Enriched categories and their centers

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Enriched monoidal categories I: centers

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An **enriched category** ${}^{\mathcal{A}|}\mathcal{L}$ consists of the following data:

- A monoidal category $\mathcal{A}$, called the base category or the **background category**;
- a set $\mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L})$, whose elements are called objects;
- an object ${}^{\mathcal{A}|}\mathcal{L}(x,y) \in \mathcal{A}$ for each $x, y \in \mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L})$, called the hom space;
- a morphism $\circ \colon {}^{\mathcal{A}|}\mathcal{L}(y,z) \otimes {}^{\mathcal{A}|}\mathcal{L}(x,y) \to {}^{\mathcal{A}|}\mathcal{L}(x,z)$ in $\mathcal{A}$ for each $x, y, z \in \mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L})$, called the composition;
- a morphism $1_x \colon \mathbb{1} = \mathbb{1}_{\mathcal{A}} \to {}^{\mathcal{A}|}\mathcal{L}(x,x)$ for each $x \in \mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L})$, called the identity;

and the composition is associative and unital.

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- Given a 1d quantum liquid phase (gapped/gapless topological phase with/without symmetries), its observables in the long wave length limit form an enriched category, called the **topological skeleton** of the phase. [Kong-Zheng: A mathematical theory of gapless edges of 2d topological orders, 2019]

An important class of examples are given by the so-called **canonical construction** [Kelly: Adjunction for enriched categories, 1969].

Let $\mathcal{B}$ be a monoidal category and $\mathcal{M}$ be a left $\mathcal{B}$-module. We say $\mathcal{M}$ is **enriched** in $\mathcal{B}$ if the right adjoint of $- \odot x\colon \mathcal{B} \to \mathcal{M}$ exists for each $x \in \mathcal{M}$, denoted by $[x, -]\colon \mathcal{M} \to \mathcal{B}$. The object $[x, y] \in \mathcal{B}$ is called the **internal hom** of x and y.

By the adjunctions $(- \odot x) \dashv [x, -]$ we have:

- for each $x \in \mathcal{M}$, there is a morphism $1_x \colon \mathbb{1} \to [x, x]$ in $\mathcal{B}$ induced by the identity morphism $1_x \colon \mathbb{1} \odot x = x \to x$ in $\mathcal{M}$;
- for every $x, y, z \in \mathcal{M}$, there is a morphism $\circ \colon [y, z] \otimes [x, y] \to [x, z]$ induced by

$$([y, z] \otimes [x, y]) \odot x \simeq [y, z] \odot ([x, y] \odot x) \to [y, z] \odot y \to z.$$

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Then there is an enriched category denoted by ${}^{\mathcal{B}}\mathcal{M}$, where:

- the set of objects is the same as $\mathcal{M}$, i.e., $\mathrm{Ob}({}^{\mathcal{B}}\mathcal{M}) := \mathrm{Ob}(\mathcal{M})$;
- the hom spaces are given by internal homs, i.e., ${}^{\mathcal{B}}\mathcal{M}(x, y) := [x, y] \in \mathcal{B}$;
- the composition and identity morphisms are given by the above constructions.

Another way to construct enriched categories is given by the operation called the **changing of the background category**:

- Recall that a lax-monoidal funtor $F\colon \mathcal{A} \to \mathcal{B}$ between two monoidal categories is a functor $F\colon \mathcal{A} \to \mathcal{B}$ equipped with a natural transformation $F^2\colon F(-) \otimes F(-) \to F(- \otimes -)$ and a morphism $F^0\colon \mathbb{1}_{\mathcal{B}} \to F(\mathbb{1}_{\mathcal{A}})$ satisfying the usual conditions for a monoidal functor.
- Suppose ${}^{\mathcal{A}|}\mathcal{L}$ is an $\mathcal{A}$-enriched category and $F\colon \mathcal{A} \to \mathcal{B}$ is a lax-monoidal functor. Then F induces a $\mathcal{B}$-enriched category ${}^{\mathcal{B}|}\mathcal{L}$ with the same objects as ${}^{\mathcal{A}|}\mathcal{L}$ and the hom spaces ${}^{\mathcal{B}|}\mathcal{L}(x, y) := F({}^{\mathcal{A}|}\mathcal{L}(x, y))$.
- In particular, when $\mathcal{B} = \mathrm{Set}$ and $F = \mathcal{A}(\mathbb{1}, -)$, the ordinary category $\mathcal{L} := {}^{\mathrm{Set}|}\mathcal{L}$ is called the **underlying category** of ${}^{\mathcal{A}|}\mathcal{L}$.

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For enriched categories, changing the background should also be viewed as a 'functor'. So we have found a new definition of an enriched functor.

An **enriched functor** ${}^{|}F\colon {}^{\mathcal{A}|}\mathcal{L} \to {}^{\mathcal{B}|}\mathcal{M}$ consists of the following data:

- a lax-monoidal functor $\hat{F}\colon \mathcal{A} \to \mathcal{B}$, called the **background changing functor**;
- a map $F\colon \mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L}) \to \mathrm{Ob}({}^{\mathcal{B}|}\mathcal{M})$;
- a morphism ${}^{|}F_{x,y}\colon \hat{F}({}^{\mathcal{A}|}\mathcal{L}(x,y)) \to {}^{\mathcal{B}|}\mathcal{M}(F(x),F(y))$ in $\mathcal{B}$ for each $x, y \in {}^{\mathcal{A}|}\mathcal{L}$;

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In particular, if $\mathcal{B} = \mathcal{A}$ and $\hat{F} = \mathrm{id}_{\mathcal{A}}$, such an enriched functor ${}^{|}F\colon {}^{\mathcal{A}|}\mathcal{L} \to {}^{\mathcal{A}|}\mathcal{M}$ is called an $\mathcal{A}$-functor. This is the traditional definition of an enriched functor (for example, see [Kelly: Basic concepts of enriched category theory, 1982]).

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Remark: Every enriched functor ${}^{|}F\colon {}^{\mathcal{A}|}\mathcal{L} \to {}^{\mathcal{B}|}\mathcal{M}$ can be decomposed as ${}^{\mathcal{A}|}\mathcal{L} \to {}^{\mathcal{B}|}\mathcal{L} \to {}^{\mathcal{B}|}\mathcal{M}$, where the first functor only changes the background, and the second is a $\mathcal{B}$-functor.

The **underlying functor** $F\colon \mathcal{L} \to \mathcal{M}$ of an enriched functor ${}^{|}F\colon {}^{\mathcal{A}|}\mathcal{L} \to {}^{\mathcal{B}|}\mathcal{M}$ is defined by the map $F\colon \mathrm{Ob}(\mathcal{L}) \to \mathrm{Ob}(\mathcal{M})$ and the map

$$F_{x,y}\colon \mathcal{L}(x,y) = \mathcal{A}(\mathbb{1}, {}^{\mathcal{A}|}\mathcal{L}(x,y)) \xrightarrow{\hat{F}} \mathcal{B}(\hat{F}(\mathbb{1}), \hat{F}({}^{\mathcal{A}|}\mathcal{L}(x,y))) \xrightarrow{\mathcal{B}(\hat{F}^0, {}^{|}F_{x,y})} \mathcal{B}(\mathbb{1}, {}^{\mathcal{B}|}\mathcal{M}(F(x),F(y))) = \mathcal{M}(F(x),F(y)).$$

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Similarly, an **enriched natural transformation** is defined by a background changing natural transformation, which is monoidal, and an underlying natural transformation.

The enriched categories, enriched functors and enriched natural transformations form a 2-category ${}^{\mathrm{lax}|}\mathbf{ECat}$. It is a symmetric monoidal 2-category with the Cartesian product defined by

- $\mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L} \times {}^{\mathcal{B}|}\mathcal{M}) := \mathrm{Ob}({}^{\mathcal{A}|}\mathcal{L}) \times \mathrm{Ob}({}^{\mathcal{B}|}\mathcal{M})$;
- $({}^{\mathcal{A}|}\mathcal{L} \times {}^{\mathcal{B}|}\mathcal{M})((x, y), (x', y')) := ({}^{\mathcal{A}|}\mathcal{L}(x, x'), {}^{\mathcal{B}|}\mathcal{M}(y, y')) \in \mathcal{A} \times \mathcal{B}$.

The 2-category of enriched categories, enriched functors with strongly monoidal background changing functors and enriched natural transformations is denoted by **ECat**.

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The 2-category of enriched categories, enriched functors with strongly monoidal background changing functors and enriched natural transformations is denoted by **ECat**.

Remark: Fix a monoidal category $\mathcal{A}$. Then there is a 2-category ${}^{\mathcal{A}|}\mathbf{Cat}$ of $\mathcal{A}$-enriched categories, $\mathcal{A}$-functors and $\mathcal{A}$-natural transformations. This is the traditional definition of a 2-category of enriched categories (for example, see [Kelly: Basic concepts of enriched category theory, 1982]).

We define an **enriched monoidal category**, an **enriched braided monoidal category** and an **enriched symmetric monoidal category** as an E_1-algebra, an E_2-algebra and an E_3-algebra in the symmetric monoidal 2-category **ECat**.

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Intuitions for E_n-algebras

- We can imagine an E_n-algebra as a collection of particles living in an n-dimensional space. So these particles can be fused together in n directions.
- Intuitively, an E_n-algebra in a symmetric monoidal ∞-category C is an object equipped with n compatible multiplications.
- In particular, an E_0-algebra in C is an object $x \in \mathrm{C}$ equipped with a morphism $\mathbb{1} \to x$.

Examples:

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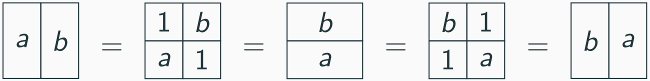

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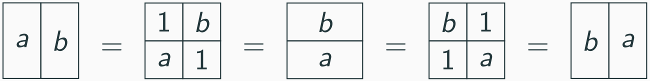

(b) In the symmetric monoidal 2-category **Cat** of categories, functors and natural tranformations, an E_1-algebra is a monoidal category, an E_2-algebra is a braided monoidal category and an E_n-algebra for $n \geq 3$ is a symmetric monoidal category.

Taking the underlying category ${}^{\mathcal{A}|}\mathcal{L} \mapsto \mathcal{L}$ is a symmetric monoidal 2-functor $\mathbf{ECat} \mapsto \mathbf{Cat}$, thus it maps an E_n-algebra to an E_n-algebra. Similarly, taking the background category ${}^{\mathcal{A}|}\mathcal{L} \mapsto \mathcal{A}$ is also a symmetric monoidal 2-functor $\mathbf{ECat} \mapsto \mathrm{Alg}_{E_1}(\mathbf{Cat})$.

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(a) Given an enriched monoidal category ${}^{\mathcal{A}|}\mathcal{L}$, its underlying category $\mathcal{L}$ is a monoidal category, and its background category $\mathcal{A}$ is a braided monoidal category.

(b) Given an enriched braided monoidal category ${}^{\mathcal{A}|}\mathcal{L}$, its underlying category $\mathcal{L}$ is a braided monoidal category, and its background category $\mathcal{A}$ is a symmetric monoidal category.

(c) Given an enriched symmetric monoidal category ${}^{\mathcal{A}|}\mathcal{L}$, its underlying category $\mathcal{L}$ and background category $\mathcal{A}$ are both symmetric monoidal categories.

The canonical construction can be extended to a symmetric monoidal 2-functor from a 2-category of monoidal categories and modules (we do not give details here) to **ECat**. Thus it maps an E_n-algebra to an E_n-algebra.

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1. Let $\mathcal{B}$ be a monoidal category and $\mathcal{M}$ be a left $\mathcal{B}$-module that is enriched in $\mathcal{B}$. The canonical construction gives an enriched category ${}^{\mathcal{B}}\mathcal{M}$.

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2. Let $\mathcal{C}$ be a braided monoidal category. A **monoidal left $\overline{\mathcal{C}}$-module** is a monoidal category $\mathcal{M}$ equipped with a braided monoidal functor $\varphi\colon \overline{\mathcal{C}} \to \mathfrak{Z}_1(\mathcal{M})$. If $\mathcal{M}$ is enriched in $\overline{\mathcal{C}}$, the canonical construction ${}^{\mathcal{C}}\mathcal{M}$ is an enriched monoidal category.

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3. Let $\mathcal{E}$ be a symmetric monoidal category. A **braided left $\mathcal{E}$-module** is a braided monoidal category $\mathcal{M}$ equipped with a symmetric monoidal functor $\phi\colon \mathcal{E} \to \mathfrak{Z}_2(\mathcal{M})$. If $\mathcal{M}$ is enriched in $\mathcal{E}$, the canonoical construction ${}^{\mathcal{E}}\mathcal{M}$ is an enriched braided monoidal category. Moreover, ${}^{\mathcal{E}}\mathcal{M}$ is symmetric if $\mathcal{M}$ is symmetric.

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Example: centers in the symmetric monoidal 1-category Vec

(a) The center of a vector space V is given by the endomorphism algebra $\mathrm{End}(V)$: a linear map $X \otimes V \to V$ is equivalent to a linear map $X \to \mathrm{Hom}(V, V) = \mathrm{End}(V)$.

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(b) In Vec, the E_1-center of a $\Bbbk$-algebra A is the usual center

$$Z(A) := \{z \in A \mid za = az, \forall a \in A\}.$$

Indeed, if $f : X \otimes A \to A$ is both an algebra homomorphism and a left unital action, then

$$f(x \otimes 1)a = f(x \otimes 1)f(1 \otimes a) = f(x \otimes a) = f(1 \otimes a)f(x \otimes 1) = af(x \otimes 1).$$

Example: centers in the symmetric monoidal 2-category **Cat**:

(a) The center of a category $\mathcal{X}$ is given by the endofunctor category $\mathrm{End}(\mathcal{X})$.

(b) The E_1-center of a monoidal category $\mathcal{A}$ is given by the Drinfeld center (or monoidal center) $\mathfrak{Z}_1(\mathcal{A})$, whose objects are objects in $\mathcal{A}$ equipped with a half-braiding.

(c) The E_2-center of a braided monoidal category $\mathcal{C}$ is given by the Müger center (or symmetric center)

$$\mathfrak{Z}_2(\mathcal{C}) := \{x \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \mathrm{id}_{x \otimes y}, \forall y \in \mathcal{C}\}.$$

Here we give the centers of enriched (monoidal) categories obtained from the canonical construction:

1. Suppose $\mathcal{B}$ is a rigid monoidal category and $\mathcal{M}$ is a left $\mathcal{B}$-module. Then $\mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{M})$ is a left $\mathfrak{Z}_1(\mathcal{B})$-module. Moreover, $\mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{M})$ is a left monoidal $\overline{\mathfrak{Z}_1(\mathcal{B})}$-module, i.e., there is a braided monoidal functor $\overline{\mathfrak{Z}_1(\mathcal{B})} \to \mathfrak{Z}_1(\mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{M}))$. The center of ${}^{\mathcal{B}}\mathcal{M}$ is given by the enriched monoidal category ${}^{\mathfrak{Z}_1(\mathcal{B})}\mathrm{Fun}_{\mathcal{B}}(\mathcal{M},\mathcal{M})$.

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3. The E_2-center of an enriched braided monoidal category ${}^{\mathcal{A}|}\mathcal{L}$ is given by ${}^{\mathcal{A}|}\mathfrak{Z}_2(\mathcal{L})$.

In particular, we have the following results.

(a) Let $\mathcal{A}$ be an indecomposable multi-fusion category and $\mathcal{L}$ be a finite semisimple left $\mathcal{A}$-module. Then

$$\mathfrak{Z}_1(\mathfrak{Z}_0({}^{\mathcal{A}}\mathcal{L})) \simeq \mathfrak{Z}_1({}^{\mathfrak{Z}_1(\mathcal{A})}\mathsf{Fun}_{\mathcal{A}}(\mathcal{L},\mathcal{L})) \simeq {}^{\mathrm{Vec}}\mathrm{Vec} = \mathrm{Vec}.$$

(b) Let $\mathcal{C}$ be a nondegenerate braided fusion category and $\mathcal{M}$ be an indecomposable multi-fusion left $\overline{\mathcal{C}}$-module defined by a braided tensor functor $\varphi\colon \overline{\mathcal{C}} \to \mathfrak{Z}_1(\mathcal{M})$. Then

$$\mathfrak{Z}_2(\mathfrak{Z}_1({}^{\mathcal{C}}\mathcal{M})) \simeq \mathfrak{Z}_2({}^{\mathrm{Vec}}\mathfrak{Z}_2(\varphi)) \simeq {}^{\mathrm{Vec}}\mathrm{Vec} = \mathrm{Vec}.$$

The center of a center is trivial. Its physical meaning is that the bulk of a bulk is trivial.

Thanks!