Introduction to Macdonald polynomials
— from viewpoint of quantum integrable systems

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§1. Introduction

In this workshop, several speakers will talk on Macdonald polynomials.

(In particular, Yamaguchi-san, Kanno-sensei, Panupong-san on Wednesday.)

So I will give an introduction, serving preliminaries of those talks.

1. Macdonald polynomials of type A and quantum integrable systems
    1.1. Schur polynomials
    1.2. Jack polynomials
    1.3. Calogero-Sutherland quantum integrable systems
    1.4. Ruijsenaars elliptic quantum integrable systems
    1.5. Macdonald polynomials of type A
2. Macdonald-Cherednik theory
3. Concluding remarks

§1.1. Schur polynomial [I.G. Macdonald (1995)]

- Notations for symmetric (Laurent) polynomials
  - $x = (x_1, \dots, x_n)$: a set of commuting n-variables.
  - $\mathbb{Z}[x^{\pm 1}]^{S_n} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$: symmetric Laurent polynomials.
  - $P_+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$: weights of type A_{n-1}.
- $s_\lambda(x) \in \mathbb{Z}[x^{\pm 1}]^{S_n}$: Schur polynomial associated to $\lambda \in P_+$.
  - Explicit form:

$$s_\lambda(x) = \frac{\det(x^{\lambda+\delta})}{\det(x^\delta)} = \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{bmatrix} \Big/ \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & & \vdots \\ x_1^0 & x_2^0 & \cdots & x_n^0 \end{bmatrix}.$$

  - They form an orthogonal basis of $\mathbb{C}[x^{\pm 1}]^{S_n}$ for

$$(f, g)_1 := \frac{1}{n!} \int_T \overline{f(x)} g(x) w_1(x)\, dx, \quad w_1(x) := \prod_{i \neq j} (1 - x_i / x_j)$$

with $T := \{x \in \mathbb{C}^n \mid |x_1| = \cdots = |x_n| = 1\}$, $dx := \prod_{i=1}^n dx_i / (2\pi\sqrt{-1}x_i)$.

§1.1. Schur polynomial (cont.)

- For $\lambda \in P_+ \cap \mathbb{N}^n = \{$partitions$\}$, $s_\lambda(x) \in \mathbb{Z}[x]^{S_n}$, a symmetric polynomial.
  - Any $\mu \in P_+ \subset \mathbb{Z}^n$ can be written as $\mu = \lambda - (\ell^n)$, λ: partition, $\ell \in \mathbb{N}$, and $s_\mu(x) = s_\lambda(x)/(x_1 \cdots x_n)^\ell$.
  - $\{s_\lambda(x) \mid \lambda$: partition$\}$ is a basis of $\mathbb{Z}[x]^{S_n}$.
- For $\lambda \in P_+ \cap \mathbb{N}^n$, $s_\lambda(x)$ has a triangular expansion

$$s_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda,\mu} m_\mu, \quad c_{\lambda,\mu} \in \mathbb{Z}.$$

  - $m_\lambda(x) := \sum_{\mu \in S_n.\lambda} x^\mu = \sum_\mu x_1^{\mu_1} \cdots x_n^{\mu_n}$: another basis of $\mathbb{Z}[x]^{S_n}$.
  - $\mu \le \lambda :\Leftrightarrow |\mu|_n = |\lambda|_n$ and $|\mu|_k \le |\lambda|_k$ $(1 \le k < n)$, $|\lambda|_k := \sum_{i=1}^k \lambda_i$.
  - Small examples:

$$s_{(1)} = m_{(1)}, \quad s_{(2)} = m_{(2)} + m_{(1,1)}, \quad s_{(1,1)} = m_{(1,1)},$$
$$s_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1^3)}, \quad s_{(2,1)} = m_{(2,1)} + 2m_{(1^3)}, \quad s_{(1^3)} = m_{(1^3)}.$$

- s_λ is uniquely determined by $\begin{cases} \text{triangular expansion} & s_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{C} m_\mu, \\ \text{orthogonality} & (s_\lambda, s_\mu)_1 = \delta_{\lambda,\mu}. \end{cases}$

§1.2. Jack polynomials [H. Jack (1972), R.P. Stanley (1989), I.G. Macdonald (1995)]

- $J_\lambda(x;\beta) \in \mathbb{C}[x]^{S_n}$: Jack polynomial assoc. to partition λ and parameter $\beta \in \mathbb{C}$.
  - Uniquely determined by $\begin{cases} \text{triangular expansion} & J_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{C} m_\mu, \\ \text{orthogonality} & (J_\lambda, J_\mu)_\beta \propto \delta_{\lambda,\mu}, \end{cases}$

$$(f,g)_\beta := \frac{1}{n!} \int_T \overline{f(x)} g(x) w_\beta(x)\, dx, \quad w_\beta(x) := \prod_{i \neq j} (1 - x_i/x_j)^\beta.$$

  - $J_\lambda(x;\beta=1) = s_\lambda(x)$, i.e., β-deformation of Schur polynomial.
  - Small examples:

$$J_{(1)} = m_{(1)}, \quad J_{(2)} = m_{(2)} + \frac{2\beta}{1+\beta} m_{(1,1)}, \quad J_{(1,1)} = m_{(1,1)}, \quad J_{(1^3)} = m_{(1^3)},$$

$$J_{(3)} = m_{(3)} + \frac{3\beta}{2+\beta} m_{(2,1)} + \frac{6\beta^2}{(1+\beta)(2+\beta)} m_{(1^3)}, \quad J_{(2,1)} = m_{(2,1)} + \frac{6\beta}{1+2\beta} m_{(1^3)}.$$

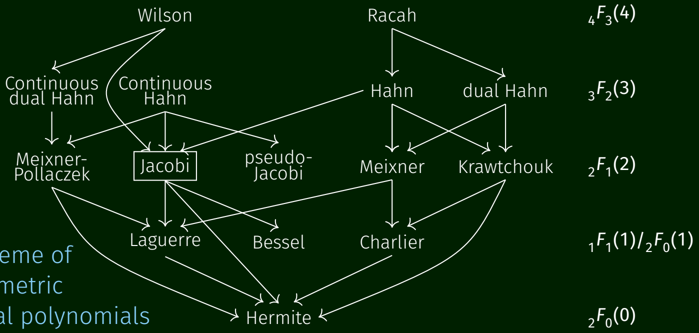
  - Unlike $s_\lambda(x) = \det(x^{\lambda+\delta}) / \det(x^\delta)$, $J_\lambda(x;\beta)$ has no simple formula. ($\exists$ vertex operator / integral presentation, a bit complicated one.)

§1.2. Jack polynomials (cont.)

- $J_l(x;\beta) = J_\lambda(x;\beta)$: Jack polynomial of one-variable ($n = 1$, $l \in P_+ \cap \mathbb{N}^n = \mathbb{N}$).
  - $J_l(x;\beta) \cong$ Jacobi polynomial $P_l^{(\alpha=\beta,\beta)}(y)$.

$$P_l^{(\alpha,\beta)}(y) := \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left[\begin{matrix}-l,\ l+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-y}{2}\right].$$

  - It sits in the Askey scheme of hypergeometric orthogonal polynomials.
    $\rightsquigarrow$ $\exists$ q-analogue in Yamaguchi-san's talk.



Askey scheme of hypergeometric orthogonal polynomials

§1.2. Jack polynomials (cont.)

- Another characterization of Jack polynomial $J_\lambda(x;\beta)$:

$$\begin{cases} \text{triangular expansion} & J_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{C} m_\mu, \\ \text{eigen property} & H_{\mathrm{CS}} J_\lambda \in \mathbb{C} J_\lambda, \end{cases}$$

where H_{CS} is the gauged Calogero-Sutherland Hamiltonian on $\mathbb{C}[x]^{S_n}$:

$$H_{\mathrm{CS}} := \sum_{1\le i\le n} \vartheta_i^2 + \beta \sum_{1\le i<j\le n} \frac{x_i + x_j}{x_i - x_j}(\vartheta_i - \vartheta_j), \quad \vartheta_i := x_i \frac{\partial}{\partial x_i}.$$

- $\exists$ mutually-commuting operators $O_x^{(1)}, O_x^{(2)} = H_{\mathrm{CS}}, O_x^{(3)}, \dots$ on $\mathbb{C}[x]^{S_n}$ with

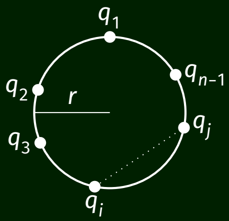
$$O_x^{(k)} := \sum_{1\le i\le n} D_i(\beta)^k, \quad D_i(\beta) := \vartheta_i + \beta \sum_{j\ne i} \frac{x_i}{x_i - x_j}(1 - s_{ij}) \quad (\text{Dunkl operator})$$

with $s_{ij} f(\dots, x_i, \dots, x_j, \dots) = f(\dots, x_j, \dots, x_i, \dots)$.
Jack polynomials $J_\lambda(x;\beta)$ are joint eigenfunctions of the operators $\{O_x^{(k)}\}_{k=1}^\infty$.

$$O_x^{(k)} J_\lambda(x;\beta) = J_\lambda(x;\beta)\epsilon_\lambda^{(k)}, \quad \epsilon_\lambda^{(k)} \in \mathbb{C} \quad (k = 1, 2, \dots).$$

§1.3. Calogero-Sutherland quantum integrable system [B. Sutherland (1971)]

- The Calogero-Sutherland quantum Hamiltonian



$$H'_{CS} := \sum_{1\le i\le n} \frac{p_i^2}{2} + \sum_{1\le i<j\le n} \frac{\hbar^2\beta(\beta-1)}{\sin^2 \frac{1}{2r}(q_i - q_j)}$$

$$q_i \in \mathbb{R}/2\pi r\mathbb{R}, \quad p_i = -\sqrt{-1}\hbar\frac{\partial}{\partial q_i}, \quad \beta \in \mathbb{R}$$

describes n quantum particles on S^1 of radius r with interaction potential $\propto$ (distance)$^{-2}$.

- H'_{CS} gives a quantization of Calogero-Moser model [F. Calogero (1969), J. Moser (1975)], which is a classical integrable system with infin. # of IMs $I_1 = P, I_2 = E, I_3, \dots$.
- H'_{CS} has the ground state $|\psi_0\rangle$ with energy E_0.

$$\psi_0(q) \propto \prod_{i<j}\left(\sin\frac{q_i - q_j}{2r}\right)^\beta = \prod_i x_i^{-\frac{\beta(n-1)}{2}} \prod_{i<j}(x_i - x_j)^\beta, \quad x_i := e^{\frac{\sqrt{-1}q_i}{r}}.$$

The gauged Hamiltonian H_{CS} is given by $\psi_0^{-1} H'_{CS}\psi_0 = \frac{2r^2}{\hbar^2}(H_{CS} + E_0)$.
The Hilbert space for H_{CS} is the symmetric polynomial space: $\mathbb{C}[x]^{S_n}$.

§1.4. Ruijsenaars elliptic quantum integrable system

[S.N.M. Ruijsenaars, Comm. Math. Phys. (1987),

Complete integrability of relativistic Calogero-Moser systems and elliptic function identities.]

- Elliptic Ruijsenaars operators: q-difference operators with elliptic func. coeffs.
  - $x = (x_1, \dots, x_n) \in \mathbb{C}^n$.
  - $T_{q,x_i} f(x) \coloneqq f(x_1, \dots, qx_i, \dots, x_n)$: shift operator with shift $q \in \mathbb{C}$, $|q| < 1$.
  - $\theta(z;p)$: modified Jacobi theta function with elliptic nome $p \in \mathbb{C}$, $|p| < 1$.

$$\theta(z;p) \coloneqq (z;p)_\infty (p/z;p)_\infty, \quad (z;p)_\infty := \prod_{i=0}^{\infty}(1 - p^i z).$$

  For $r = 1, \dots, n$ and another parameter $t \in \mathbb{C}$, the operator is defined to be

$$D_x^{(r)}(q,t,p) \coloneqq \sum_{\substack{I \subset \{1,\dots,n\} \\ |I| = r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{\theta(tx_i/x_j;p)}{\theta(x_i/x_j;p)} \prod_{i \in I} T_{q,x_i}.$$

  They commute with each other: $[D_x^{(r)}, D_x^{(s)}] = 0$.
- A long-standing problem is to find and study joint eigenfunctions of $D_x^{(r)}$'s.
  - Some existence theorems are known.

    [Ruijsenaars (2009), Noumi-Shiraishi-Stokman (2021)]

§1.5. Macdonald polynomials of type A_{n-1} [I.G. Macdonald (around 1987)]

- Macdonald q-difference operators of type A_{n-1} are trigonometric degeneration of elliptic Ruijsenaars operators.
  - $q,t\in\mathbb{C},\ |q|<1,\ x=(x_1,\dots,x_n),\ T_{q,x_i}f(x) := f(x_1,\dots,qx_i,\dots,x_n)$.

$$D_x^{(r)}(q,t) := \lim_{p\to 0} D_x^{(r)}(q,t,p) = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{1-tx_i/x_j}{1-x_i/x_j} \prod_{i\in I} T_{q,x_i}.$$

- $[D_x^{(r)}(q,t), D_x^{(s)}(q,t)]=0$ and $D_x^{(r)}(q,t) \curvearrowright \mathbb{C}[x^{\pm1}]^{S_n}$.
- $\exists!$ a basis $\{P_\lambda(x;q,t)\mid \lambda\in P_+\}$ of $\mathbb{C}[x^{\pm1}]^{S_n}$ such that, for partition λ,

$$\begin{cases} \text{triangularity} & P_\lambda \in m_\lambda + \sum_{\mu<\lambda}\mathbb{C}m_\mu, \\ \text{joint eigen property} & D_x^{(r)}(q,t)P_\lambda(x) = P_\lambda(x)e_r(q^\lambda t^\rho). \end{cases}$$

  $P_\lambda(x;q,t)$ is called the Macdonald polynomial of type A.
- $\exists$ another characterization: $\begin{cases} \text{triangularity} & P_\lambda \in m_\lambda + \sum_{\mu<\lambda}\mathbb{C}m_\mu, \\ \text{orthogonality} & (P_\lambda,P_\mu)_{q,t} \propto \delta_{\lambda,\mu}. \end{cases}$

$$(f,g)_{q,t} := \frac{1}{n!}\int_T \overline{f(x)}g(x)w_{q,t}(x)\,dx, \quad w_{q,t}(x) := \prod_{i\neq j}\frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}.$$

- Examples of $P_\lambda(x;q,t)$ for small λ:

$$P_{(1)} = m_{(1)}, \quad P_{(2)} = m_{(2)} + \frac{(1+q)(1-t)}{1-qt} m_{(1,1)}, \quad P_{(1,1)} = m_{(1,1)},$$
$$P_{(3)} = m_{(3)} + \frac{(1-q^3)(1-t)}{(1-q)(1-q^2t)} m_{(2,1)} + \frac{(1-q^2)(1-q^3)(1-t)^2}{(1-q)^2(1-qt)(1-q^2t)} m_{(1^3)},$$
$$P_{(2,1)} = m_{(2,1)} + \frac{(1-t)(2+q+t+2qt)}{(1-qt^2)} m_{(1^3)}, \quad P_{(1^3)} = m_{(1^3)}.$$

- Parameter degeneration of Macdonald polynomials $P_\lambda(x;q,t)$.
  - $P_\lambda(q, t=q) = s_\lambda$: Schur polynomial.
  - $\lim_{q\to 1} P_\lambda(q, t=q^\beta) = J_\lambda(\beta)$: Jack polynomial.

  These can be checked directly by triangular expansions,
  and also by the inner product $(\cdot,\cdot)_{q,t=q} = (\cdot,\cdot)_1$, $(\cdot,\cdot)_{q,t=q^\beta} \xrightarrow{q\to 1} (\cdot,\cdot)_\beta$.
- Parameter degeneration of Macdonald operators $D_x^{(r)}$ needs technical care.
  We have a stable limit with infinite-variable $x = (x_1, x_2, \dots)$,
  where the operator algebra $\mathrm{U}(q,t)$ is the quantum toroidal gl_1, also called Ding-Iohara-Miki algebra (c.f. [F+09]). $\rightsquigarrow$ Panupong-san's talk.
  By vast calculation, one can show that the limit algebra $\lim_{q\to 1} \mathrm{U}(q, t=q^\beta)$ contains the infinite family $O_x^{(1)}, O_x^{(2)} = H_{\mathrm{CS}}, O_x^{(3)}, \dots$ of commuting operators including the Calogero-Sutherland Hamiltonian H_{CS}.

§2. Macdonald-Cherednik theory

In the previous §1, I introduced the Macdonald polynomials of “type A”.

In this §2, I will introduce an extension of the theory to general root systems, established by I.G. Macdonald and I. Cherednik.

In the end, it will be clarified that the type A polynomials are associated to the extended affine Weyl group $W(\mathrm{A}_{n-1})$.

1. Macdonald polynomials of type A and quantum integrable systems
2. Macdonald-Cherednik theory
   - 2.1. An observation in type A
   - 2.2. Extended affine Weyl groups and Hecke algebras.
   - 2.3. Macdonald-Cherednik theory
3. Concluding remarks

§2.1. An observation in type A

- Recall the eigen property $D_x^{(1)} P_\lambda = P_\lambda \epsilon_\lambda$ for $P_\lambda = P_\lambda(x;q,t)$, $\epsilon_\lambda \in \mathbb{C}$.
  In the case $\lambda = \varnothing = (0^n)$, we have $P_\varnothing = 1$, and the eigen property yields

$$\sum_{w \in S_n} \prod_{1 \le i < j \le n} \frac{1 - t x_{w(j)}/x_{w(i)}}{1 - x_{w(j)}/x_{w(i)}} = \text{const.} \qquad (\text{without } x_i\text{'s}).$$

- There is known an extension of the above relation to any finite Weyl group W_{fin}. For an irreducible finite root system R with $W_{\mathrm{fin}} = W_{\mathrm{fin}}(R)$, one has

$$\sum_{w \in W_{\mathrm{fin}}} \prod_{\alpha \in \Pi} \frac{1 - t_\alpha e^{-w\alpha}}{1 - e^{-w\alpha}} = \sum_{w \in W_{\mathrm{fin}}} \prod_{\alpha \in \Pi \cap w^{-1}(-\Pi)} t_\alpha,$$

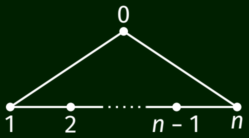
  where $\Pi \subset R$ is a chosen set of positive roots. [I.G. Macdonald (1972)]
  The first equality is recovered by taking
  - $R = \mathrm{A}_{n-1} := \{\alpha_{i,j} = \varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\} \subset \sum_{i=1}^n \mathbb{R}\varepsilon_i$,
  - $x_i = e^{\alpha_{i,i+1}} = e^{\varepsilon_i - \varepsilon_{i+1}}$, and $u_\alpha = t$ for any $\alpha \in \Pi := \{\alpha_{i,j} \mid i < j\} \subset R$.

  So it might be possible to extend the polynomials P_λ to general root systems.
- The point is the q-difference operators $D_x^{(r)}$'s form the center of extended affine Hecke algebra, which can be generalized to other root systems.

§2.2 Extended affine Weyl groups and Hecke algebras

- $W_{\mathrm{fin}} = W_{\mathrm{fin}}(\mathrm{A}_{n-1}) = S_n = \langle s_1, \dots, s_{n-1} \rangle$: finite Weyl group
  $\subset \mathrm{W}_{\mathrm{aff}}(\mathrm{A}_{n-1}) := Q \rtimes W_{\mathrm{fin}} = \langle s_0, \dots, s_{n-1} \rangle$: affine Weyl group
  $Q := \sum_{i=1}^{n-1} \mathbb{Z}\alpha_i$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$: root lattice
  $\subset W = W(\mathrm{A}_{n-1}) := P \rtimes W_{\mathrm{fin}} = \langle s_0, \dots, s_{n-1}, \omega \rangle$: extended affine Weyl group
  $P := \sum_{i=1}^{n} \mathbb{Z}\varepsilon_i$: weight lattice
- Fundamental relations of W

$$\begin{cases} s_i^2 = 1 & (i = 0, \dots, n-1) \\ s_i s_j = s_j s_i & (|i-j| > 1) \\ s_i s_j s_i = s_j s_i s_j & (|i-j| = 1) \\ \omega s_i = s_{i-1}\omega & (i = 0, \dots, n-1) \end{cases}$$



- Lusztig operators T_i acting on $\mathbb{C}(x)$ [G. Lusztig (1989)]

$$T_i := t^{-1/2} \frac{1 - t x^{\alpha_i}}{1 - x^{\alpha_i}} s_i + \frac{t^{1/2} - t^{-1/2}}{1 - x_i^{\alpha}} \quad (i = 0, \dots, n-1),$$

where $x^{\alpha_i} = x^{\varepsilon_i - \varepsilon_{i+1}} := x_i / x_{i+1}$ $(i \geq 1)$ and $x^{\alpha_0} = x^{\delta} x^{\varepsilon_n - \varepsilon_1} := q x_n / x_1$.
T_i's and $\tilde{\omega} := s_{n-1} \cdots s_1 T_{q, x_1}$ generate the extended affine Hecke algebra $H(W)$.

§2.3. Macdonald-Cherednik theory [I. Cherednik (1992–95), I.G. Macdonald (2003)]

- q-Dunkl operators $Y_i \in H(W)$:

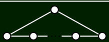
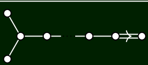
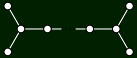
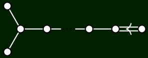
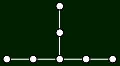
$$Y_1 := T_1 T_2 \cdots T_{n-1}\tilde{\omega}, \quad Y_2 := T_2 \cdots T_{n-1}\tilde{\omega}T_1^{-1}, \quad \ldots, \quad Y_n := \tilde{\omega}T_1^{-1}\cdots T_{n-1}^{-1}.$$

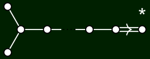
  These are centers: $Z\big(H(W)\big) = \mathbb{C}[Y_1, \ldots, Y_n]$. Moreover, on $\mathbb{C}[x^{\pm 1}]^{S_n}$,

$$D_x^{(r)}(q,t) = \sum_{1 \le i_1 < \cdots < i_r \le n} Y_{i_1} \cdots Y_{i_r}.$$

- The above arguments can be extended to other root systems and W.
  A general theory is built for some pairs (S, S') of affine root systems.
  - Affine root systems in the sense of [I.G. Macdonald (1972)], see next page.
- There are three classes of such pairs (S, S').
  - (I) $(S, S') = (S(R), S(R^\vee))$, where R is an irreducible finite root system.
  - (II) $S = S' = S(R)^\vee$, where R is an irreducible finite root system.
  - (III) $S = S'$ is a non-reduced affine root system of type (X, Y).
    - Type A case corresponds to (I), $(S, S') = (S(\mathrm{A}_{n-1}), S(\mathrm{A}_{n-1}))$.
    - Type $(C_n^\vee, C_n)$ in (III) is Koornwinder polynomial $\rightsquigarrow$ Yamaguchi-san's talk.

Irreducible affine root systems

reduced, simply-laced		reduced, non-simply-laced	
A_n		B_n	
D_n		$B_n^\vee$	
E_6		C_n	
		$C_n^\vee$	
		BC_n	
E_7		F_4	
		$F_4^\vee$	
E_8		G_2	
		$G_2^\vee$	

non-reduced			
(BC_n, C_n)		$(C_n^\vee, BC_n)$	
$(B_n^\vee, B_n)$		$(C_n^\vee, C_n)$	

- Given a pair (S, S') of affine root systems, we can construct
  - the extended affine Weyl group $W := W_{\mathrm{fin}} \ltimes P'$
    (W_{fin} is the finite Weyl group coming from S, and P' is the weight lattice coming from S')
  - the extended affine Hecke algebra $H(W)$ and q-diff. operator realization
  - q-Dunkl operators $\rightsquigarrow$ Macdonald q-difference operators of type (S, S') acting on the W_{fin}-invariant Laurent polynomials $\mathbb{C}[x^{\pm 1}]^{W_{\mathrm{fin}}}$.
  - $\exists!$ basis of $\mathbb{C}[x^{\pm 1}]^{W_{\mathrm{fin}}}$ satisfying triangularity and joint eigen property $\rightsquigarrow$ Macdonald polynomials $P_\lambda(x; q, t_*)$ of type (S, S')
    - λ runs over dominant weights $P'_+ \subset P'$.
    - (# of t-parameters) = (# of W-orbits in the affine root system S).
      A,D,E: t; B,C,F,G: t_l, t_s; BC: t_l, t_m, t_s; $(C^\vee, C)$: $t, t_0, t_0^\vee, t_n, t_n^\vee$
    - Koornwinder has 6 parameters q and t's. $\rightsquigarrow$ Yamaguchi-san's talk.
- Using the double affine Hecke algebra (DAHA) associated to (S, S'), one can study detailed properties of Macdonald polynomials.
  - Orthogonality and the norm formula
  - Duality: $P_\lambda(q^\mu t_*^\rho)/P_\lambda(t_*^\rho) = P_\mu(q^\lambda t_*^\rho)/P_\mu(t_*^\rho)$. $\rightsquigarrow$ Kanno-sensei's talk
  - Non-symmetric Macdonald polynomials

§3. Concluding remarks

- Orthogonality of Schur/Jack/Macdonald symmetric functions can be reformulated by using power-sum symmetric functions $p_k \coloneqq \sum_i x_i^k$.
  - $(s_\lambda, s_\mu)'_{\beta=1} = \delta_{\lambda,\mu}$ with $(p_k, p_l)'_{\beta=1} \coloneqq \frac{1}{n}\delta_{m,n}$.
  - $(J_\lambda, J_\mu)'_\beta \propto \delta_{\lambda,\mu}$ with $(p_k, p_l)'_\beta \coloneqq \frac{1}{n}\frac{1}{\beta}\delta_{m,n}$.
  - $(P_\lambda, P_\mu)'_{q,t} \propto \delta_{\lambda,\mu}$ with $(p_k, p_l)'_{q,t} \coloneqq \frac{1}{n}\frac{1-q^n}{1-t^n}\delta_{m,n}$. $\rightsquigarrow$ Kanno-sensei's talk
- Bispectral property of Macdonald q-difference operators $D_x^{(r)}$ of type A:
  - $\exists$ series solution $P(x,s;q,t)$ s.t. $\begin{cases} D_x^{(r)}P(x,s) = P(x,s)\epsilon_x(s), \\ D_s^{(r)}P(x,s) = P(x,s)\epsilon_s(x). \end{cases}$

    $\rightsquigarrow$ Kanno-sensei's talk
- The ring structure of the invariant polynomials $\mathbb{C}[x^{\pm 1}]^{W_{\mathrm{fin}}}$.
  - Littlewood-Richardson rule of Schur: $s_\lambda s_\mu = \sum_{\nu:\text{partitions}} \mathrm{LR}_{\lambda,\mu}^\nu s_\nu$ with $\mathrm{LR}_{\lambda,\mu}^\nu \coloneqq \#\{\text{semistandard tableaux } T \text{ of shape } \lambda/\mu \text{ and weight } \nu\}$.
  - LR rules for Jack/Macdonald polynomials of type A/general type (S, S').

    $\rightsquigarrow$ Yamaguchi-san's talk on Koornwinder case (type $(C_n^\vee, C_n)$)
  - Deformed ring structure $\rightsquigarrow$ Kanno-sensei's talk.

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Thank you.