n,\ M := m - l \ge 0$ and $N := n - m - k + l \ge 0.\ x = 0, 1, \dots, n.)$

Theorem 2

The cumulative distribution function (cdf) satisfies

$$P_{n,m,k,l}[X \le x] = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} {}_4F_3\left[\begin{matrix} -x, & x-n, & -M, & -N \\ -m, & m-n, & -M-N \end{matrix}; 1\right].$$

Moreover, we have $P[X \le m] = P[X \le m+1] = \cdots = P[X \le n] = 1$.

- The ${}_4F_3$-part in the cdf $= \sum_{i=0}^{M \wedge N} (-1)^i \frac{\binom{x}{i}\binom{n-x}{i}\binom{M}{i}\binom{N}{i}}{\binom{m}{i}\binom{n-m}{i}\binom{M+N}{i}}$.
- Theorem 2 can be regarded as a kind of hypergeometric summation formula

$$\sum_{x=0}^{n} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ -m, & m-n, & -M-N \end{matrix}; 1\right] = {}_4F_3\left[\begin{matrix} -x, & x-n, & -M, & -N \\ -m, & m-n, & -M-N \end{matrix}; 1\right].$$

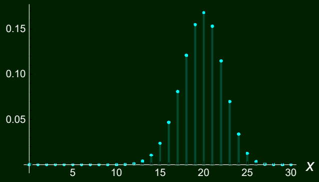
Summary:

- Our discrete distribution $P_{n,m,k,l}$ has four integer parameters n, m, k, l.
- For our distribution $P_{n,m,k,l}$, both pdf and cdf are ${}_4F_3$-polynomials. Such a distribution seems to be new.

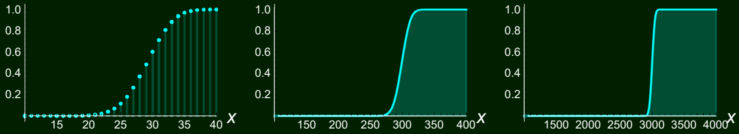
distribution	pdf $\mathrm{Pr}[X = x]$	cdf $\mathrm{Pr}[X \le x]$
binomial	$\binom{n}{x} p^x (1-p)^x$	$\sim {}_1F_0\left[{-n \atop }; \frac{p}{1-p}\right]_{\le x}$
hypergeometric	$\binom{m}{x}\binom{n-m}{l-x}/\binom{n}{l}$	$\sim {}_3F_2\left[{1,\ x+1-m,\ x+1-l \atop x+2,\ n+x+2-m-l}; 1\right]$
our distribution	$\sim {}_4F_3\left[{-x,\ x-n-1,\ -M,\ -N \atop -m,\ m-n,\ -M-N}\right]$	$\sim {}_4F_3\left[{-x,\ x-n,\ -M,\ -N \atop -m,\ m-n,\ -M-N}; 1\right]$

(~ denotes that some factor is suppressed.)

1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$ (4/4)



pdf $P_{n,m,k,l}[X = x]$ with $(n, m, k, l) = (100, 30, 40, 20)$ in left and $(100, 40, 60, 30)$ in right.



cdf $P_{n,m,k,l}[X \le x]$ with $(\frac{m}{n}, \frac{k}{n}, \frac{l}{n}) = (0.4, 0.6, 0.3)$, $n = 100$ (left), 10^3 (middle) and 10^4 (right).

Consider the classical Schur-Weyl duality of SU(2) and $\mathfrak{S}_n$.

- $\mathrm{SU}(2) \curvearrowright \mathbb{C}^2$: the vector repr. of the special unitary group SU(2).
  $\mathrm{SU}(2) \curvearrowright (\mathbb{C}^2)^{\otimes n}$: the n-th fold tensor representation.
- $(\mathbb{C}^2)^{\otimes n} \curvearrowleft \mathfrak{S}_n$: permuting tensor factors by the symmetric group $\mathfrak{S}_n$.
- These two actions of SU(2) and $\mathfrak{S}_n$ commute:

$$\mathrm{SU}(2) \curvearrowright \mathcal{H} := (\mathbb{C}^2)^{\otimes n} \curvearrowleft \mathfrak{S}_n,$$

- The irreducible decomposition of the bimodule is

$$(\mathbb{C}^2)^{\otimes n} = \bigoplus_{x=0}^{\lfloor n/2 \rfloor} \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)}.$$

  where $\mathcal{U}_r$ is the highest weight SU(2)-irrep of dimension r,
  and $\mathcal{V}_{(n-x,x)}$ is $\mathfrak{S}_n$-irrep corresponding to the partition $(n-x,x)$.

- The decomp. $(\mathbb{C}^2)^{\otimes n} = \bigoplus_{x=0}^{\lfloor n/2 \rfloor} \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)}$ gives projectors

$$\pi_x : \mathcal{H} = (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \dots, \lfloor n/2 \rfloor).$$

  Then any normalized element $|v\rangle \in (\mathbb{C}^2)^{\otimes n}$ with respect to the standard hermitian pairing gives rise to a discrete probability by

$$\Pr[X = x] := \langle v | \pi_x | v \rangle \quad (x = 0, 1, \dots, \lfloor n/2 \rfloor).$$

- Our choice of the normalized element: using the basis $\mathbb{C}^2 = \mathbb{C}\,|0\rangle + \mathbb{C}\,|1\rangle$,

$$|\Xi_{n,m|k,l}\rangle := |1^l\, 0^{k-l}\rangle \otimes |\Xi_{n-k,m-l}\rangle \ \in (\mathbb{C}^2)^{\otimes n},$$

$$|1^l\, 0^{k-l}\rangle \in (\mathbb{C}^2)^{\otimes k}, \quad |\Xi_{n-k,m-l}\rangle := \frac{1}{\binom{n-k}{m-l}^{1/2}} \sum_{w \in |1^{m-l} 0^{n-m-k+l}\rangle . \mathfrak{S}_{n-k}} w \ \in (\mathbb{C}^2)^{\otimes (n-k)}.$$

  We have the natural conditions

$$l \geq 0, \quad k - l \geq 0, \quad M := m - l \geq 0 \text{ and } N := n - m - k + l \geq 0.$$

1.2. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $(\mathbb{C}^2)^{\otimes n}$ (3/3)

Definitions again:

$$\pi_x : (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

$$|\Xi_{n,m|k,l}\rangle := \left|1^l\, 0^{k-l}\right\rangle \otimes \frac{1}{\binom{n-k}{m-l}^{1/2}} \sum_{w \in |1^{m-l} 0^{n-m-k+l}\rangle . \mathfrak{S}_{n-k}} w \quad \in (\mathbb{C}^2)^{\otimes n}.$$

Main Theorem (coincise form of Theorem 1)

The discrete probability associated to $|\Xi_{n,m|k,l}\rangle$ is $P_{n,m,k,l}$ in Theorem 1, i.e.,

$$\langle \Xi_{n,m|k,l} |\, \pi_x \,| \Xi_{n,m|k,l} \rangle = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \, {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ -m, & m-n, & -M-N & \end{matrix} ; 1\right]$$

for $x = 0, 1, \ldots, \lfloor n/2 \rfloor$. $(M := m-l,\ N := n-m-k+l,\ M+N = n-k.)$

End of first half.

2. How to prove Main Theorem

1. Conclusion and setting
2. How to prove Main Theorem (5 pages), based on §4 of our paper [HHY].
   - 2.1. Projector formula
   - 2.2. Gelfand pairs and zonal spherical functions.
   - 2.3. Hahn summation formula
   - 2.4. Main Theorem — Racah formula
3. Asymptotic behavior of $P_{n,m,k,l}$
4. Concluding remarks

A. q-analogue

2.1. Projector formula (1/1)

Recollection of Main Theorem: Using $M := m - l$ and $N := n - m - k + l$, define

$$\pi_x : (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \dots, \lfloor n/2 \rfloor).$$

$$|\Xi_{n,m|k,l}\rangle := \left|1^l\, 0^{k-l}\right\rangle \otimes \binom{M+N}{M}^{-1/2} \sum_{w \in |1^M 0^N\rangle . \mathfrak{S}_{M+N}} w \quad \in (\mathbb{C}^2)^{\otimes n}.$$

Then we have

$$\langle \Xi_{n,m|k,l} |\, \pi_x \,| \Xi_{n,m|k,l} \rangle = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1}\, {}_4F_3\!\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ -m, & m-n, & -M-N & \end{matrix}; 1\right].$$

We will calculate $\langle \Xi_{n,m|k,l} |\, \pi_x \,| \Xi_{n,m|k,l} \rangle$ by regarding the decomposition as $\mathfrak{S}_n$-representation:

$$\pi_x : (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{V}_{(n-x,x)}{}^{\otimes \dim_{\mathbb{C}} \mathcal{U}_{n-2x+1}} = \mathcal{V}_{(n-x,x)}{}^{\otimes (n-2x+1)}.$$

To calculate $\langle \Xi_{n,m|k,l} |\, \pi_x \,| \Xi_{n,m|k,l} \rangle$, we want some formula for π_x.

Fact (projector formula)

Denoting by φ the $\mathfrak{S}_n$-action, we have

$$\pi_x = \frac{\dim_{\mathbb{C}} \mathcal{V}_{(n-x,x)}{}^{\otimes (n-2x+1)}}{|\mathfrak{S}_n|} \sum_{\sigma \in \mathfrak{S}_n} \chi^{(n-x,x)}(\sigma)\, \varphi(\sigma)$$

with $\chi^{(n-x,x)}$ the character of the irreducible representation $\mathcal{V}_{(n-x,x)}$.

$\dim_{\mathbb{C}} \mathcal{V}_{(n-x,x)}$ is given by the hook length formula. What about $\sum_\sigma \cdots \chi^{(n-x,x)}(\sigma)\varphi(\sigma)$?

2.2. Gelfand pairs and zonal spherical functions

Consider the subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{S}_n$.

The pair $(G, K) := (\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ is a Gelfand pair, i.e., the induced representation $\mathrm{Ind}_K^G \mathbb{C}_{\mathrm{triv}}$ has multiplicity free irreducible decomposition.

For this Gelfand pair, zonal spherical function $\omega_{(n-x,x)} : G \to \mathbb{C}$ is

$$\omega_{(n-x,x)}(g) := \frac{1}{|K|} \sum_{k \in K} \chi^{(n-x,x)}(kg^{-1}).$$

The value $\omega_{(n-x,x)}(g)$ depends only on the double coset KgK,
and we have the induced $\omega_{(n-x,x)} : K \backslash G / K \to \mathbb{C}$.

Fact [Delsarte 1973, 1978]

The set G/K, equipped with a certain distance function, has the structure of Johnson graph $J(n, m)$, which induces bijections

$$K \backslash G / K = \{K\text{-orbits of } J(n, m)\} = \{0, 1, \dots, m\}.$$

2.3. Hahn summation formula

The zonal spherical function $\omega_{(n-x,x)} : K\backslash G/K \to \mathbb{C}$ is now totally determined by the values $\{\omega_{(n-x,x)}(i) \mid i = 0, 1, \dots, m\}$.

Fact [Delsarte]

The value $\omega_{(n-x,x)}(i)$ is given by

$$\omega_{(n-x,x)}(i) = {}_3F_2\left[\begin{matrix} -i, & -x, & x-n-1 \\ & -m, & m-n \end{matrix}; 1\right] := \sum_{a \geq 0} \frac{(-i)_a(-x)_a(x-n-1)_a}{(1)_a(-m)_a(m-n)_a}.$$

The RHS is the Hahn polynomial with variable i and degree x.

Hahn summation formula [Hayashi-Hora-Y., Theorem 4.1.1]

Using $M := m - l$ and $N := n - m - k + l$, we have

$$\langle \Xi_{n,m|k,l} | \, \pi_x \, | \Xi_{n,m|k,l} \rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \sum_{i=0}^{M \wedge N} \binom{M}{i}\binom{N}{i} \omega_{(n-x,x)}(i).$$

The Hahn summation formula is a double sum, and difficult to use for analysis.

$$\langle \Xi |\, \pi_x \,| \Xi \rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \sum_{i=0}^{M\wedge N} \binom{M}{i}\binom{N}{i} {}_3F_2\left[\begin{matrix} -i, & -x, & x-n-1 \\ & -m, & m-n \end{matrix}; 1\right].$$

$$(M \coloneqq m-l,\ N \coloneqq n-m-k+l.)$$

Racah formula (Main Theorem) [Hayashi-Hora-Y., Theorem 4.2.1]

We have the following hypergeometric summation formula

$$\sum_{i=0}^{M\wedge N} \binom{M}{i}\binom{N}{i} {}_3F_2\left[\begin{matrix} -i, & -x, & x-n-1 \\ & -m, & m-n \end{matrix}; 1\right] = \binom{M+N}{M} {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ & -m, & m-n, & -M-N \end{matrix}; 1\right],$$

where $R_x(M) = {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ & -m, & m-n, & -M-N \end{matrix}; 1\right]$ is Racah polynomial. It yields Main Theorem:

$$\langle \Xi_{n,m|k,l} |\, \pi_x \,| \Xi_{n,m|k,l} \rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \binom{n-k}{m-l} R_x(m-l).$$

3. Asymptotic behavior of $P_{n,m,k,l}$

1. Conclusion and setting
2. How to prove Main Theorem (Racah formula)

$$P_{n,m,k,l}[X = x] = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n - 2x + 1}{n - x + 1} \binom{n-k}{m-l} {}_4F_3\left[\begin{matrix} -x, & x - n - 1, & -M, & -N \\ & -m, & m - n, & -M - N \end{matrix}; 1\right].$$

$(n, m, k, l \in \mathbb{Z}, \quad 0 \le 2m, k, l \le n, \quad M := m - l \ge 0, \quad N := n - m - k + l \ge 0, \quad x \in \{0, 1, \dots, n\}.)$

3. Asymptotic behavior of $P_{n,m,k,l}$ (3 pages), based on §5 of our paper [HHY].
   - 3.1. What is Racah formula useful for?
   - 3.2. Central limit theorem
4. Concluding remarks

A. q-analogue

3.1. What is Racah formula useful for?

Racah polynomial R_x of degree x (and variable M) in Main Theorem

$$P_{n,m,k,l}[X = x] = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \binom{n-k}{m-l} R_x, \quad R_x := {}_4F_3\left[\begin{matrix} -x, & \cdots \\ -m, & \cdots \end{matrix}; 1\right]$$

is an orthogonal polynomial, and satisfies three-term recursive formula of the form $a_x R_{x+1} + b_x R_x + c_x R_{x-1} = 0$. It is rewritten as:

Three-term recursive formula

$p(x) = P_{n,m,k,l}[X = x]$ satisfies the recursive formula

$$A_x p(x+1) + B_x p(x) + C_x p(x-1) = 0,$$

$$A_x \coloneqq \frac{(m-x)(n-m-x)(n-k-x)(n-x+1)}{(n-2x)(n-2x+1)} \frac{n-2x-1}{n-x} \frac{x+1}{n-x},$$

$$C_x \coloneqq \frac{x(x-k-1)(m-x+1)(n-m-x+1)}{(n-2x+1)(n-2x+2)} \frac{n-2x+3}{n-x+2} \frac{x-1}{n-x+1},$$

$$B_x \coloneqq A_x + C_x - MN.$$

It enables us to do asymptotic analysis for $P_{n,m,k,l}$, $n \to \infty$.

3.2. Central limit theorem (1/2)

Consider the limit $n \to \infty$ with the ratios $\frac{m}{n}, \frac{k}{n}, \frac{l}{n}$ fixed. We use

$$\alpha = \frac{l}{n}, \quad \beta = \frac{m-l}{n}, \quad \gamma = \frac{k-l}{n}, \quad \delta = \frac{n-m-k+l}{n}.$$

Central limit theorem for generic type II limit [Hayashi-Hora-Y., Thm 5.2.9]

In the above limit $n \to \infty$ with $\alpha+\gamma, \beta, \delta > 0$, we have

$$\lim_{n\to\infty} P_{n,m,k,l}\left[r \le \frac{X - n\mu}{\sqrt{n}\sigma} \le s\right] = \frac{1}{\sqrt{2\pi}} \int_r^s e^{-u^2/2}\, du$$

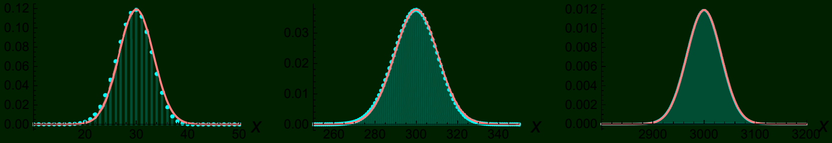
with μ and σ given by

$$\mu := \frac{1-\sqrt{D}}{2}, \quad \sigma := \sqrt{\frac{(\alpha+\gamma)\beta\delta}{D}}, \quad D := 1 - 4(\alpha\gamma + \alpha\delta + \beta\gamma).$$

We guessed the expectation value μ and the variance σ by taking a formal limit of the recursive formula $A_x p(x+1) + B_x p(x) + C_x p(x-1) = 0$ to get a differential equation

$$\frac{d}{dt} \log p(nt) \approx -\frac{t-\mu}{\sigma/\sqrt{n}} \quad (n \to \infty).$$

3.2. Central limit theorem (2/2)



Pdf $P_{n,m,k,l}[X = x]$ by cyan dots and the limit normal distribution by pink lines with $(\frac{m}{n}, \frac{k}{n}, \frac{l}{n}) = (0.4, 0.6, 0.3)$ fixed and $n = 100$ (left), 1000 (middle), 10000 (right). The limit distribution has $\mu = 0.3$ and $\sigma = 0.3354...$.

4. Concluding remarks (1/2)

Conclusions again:

- We found a discrete probability distribution $P_{n,m,k,l}$ whose pdf is a Racah ${}_4F_3$-polynomial, and cdf is a ${}_4F_3$-polynomial. $\leftarrow$ the first (?) example of distribution whose pdf and cdf are higher HG polynomials.
- Central limit theorem holds for generic type II limit:
  $n \to \infty$ with ratios $\frac{m}{n}, \frac{k}{n}, \frac{l}{n}$ fixed, satisfying a generic condition.

Topics in [HHY] not explained in this talk:

- Asymptotic analysis beyond central limit theorem [§5.5]
- Another limit of $P_{n,m,k,l}$: $n \to \infty$ with $\frac{m}{n}, k, l$ fixed. [§5.1]
- Meanings and applications in quantum information theory. [§1, §3]
- Computation using $\mathfrak{sl}_2$-Casimir operator. [§4.4, §5.5]
- q-analogue of the distribution $P_{n,m,k,l}$. [Appendix C]

Logically we started with the distinguished element

$$|\Xi_{n,m|k,l}\rangle := \left|0^l 1^{k-l}\right\rangle \otimes |\Xi_{n-k,m-l}\rangle \in \mathcal{H} = (\mathbb{C}^2)^{\otimes n}$$

and succeeded in the computation of $\langle\Xi_{n,m|k,l}|\,\pi_x\,|\Xi_{n,m|k,l}\rangle$, obtaining explicit and useful hypergeometric formulas.

However, at this moment, we do not have a conceptual reason why we were able to get nice formulas of the distribution.

Naive open problem

What property of the state $|\Xi_{n,m|k,l}\rangle$ enabled us to get nice formulas?

Is there some characterization of $|\Xi_{n,m|k,l}\rangle$ among all the normalized states of $\mathcal{H}$ so that the associated distribution can be expressed by a hypergeometric orthogonal polynomial?

(I expect some hidden "integrability" of the state $|\Xi_{n,m|k,l}\rangle$.)

Appendix: q-analogue of the distribution $P_{n,m,k,l}$

q-hypergeometric series and q-binomial coefficient:

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad [n]_q := 1+q+\cdots+q^{n-1},$$

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix}; q, z\right] := \sum_{i\geq 0} \frac{(a_1,\dots,a_r;q)_i}{(b_1,\dots,b_s;q)_i} z^i, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}.$$

[Hayashi-Hora-Y., Theorems C.3.1, C.3.2]

Let $n, m, k, l \in \mathbb{Z}$ s.t. $0 \leq 2m, k, l \leq n$, $M := m - l, N := n - m - k + l \geq 0$.
Then, for $q \in \mathbb{R}$, $0 < q < 1$, the function having the q-Racah polynomial part

$$p(x;q) := \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q}{\begin{bmatrix} n \\ m \end{bmatrix}_q} q^x \frac{[n-2x+1]_q}{[n-x+1]_q} \, {}_4\phi_3\left[\begin{matrix} q^{-x}, & q^{x-n-1}, & q^{-M}, & q^{-N} \\ & q^{-m}, & q^{m-n}, & q^{-M-N} \end{matrix}; q, q\right]$$

defines a discrete probability distribution for $x \in \{0, 1, \dots, n\}$.
Moreover, the cdf is also expressed by a ${}_4\phi_3$-polynomial:

$$\sum_{u=0}^{x} p(u;q) = \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q}{\begin{bmatrix} n \\ m \end{bmatrix}_q} \, {}_4\phi_3\left[\begin{matrix} q^{-x}, & q^{x-n}, & q^{-M}, & q^{-N} \\ & q^{-m}, & q^{m-n}, & q^{-M-N} \end{matrix}; q, q\right].$$

Thank you for your attention.