Deformation theory and vertex algebras

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Topological Field Theories, String theory and Matrix Models

§0 Introduction

(1) (Derived) deformation theory

- In classical algebraic geometry (:= scheme theory), moduli problems of algebro-geometric objects are formulated s.t. the tangent space of moduli space M at an object E describes 1st order deformations of E:

$$T_E M \simeq \{\text{1st order deformations of } E\}.$$

  The tangent space has a structure of Lie algebra, so we may restate:

  A Lie algebra controls 1-st order deformation theory.

- In derived algebraic geometry, a higher-order completion of the statement above is known as the Deligne-Drinfeld-B.Feigin-... principle:

  A dg Lie algebra controls derived deformation theory.

§0 Introduction

(2) Chiral algebras

- The theory of vertex algebras give an algebraic formulation of chiral two-dimensional conformal field theories.
- Beilinson and Drinfeld introduced a geometric reformulation of vertex algebras, which is called the theory of chiral algebras.
- Chiral algebras are defined to be “Lie objects" in the category of D-modules on Ran spaces.

(3) Question

- Now let me consider the following question:

  What do chiral algebras control?

  or

  Is there any good notion of chiral deformation theory controlled by chiral algebras?

Contents

Based on the papers

- S.Y., Jacobi complexes on the Ran space arXiv:1608.07472.
- S.Y., Boson-fermion correspondence from factorization spaces, 1611.06100.
- S.Y., Factorization spaces & moduli spaces over curves, Josai Math. Mon. (2017)

and a work in progress.

§1. Derived deformation theory & dg Lie algebras

§1.1 Classical deformation theory

In classical algebraic geometry, moduli problems are formulated as functors of certain type which are represented by schemes or stacks.

Example (Moduli problem of line bundles and Picard variety)

X: a scheme over a field k [algebraic variety over $\mathbb{C}$].
Sch: the category of schemes over k.
Set: the category of sets.
$\mathfrak{Pic}_X : (\mathsf{Sch})^{\mathsf{op}} \to \mathsf{Set}$: the Picard functor

$$\mathfrak{Pic}_X(S) = \{\mathcal{L} \mid \text{line bundle over } X \times S, \text{ flat over } S\}.$$

It is represented by the Picard scheme $\operatorname{Pic} X$ of X:

$$\mathfrak{Pic}_X(-) \simeq \operatorname{Hom}_{\mathsf{Sch}}(-, \operatorname{Pic} X).$$

Thus $\operatorname{Pic} X$ is the moduli space of line bundles on X.

I focus on infinitesimal study of moduli problem, i.e., deformation theory.
Replace **Sch** in $\mathfrak{F} : (\mathbf{Sch})^{\mathrm{op}} \to \mathbf{Set}$ by the category of "infinitesimally small affine schemes", i.e., the (opposite) category of artinian rings.

- **Com**: category of commutative rings
  $\mathbf{Com}^{\mathrm{op}}$ is equiv. to cat. of affine schemes via $R \leftrightarrow \operatorname{Spec} R$
- **Art** $\subset$ **Com**: full subcat. of local artinian k-alg. with res. field k
  [e.g. $I_n := \mathbb{C}[\varepsilon]/(\varepsilon^{n+1})$ for $k = \mathbb{C}$]

A prorepresentable functor is a functor $\mathfrak{F} : \mathbf{Art} \to \mathbf{Set}$ such that

(i) $\mathfrak{F}(k)$ is a one-point set.

(ii) $\exists$ complete local k-algebra R s.t. $\mathfrak{F} \simeq \mathrm{Hom}_{\mathbf{Com}}(R, -)$.

- The algebra R can be regarded as a formal neighborhood of the moduli space corresponding to $\mathfrak{F}$ at the point corresponding to $\mathfrak{F}(k)$.

Example (The case of Picard variety)

Fix a line bundle $\mathcal{L}_0$ on X.
Consider the following “infinitesimal" Picard functor:

$$\mathfrak{Pic}_{X,\mathcal{L}_0} : \mathbf{Art} \longrightarrow \mathbf{Set},$$
$$A \longmapsto \left\{ \mathcal{L} \,\middle|\, \begin{array}{l} \text{line bundle on } X \times \operatorname{Spec} A, \\ \text{flat over } \operatorname{Spec} A, \ \ \mathcal{L}|_{X\times\{0\}} \simeq \mathcal{L}_0 \end{array} \right\}.$$

Then $\mathfrak{Pic}_{X,\mathcal{L}_0} = \{\mathcal{L}_0\}$, $\mathfrak{Pic}_{X,\mathcal{L}_0}$ is a prorepresentable functor, and

$$\begin{aligned} \mathfrak{Pic}_{X,\mathcal{L}_0}(I_1) = \mathfrak{Pic}_{X,\mathcal{L}_0}(k[\varepsilon]/(\varepsilon^2)) &\simeq \{\text{1st order deformations of } \mathcal{L}_0\} \\ &\simeq \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{L}_0) \end{aligned}$$

- A small extension is a surjection $f : B \twoheadrightarrow A$ in Art such that $J := \mathrm{Ker}\, f$ satisfies $J^2 = 0$ in B and $J \simeq k$.
  [e.g. $I_1 = \mathbb{C}[\varepsilon]/(\varepsilon^2) \twoheadrightarrow I_0 = \mathbb{C}$]
- For a functor $\mathfrak{F} : \mathrm{Art} \to \mathrm{Set}$ and a cartesian diagram

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

  in Art, we have a map $\alpha : \mathfrak{F}(B \times_A C) \longrightarrow \mathfrak{F}(B) \times_{\mathfrak{F}(A)} \mathfrak{F}(C)$.

Proposition/Definition (Schlessinger, 1968)

A prorepres. functor $\mathfrak{F} : \mathrm{Art} \to \mathrm{Set}$ is a formal moduli functor, i.e.

(1) $\mathfrak{F}(k)$ is a one-point set.

(2) α is surjective if $B \to A$ is a small extension.

(3) α is an isomorphism if $A = k$.

§1.2. Quick intro. to derived algebraic geometry

Let me explain why classical algebraic geometry is not satisfactory and why I need to work in derived algebraic geometry.

- Most of interesting moduli problems are NOT representable by schemes. Some of them are repr. by stacks, but not all of them. By the recent progress of derived algebraic geometry, we have the notion of derived stacks, although there are several versions ([Toën-Vezzosi], [Lurie], ...).
  Derived stacks represent many moduli problems which cannot be treated in classical algebraic geometry.
- The relation between deformation theory and dg Lie algebras cannot be stated in the classical algebraic geometry.

Derived algebraic geometry is built using the theory of ∞-categories. Very roughly speaking, one replaces "sets" in the ordinary category theory by "simplicial sets" or "topological spaces".

- **sSet**: cat. of simplicial sets and simpl. maps with Kan model str.
- Each simplicial set K has homotopy groups $\pi_n K$.
- Quillen adjunction between model categories
  $|-| : \mathbf{sSet} \leftrightarrows$ (compactly generated Hausdorff spaces) $: \mathrm{Sing}$

Definition

An ∞-category is a simplicial set K s.t. $\forall\, n \in \mathbb{N}$, $\forall\, 0 < i < n$, any simplicial map $f_0 : \Lambda^n_i \to K$ admits an extension $f : \Delta^n \to K$.

- $\Lambda^n_j \subset \Delta^n$: the j-th horn of the n-simplex Δ^n $(0 \le j \le n)$.

An ∞-category is defined to be a nice simplicial set.

- Relation to the ordinary category theory:
    - **A vertex of an ∞-category K is called an object of K,**
    - **An edge of an ∞-category K is called a morphism of K.**
    - **$\mathrm{N(C)}$: nerve of a category C, an ∞-category obtained canonically.**
      objects [morphisms] of $\mathrm{N(C)}$ = objects [morphisms] of C
- Difference from the ordinary category theory:
    - **For objects V, W of an ∞-category K, $\exists$ the mapping space $\mathrm{Map}_K(V, W)$ s.t. $\pi_0 \mathrm{Map}_K(V, W) = \{$morphisms $V \to W\}$.**

Functors between ∞-categories

- A functor $K \to L$ of ∞-cat. is defined to be a simplicial map.
- **Fun**$_\infty(K, L)$: the ∞-category of functors $K \to L$ of ∞-categories.

Definition of affine derived scheme:
Recall that the category of affine schemes is equivalent to $(\mathbf{Com})^{\mathrm{op}}$.

- **sCom**: cat. of simplicial comm. alg. over the base field k.
- **Com$_\infty$**: ∞-category obtained by localizing **sCom** by the set of weak equivalences in $\mathbf{sCom} \subset \mathbf{sSet}$.
  - An object $A \in \mathrm{Com}_\infty$ is a simplicial commutative algebra depicted as $A = (\cdots A_2 \rightrightarrows A_1 \rightrightarrows A_0)$.
  - $\pi_0(A)$ is a com. ring and $\pi_n(A)$ is a $\pi_0(A)$-module.

Definition

$\mathbf{dAff}_\infty := (\mathrm{Com}_\infty)^{\mathrm{op}}$: the ∞-category of affine derived schemes.

A rough explanation of derived schemes:

- $\mathsf{Com}_\infty(X)$: ∞-category of sheaves on a topological space X with coefficients in Com_∞.
- dRgSp_∞: ∞-cat. of derived ringed spaces whose object is a pair $(X, \mathcal{O}_X)$ of a topological space X and $\mathcal{O}_X \in \mathsf{Com}_\infty(X)$.
- $\mathsf{dSch}_\infty \subset \mathsf{dRgSp}_\infty$: ∞-subcat. of derived schemes $(X, \mathcal{O}_X)$ s.t.
    - the truncation $(X, \pi_0(\mathcal{O}_X))$ is a scheme,
    - $\pi_n(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-modules $\forall n \in \mathbb{N}$.

Turn to the definition of derived stacks.

Definition (Toën-Vezzosi)

τ: (good) Grothendieck topology on dAff_∞

∞-category of derived stacks:

$$\mathsf{dSt}_\infty := \mathsf{Sh}_{\infty,\tau}(\mathsf{dAff}_\infty)^\wedge \subset \mathsf{Fun}_\infty((\mathsf{dAff}_\infty)^{\mathrm{op}}, \mathsf{Space}_\infty).$$

- Space_∞: ∞-category of spaces

Example of derived stacks ($\tau :=$ étale topology):

- Moduli space of complexes of sheaves on a fixed scheme X
- Moduli space of local systems on a fixed scheme X
- Moduli space of maps $X \to Y$ between fixed schemes X, Y

§1.3. Derived deformation theory

- Let me explain the deformation theory in derived algebraic geometry following Lurie's work [DGA-X].
  - It goes back to the deformation theory using dg-schemes due to Drinfeld, Kapranov, Kontsevich and others.
- Recall that classical deformation theory is formulated via Artin rings and their small extensions.
  Now introduce its derived analogue.

Definition (Derived analogue of local Artin algebra)

$f : B \to A$: a morphism in Com_∞

- f is called elementary if $\exists n \in \mathbb{Z}_{>0}$ and a pullback diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

- f is called small if it is a composition of elementary morphisms.
- An object $A \in \mathrm{Com}_\infty$ is called small if $A \to k$ is small.
- $\mathrm{Com}_\infty^{\mathrm{sm}} \subset \mathrm{Com}_\infty$: full subcategory spanned by small objects.

Example

The small extension $I_1 = \mathbb{C}[t]/(t^2) \twoheadrightarrow I_0 = \mathbb{C}$ in Com corresponds to $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}[0]$ in Com_∞.

Definition

- A derived deformation functor is a functor $D : \mathbf{Com}_\infty^{\mathrm{sm}} \to \mathbf{Space}_\infty$ of ∞-categories such that
  - (i) The space $D(k)$ is contractible.
  - (ii) If we have a pullback diagram in $\mathbf{Com}_\infty$ with f small

$$\begin{array}{ccc} B' & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & A \end{array}$$

    then its image under D is a pullback diagram in $\mathbf{Space}_\infty$.
- $\mathbf{Def}_\infty^{\mathrm{der}}$: the ∞-category of derived deformation functors.

Proposition

A functor $(\mathbf{dSt}_\infty)^{\mathrm{op}} \to \mathbf{Space}_\infty$ represented by a (geometric) derived stack X gives rise to a derived deformation functor.

§1.4. Dg Lie algebras

Recall the notion of a dg Lie algebra.

- It is a chain complex $(\mathfrak{g}_*, d)$ of linear spaces together with a graded Lie bracket $[\,,\,]$ such that d is a derivation.

Recall the notion of Chevalley-Eilenberg complex.

- $C_*(\mathfrak{g}_*)$: homological Chevalley-Eilenberg complex of $\mathfrak{g}_*$.
  - It is a dg associative algebra.
  - As a graded vector space $C_n(\mathfrak{g}_*) \simeq \mathrm{Sym}^n(\mathfrak{g}_*[-1])$.
- $C^*(\mathfrak{g}_*) := \mathrm{Hom}_k(C_*(\mathfrak{g}_*), k)$: cohom. Chevalley-Eilenberg cpx.
  - It is a dg commutative algebra (with the cup product).
  - It is an augmented algebra, i.e., having a dg morphism $C^*(\mathfrak{g}_*) \to k$.

The cohomological Chevalley-Eilenberg complex gives rise to a dg functor

$$C^* : \mathrm{Lie}_{\mathrm{dg}} \longrightarrow (\mathrm{Com}^{\mathrm{aug}}_{\mathrm{dg}})^{\mathrm{op}}.$$

Enhanced to ∞-categories, we have an adjunction

$$C^* : \mathrm{Lie}_{\infty} \leftrightarrows (\mathrm{Com}^{\mathrm{aug}}_{\infty})^{\mathrm{op}} : K.$$

Restricting to certain ∞-subcategories,
we have an equivalence called Koszul duality.

Theorem (Deligne-Drinfeld-Feigin-... principle [Lurie, 2011])

Assume char $k = 0$.

(1) For each $\mathfrak{g}_* \in \mathsf{Lie}_\infty$ the composition

$$\mathsf{Com}_\infty \xrightarrow{K^{\mathrm{op}}} (\mathsf{Lie}_\infty)^{\mathrm{op}} \xrightarrow{j(\mathfrak{g}_*)} \mathsf{Space}_\infty$$

is a derived deformation functor.

- $K^{\mathrm{op}} : \mathsf{Com}_\infty^{\mathrm{aug}} \leftrightarrows (\mathsf{Lie}_\infty)^{\mathrm{op}} : (C^*)^{\mathrm{op}}$: opposite of Koszul duality.
- $j : \mathsf{Lie}_\infty \to \mathsf{Fun}_\infty((\mathsf{Lie}_\infty)^{\mathrm{op}}, \mathsf{Space}_\infty)$: ∞-cat. Yoneda emb.

(2) Moreover, the resulting functor

$$\Psi : \mathsf{Lie}_\infty \longrightarrow \mathsf{Def}_\infty^{\mathrm{der}}, \quad \mathfrak{g}_* \longmapsto j(\mathfrak{g}_*) \circ K^{\mathrm{op}}$$

is an equivalence.

Examples of $\Psi : \mathsf{Lie}_\infty \xrightarrow{\sim} \mathsf{Def}^{\mathsf{der}}_\infty$

- X: scheme, $\mathcal{L}_0$: line bundle on X,
  $\mathfrak{g}_{\mathcal{L}_0} := \wedge^* \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_0, \mathcal{L}_0)$ with trivial Lie bracket.

  $$\Psi(\mathfrak{g}_{\mathcal{L}_0}) = \text{inifinitesimal Picard functor } \mathfrak{Pic}_{X,\mathcal{L}_0}.$$

- X: complex manifold,
  $\mathfrak{g}_X := T_X[-1] \otimes_{\mathcal{O}_X} \Omega^*_X$ with some dg Lie algebra str.

  $$\Psi(\mathfrak{g}_X) = \text{deformation functor of complex structures of } X.$$

§2. Chiral algebras

§2.1. The definition of chiral algebras

- X: smooth scheme over k [complex manifold].
- $\mathsf{D}_\infty(X)$: stable ∞-category of D-modules over X.
  (∞-cat. counterpart of derived category of D-modules)
- $f^! : \mathsf{D}_\infty(Y) \to \mathsf{D}_\infty(X)$: pullback of D-modules for $f : X \to Y$.

- $\mathsf{Set}^{\mathsf{fin}}$: category of finite sets.
- For $\pi : I \twoheadrightarrow J$ in $\mathsf{Set}^{\mathsf{fin}}$ we set

$$\Delta(\pi) : X^J \hookrightarrow X^I, \quad (x_j) \longmapsto (y_i), \quad y_i := x_j \text{ for } \pi(i) = j.$$

- $\Delta(\pi)^! : \mathsf{D}_\infty(X^I) \to \mathsf{D}_\infty(X^J)$: corresponding pullback.

- $\mathsf{D}_\infty(\mathrm{Ran}\,X)$: ∞-cat. of D-modules on Ran space of X,
  the colimit of $\mathsf{D}_\infty(X^I)$ over morphisms in $\mathsf{Set}^{\mathrm{fin}}$.
    - Its object is a collection $\mathcal{M} = \{\mathcal{M}_I \in \mathsf{D}_\infty(X^I)\}_{I \in \mathsf{Set}^{\mathrm{fin}}}$ together with equivalences $\Delta(\pi)^!(\mathcal{M}_I) \simeq \mathcal{M}_J$ $\forall \pi : I \twoheadrightarrow J$.
- $(\Delta^{\mathrm{main}})_* : \mathsf{D}_\infty(X) \to \mathsf{D}_\infty(\mathrm{Ran}\,X)$.

The chiral tensor product $\otimes^{\mathrm{ch}}$ on $\mathsf{D}_\infty(\mathrm{Ran}\,X)$:
for $\mathcal{M} = \{\mathcal{M}_I\}_I$ and $\mathcal{N} = \{\mathcal{N}_I\}_I$,

$$\mathcal{M} \otimes^{\mathrm{ch}} \mathcal{N} := \{(\mathcal{M} \otimes^{\mathrm{ch}} \mathcal{N})_I\}_I, \quad (\mathcal{M} \otimes^{\mathrm{ch}} \mathcal{N})_I := \bigoplus_{I = J \sqcup K} \mathcal{M}_J \otimes \mathcal{N}_K.$$

Definition

- $\mathrm{Lie}^{\mathrm{ch}}_{\infty}(\mathrm{Ran}\,X)$: ∞-cat. of simplicial Lie algebras in $\mathrm{D}_{\infty}(\mathrm{Ran}\,X)$
- $\mathrm{Lie}^{\mathrm{ch}}_{\infty}(X) \subset \mathrm{Lie}^{\mathrm{ch}}_{\infty}(\mathrm{Ran}\,X)$: ∞-subcategory spanned by objects in the image of $(\Delta^{\mathrm{main}})_* : \mathrm{D}_{\infty}(X) \to \mathrm{D}_{\infty}(\mathrm{Ran}\,X)$.
  Its object is called a chiral (Lie) algebra on X.

Theorem (Beilinson-Drinfeld (late 1990s))

$X = \Sigma$: smooth curve over $\mathbb{C}$ [Riemann surface]
A VOA V gives rise to a chiral algebra $\mathcal{A}_V$ on Σ .

Construction of a chiral algebra from a VOA:

- (V, Y): VOA w. state-field corresp. $Y(-, z) : V \to \mathrm{End}(V)[[z^{\pm 1}]]$
- Virasoro element in V $\twoheadrightarrow$ action of $\mathrm{Aut}\,\mathbb{C}[[z]]$ on V.
- $\mathcal{A}ut_\Sigma$: the principal $\mathrm{Aut}\,\mathbb{C}[[z]]$-bundle on Σ with the stalk $\mathcal{A}ut_{\Sigma,x} \simeq \mathrm{Aut}\,\mathcal{O}_{\Sigma,x} \simeq \mathrm{Aut}\,\mathbb{C}[[z]]$
- $\mathcal{V} := \mathcal{A}ut_\Sigma \times_{\mathrm{Aut}\,\mathbb{C}[[z]]} V$ with the left D-module str. corresponding to $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_\Sigma$, $\nabla_{\partial_z} := \partial_z + L_{-1}$
- Define $\mu : \mathcal{V} \otimes^{\mathrm{ch}} \mathcal{V} \to \mathcal{V}$ by

$$\mu(f(z,w)A \otimes B) := f(z,w)Y(A, z-w)B \text{ mod } V[[z,w]].$$

$(\mathcal{V}, \mu)$ gives rise to a chiral algebra with chiral Lie bracket μ.

§2.2. Chiral Koszul duality

Theorem ([Francis-Gaitsgory, 2011])

The functor C^{ch} of taking chiral Chevalley-Eilenberg cpx. gives

$$C^{\mathrm{ch}} : \mathrm{Lie}^{\mathrm{ch}}_{\infty}(\mathrm{Ran}\,X) \xrightarrow{\sim} \left(\mathrm{Com}^{\mathrm{ch}}_{\infty}(\mathrm{Ran}\,X)\right)^{\mathrm{op}}.$$

Restricting to $\mathrm{Lie}^{\mathrm{ch}}_{\infty}(X)$ of chiral algebras, we have an equivalence

$$C^{\mathrm{ch}} : \mathrm{Lie}^{\mathrm{ch}}_{\infty}(X) \xrightarrow{\sim} \left(\mathrm{Fact}_{\infty}(X)\right)^{\mathrm{op}}.$$

$\mathrm{Fact}_{\infty}(X) \subset \mathrm{Com}^{\mathrm{ch}}_{\infty}(\mathrm{Ran}\,X)$: ∞-subcat. of factorization algebras, i.e., those objects $\mathcal{B}$ s.t. for any decomposition $I = J \sqcup K$ in $\mathrm{Set}^{\mathrm{fin}}$ the algebra structure map $\mathcal{B}_J \otimes^{\mathrm{ch}} \mathcal{B}_K \to \mathcal{B}_I$ is an equivalence.

§3. Chiral deformation theory

Summary of the key points so far.

- Deligne-Drinfeld-Feigin-... principle:

$$\Psi : \mathsf{Lie}_\infty \xrightarrow{\sim} \mathsf{Def}^{\mathsf{der}}_\infty, \quad \mathfrak{g}_* \longmapsto j(\mathfrak{g}_*) \circ K^{\mathsf{op}}$$

  with $\mathsf{Def}^{\mathsf{der}}_\infty \subset \mathsf{Fun}_\infty(\mathsf{Com}^{\mathsf{sm}}_\infty, \mathsf{Set})$ derived deformation functors and K the Koszul duality in

$$C^* : \mathsf{Lie}_\infty \leftrightarrows (\mathsf{Com}^{\mathsf{aug}}_\infty)^{\mathsf{op}} : K$$

- Chiral Koszul duality

$$C^{\mathsf{ch}} : \mathsf{Lie}^{\mathsf{ch}}_\infty(X) \leftrightarrows (\mathsf{Fact}_\infty(X))^{\mathsf{op}} : K^{\mathsf{ch}}$$

Now I want to make a chiral analogue of $\mathsf{Def}^{\mathsf{der}}_\infty$ and Ψ.

First we need “chiral moduli theory".

Definition (slight generalization of [Kapranov-Vasserot, '04])

X: a scheme over k.

A factorization space $F = \{F_I\}_{I \in \mathsf{Set}^{\mathrm{fin}}}$ is a family of derived stacks over X^I together with isomorphisms

$$\Delta(\pi)^* F_I \xrightarrow{\sim} F_J, \quad u(\pi)^* \Big(\prod_{j \in J} F_{\pi^{-1}(j)} \Big) \xrightarrow{\sim} u(\pi)^* F_I$$

for $\pi : I \twoheadrightarrow J$.

- $\Delta(\pi) : X^J \hookrightarrow X^I$, $(x_j) \mapsto (y_i)$, $y_i := x_j$ for $i \in \pi^{-1}(j)$.
- $u(\pi) : \{(x_i) \in X^I \mid x_i \neq x_{i'} \text{ if } \pi(i) \neq \pi(i')\} \hookrightarrow X^I$.

Proposition/Definition (Chiral analogue of Schlessinger)

Restriction of a factorization space F to $(\mathsf{Com}_\infty^{\mathsf{sm}})^{\mathsf{op}} \subset \mathsf{dAff}_\infty$ gives rise to a chiral deformation functor, i.e., a functor

$$D : \mathsf{Fact}_\infty^{\mathsf{sm}}(X) \longrightarrow \mathsf{Space}_\infty$$

of ∞-categories such that

(i) The space $D(k)$ is contractible.

(ii) If we have a pullback diagram in $\mathsf{Fact}_\infty^{\mathsf{sm}}(X)$ with f small

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow f \\ A' & \longrightarrow & A \end{array}$$

then its image under D is a pullback diagram in Space_∞.

$\mathsf{Def}_\infty^{\mathsf{ch}}(X)$: the ∞-category of chiral deformation functors.

Definition

X: a smooth variety over k.

$f : \mathcal{B} \to \mathcal{A}$: a morphism in $\mathbf{Fact}_\infty(X) \subset \mathbf{Com}^{\mathrm{ch}}_\infty(\mathrm{Ran}\, X)$.

- f is called elementary if $\exists n \in \mathbb{Z}_{>0}$ and a pullback diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{f} & \mathcal{A} \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n] \end{array}$$

- f is called small if it is a composition of elementary morphisms.
- An object $\mathcal{A} \in \mathbf{Fact}_\infty(X)$ is called small if $\mathcal{A} \to k$ is small.
- $\mathbf{Fact}^{\mathrm{sm}}_\infty(X) \subset \mathbf{Fact}_\infty(X)$: full subcat. spanned by small objects.

§3.2. Main theorem

Theorem

(1) For each $\mathcal{A} \in \mathsf{Lie}^{\mathsf{ch}}_{\infty}(X)$ the composition

$$\mathfrak{A} : \mathsf{Fact}_{\infty}(X) \xrightarrow{(K^{\mathsf{ch}})^{\mathsf{op}}} (\mathsf{Lie}^{\mathsf{ch}}_{\infty}(X))^{\mathsf{op}} \xrightarrow{j(\mathcal{A})} \mathsf{Space}_{\infty}$$

is a derived deformation functor.

- $C^{\mathsf{ch}} : \mathsf{Lie}^{\mathsf{ch}}_{\infty}(X) \leftrightarrows (\mathsf{Fact}_{\infty}(X))^{\mathsf{op}} : K^{\mathsf{ch}}$: chiral Koszul duality.
- $j : \mathsf{Lie}^{\mathsf{ch}}_{\infty}(X) \to \mathsf{Fun}_{\infty}((\mathsf{Lie}^{\mathsf{ch}}_{\infty}(X))^{\mathsf{op}}, \mathsf{Space}_{\infty})$: Yoneda embedding.

(2) The resulting functor

$$\Psi^{\mathsf{ch}} : \mathsf{Lie}^{\mathsf{ch}}_{\infty}(X) \longrightarrow \mathsf{Def}^{\mathsf{ch}}_{\infty}(X), \quad \mathcal{A} \longmapsto \mathfrak{A}$$

is an equivalence.

Example (Beilinson-Drinfeld Grassmannian and affine VOA)

Σ: smooth curve, G: reductive algebraic group.
$\mathrm{Gr}(\Sigma, G) = \{\mathrm{Gr}(\Sigma, G)_I\}_{I \in \mathsf{Set}^{\mathsf{fin}}}$: Beilinson-Drinfeld Grassmannian

$$\mathrm{Gr}(\Sigma, G)_I := \left\{ (\mathcal{P}, \{s_i\}_{i \in I}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{P} : \text{principal } G\text{-bundle on } \Sigma, \\ s_i \in \Sigma, \\ \varphi : \text{trivialization of } \mathcal{P}|_{\Sigma \setminus \{s_i\}_{i \in I}} \end{array} \right\}$$

$\mathrm{Gr}(\Sigma, G)$: factorization space $\twoheadrightarrow$ chiral deformation functor $\mathfrak{Gr}(\Sigma, G)$

$$\Psi^{\mathsf{ch}}(\mathfrak{Gr}(\Sigma, G)) \simeq \text{affine VOA } \widehat{\mathfrak{g}} \text{ with level } 0.$$

$G = \mathrm{GL}(1)$: $\Psi^{\mathsf{ch}}(\mathfrak{Gr}(\Sigma, \mathrm{GL}(1))) \simeq$ Heisenberg VOA.
$\exists$ morphism $\mathrm{Gr}(\Sigma, \mathrm{GL}(1))_I \to \mathrm{Pic}(\Sigma)$.

Example (Moduli of stable curves and Virasoro VOA)

Σ: smooth projective curve with genus ≥ 2

$M(\Sigma) = \{M(\Sigma)_I\}_{I \in \mathsf{Set}^{\mathrm{fin}}}$; factorization space of stable curves

$$M(\Sigma)_I := \left\{ (\Sigma', \{s'_i\}_{i\in I}, \{s_i\}_{i\in I}, \varphi) \;\middle|\; \begin{array}{l} \Sigma' : \text{smooth projective curve}, \\ s'_i \in \Sigma', s_i \in \Sigma, \\ \varphi : \Sigma \setminus \{s_i\} \xrightarrow{\sim} \Sigma' \setminus \{s'_i\} \end{array} \right\}$$

$M(\Sigma) \rightsquigarrow$ chiral deformation functor $\mathfrak{M}(\Sigma)$

$$\Psi^{\mathrm{ch}}(\mathfrak{M}(\Sigma)) \simeq \text{Virasoro VOA with } c = 0,\ h = 0.$$

Example (Moduli of maps from elliptic curves and CDO)

E: elliptic curve, X: smooth variety.

$T^*\mathrm{Map}(E,X) = \{M_I\}_I$: cotangent factoriz. space of maps $E \to X$.

$$M_I := \left\{ (f, \{s_i\}_{i\in I}, \varphi) \;\middle|\; \begin{array}{l} f : E \to X,\ s_i \in E, \\ \varphi : \text{trivialiation of } f^*(T^*X)|_{E\setminus\{s_i\}} \end{array} \right\}$$

$T^*\mathrm{Map}(E,X) \twoheadrightarrow$ chiral deformation functor $\mathfrak{T}^*\mathfrak{M}ap(E,X)$.

$$\Psi^{\mathrm{ch}}(\mathfrak{T}^*\mathfrak{M}ap(E,X)) \simeq \text{sheaf of chiral differential operators on } X.$$

Speculation

$\exists$? Relation to elliptic genus of X...

Summary

- Introduced the notion of chiral deformation functor, typical examples of which come from factorization spaces.
- Established Deligne-Drinfeld-Feigin-...-type principle:

$$\Psi^{\mathrm{ch}} : \mathrm{Lie}^{\mathrm{ch}}_{\infty}(X) \xrightarrow{\sim} \mathrm{Def}^{\mathrm{ch}}_{\infty}(X).$$

- Examples related to typical VOAs.
  - Beilinson-Drinfeld Grassmannian $\mathrm{Gr}(X,G) \twoheadrightarrow$ affine VOA.
  - Moduli of stable curves $M(\Sigma) \twoheadrightarrow$ Virasoro VOA.
  - Moduli of maps from elliptic curve $T^* \mathrm{Map}(E,X) \twoheadrightarrow$ CDO of X.

Thank you