The moduli space of spatial polygons and geometric quantization.

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§1. Introduction

- (M, ω) : a compact symplectic mfd ($\dim M = 2n$)
- L : a prequantum line bdl over (M, ω)

  (a complex line bdl over M such that $c_1(L) = [\omega]$)

① A compatible complex structure of (M, ω) i.e. a Kähler structure

$$\rightsquigarrow \alpha := \dim H^0(M, \mathcal{O}_L)$$

② An integrable system $F = (f_1, \cdots, f_n) : M \longrightarrow \mathbb{R}^n$

$$\rightsquigarrow \beta := \# \operatorname{Im}(F) \cap \mathbb{Z}^n$$

Then $\alpha = \beta$ is observed in several cases:

e.g.
- toric mfds
- complex flag mfds [Guillemin-Sternberg]
- the moduli space of flat $SU(2)$-bdls on Riem. surf [Jeffrey-Weitsman]

In this talk, we focus on $\alpha = \beta$ in the case of

the moduli space of spatial polygons.

The moduli space of spatial polygons

$n \geq 4$, $\vec{r} = (r_1, \cdots, r_n) \in \mathbb{R}^n_{>0}$

$S^2(r_i)$: the sphere of radius r_i in $\mathbb{R}^3$

<u>Def.</u> The moduli space of spatial n-gons with edge lengths $\vec{r}$ is the following:

$$\mathcal{M}(\vec{r}) := \{ (u_1, \cdots, u_n) \in S^2(r_1) \times \cdots \times S^2(r_n) \mid u_1 + \cdots + u_n = 0 \} / SO(3)$$

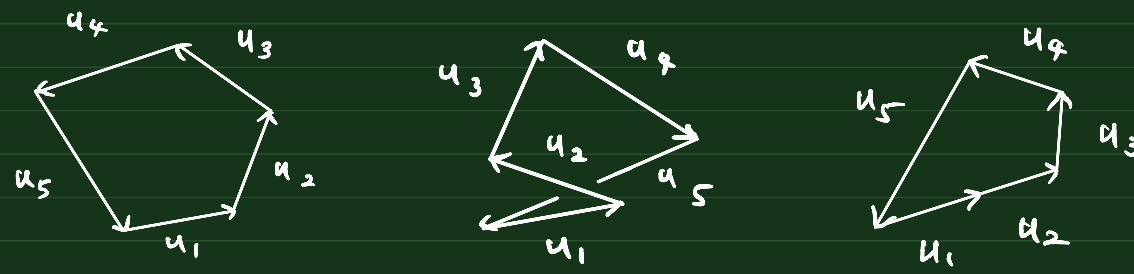
$$= \mu^{-1}(0) / SO(3)$$

$\left(\mu : S^2(r_1) \times \cdots \times S^2(r_n) \longrightarrow \mathfrak{so}(3)^* \cong \mathbb{R}^3 \text{ : the momentum mapping} \right)$

$\mathbb{R}^3$

$n = 5$

(spatial hexagons)



① Conditions on $\vec{r} = (r_1, \cdots, r_n)$:

- $\pm r_1 \pm \cdots \pm r_n \neq 0$ ($\Leftrightarrow$ 0 is a regular value of μ)

  $\Rightarrow$ $\mathcal{M}(\vec{r}) = \mu^{-1}(0)/SO(3)$: a Kähler mfd of $\dim_{\mathbb{C}} = n-3$

- $\vec{r} \in \mathbb{Z}^n_{>0}$

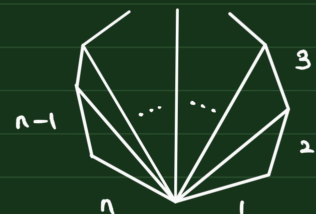
  $\Rightarrow$ We can construct a prequantum line bdl $\mathcal{L}(\vec{r}) \longrightarrow \mathcal{M}(\vec{r})$ by using the line bdls $\mathcal{O}(r_i)$ on $\mathbb{P}^1 \cong S^2(r_i)$.

② There exists an integrable system on $\mathcal{M}(\vec{r})$ called **the bending system** :

$$\pi^{\vec{r}}_{T} : \mathcal{M}(\vec{r}) \longrightarrow \mathbb{R}^{n-3} \quad \text{for } T : \text{a triangulation of } n\text{-gon}$$

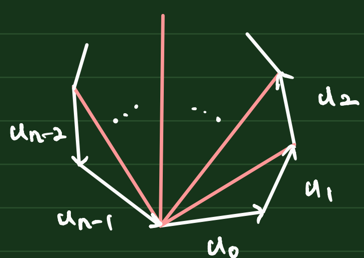
··· [Kapovich – Millson]

<u>Example</u>

When T = 

: the caterpillar ,

$$\begin{array}{cccc} \pi^{\vec{r}}_{T} : & \mathcal{M}(\vec{r}) & \longrightarrow & \mathbb{R}^{n-3} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & [(u_1, \cdots, u_n)] & \longmapsto & (\|u_1 + u_2\|, \cdots, \|u_1 + \cdots + u_{n-2}\|) \end{array}$$



the lengths of $n-3$ diagonals given by T

- In this way, $\mathcal{M}(\vec{r})$ has a Kähler structure and an integrable system.

<u>Thm. (Kamiyama, 2000)</u>

$n \geq 5$: odd, $\vec{r} = \overbrace{(1, \cdots, 1)}^{n}$, T : the Caterpillar

$$\Longrightarrow \underbrace{\dim H^0(\mathcal{M}(\vec{r}), \mathcal{O}_{\mathcal{L}(\vec{r})})}_{\alpha} = \underbrace{\# \operatorname{Im}(\pi_T^{\vec{r}}) \cap \mathbb{Z}^{n-3}}_{\beta}$$

In this talk,

we generalize Kamiyama's theorem to $\forall n \geq 4$, $\forall \vec{r} \in \mathbb{Z}_{>0}^n$, $\forall T$

by using the notion of operads.

Main theorem

(1) In case of the moduli space of spatial polygons,

We can describe recurrence structures of α and β as morphisms of operads

$$f_{\text{käh}} : \mathrm{Corolla} \longrightarrow \mathcal{W}(\mathbb{Z}_{\geq 0}), \quad f_{\text{re}} : \mathrm{RibTree}^3 \longrightarrow \mathcal{W}(\mathbb{Z}_{\geq 0}).$$

(2)

$$\begin{tikzcd}
\mathrm{RibTree}^3 \arrow[r, "f_{\text{re}}"] \arrow[d, "\mathrm{cont}"'] & \mathcal{W}(\mathbb{Z}_{\geq 0}) \\
\mathrm{Corolla} \arrow[ur, "f_{\text{käh}}"'] \arrow[u, phantom, shift left=6ex, "\circlearrowright"] &
\end{tikzcd}$$

Cor. $\forall n \geq 4$, $\forall \vec{r} = (r_1, \cdots, r_n) \in \mathbb{Z}_{>0}^n$ s.t. $\pm r_1 \pm \cdots \pm r_n \neq 0$

$\forall T$: a triangulation of n-gon.

$$\Longrightarrow \dim H^0(\mathcal{M}(\vec{r}), \mathcal{O}_{\mathcal{L}(\vec{r})}) = \# \operatorname{Im}(\pi_T^{\vec{r}}) \cap \mathbb{Z}^{n-3}$$

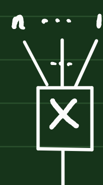
§2. Operads

Def. An *operad* is a sequence $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 1}$ of sets, togther with maps

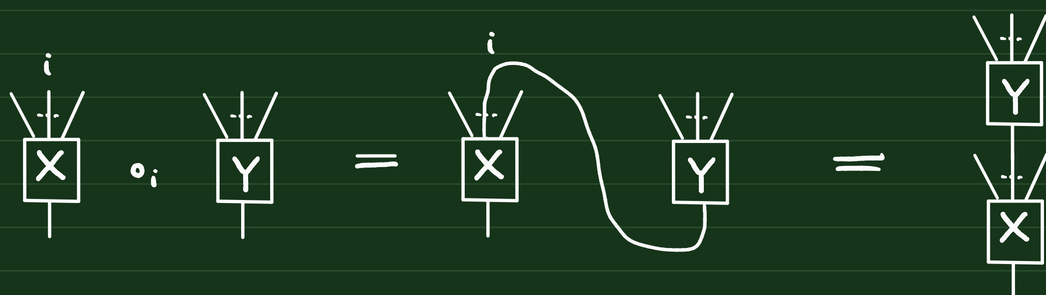
$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1) \qquad 1 \leq i \leq n ,$$

(operadic compositions)

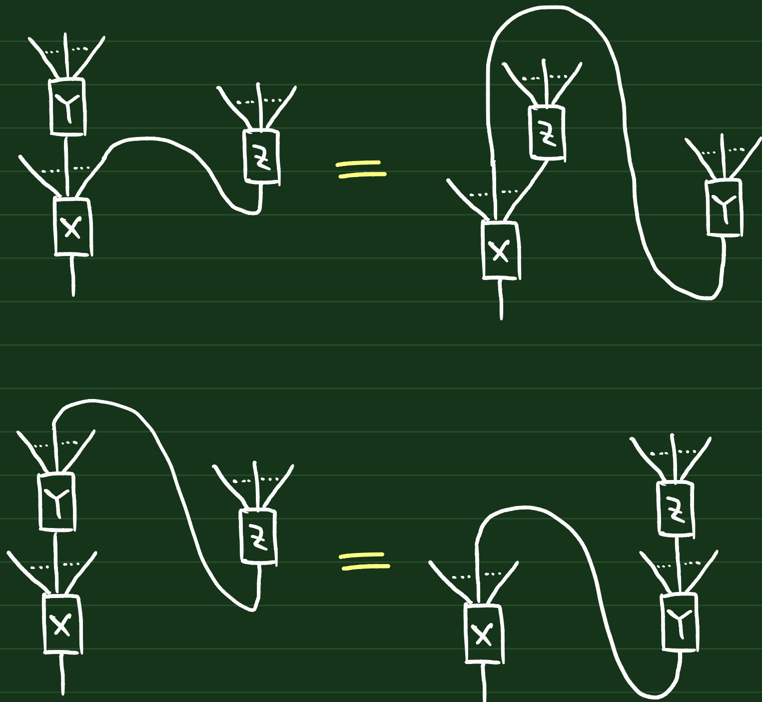
satisfying the associativity and unit axioms.

- $\mathcal{O}(n) = \Big\{$ 

n inputs
one output $\Big\}$

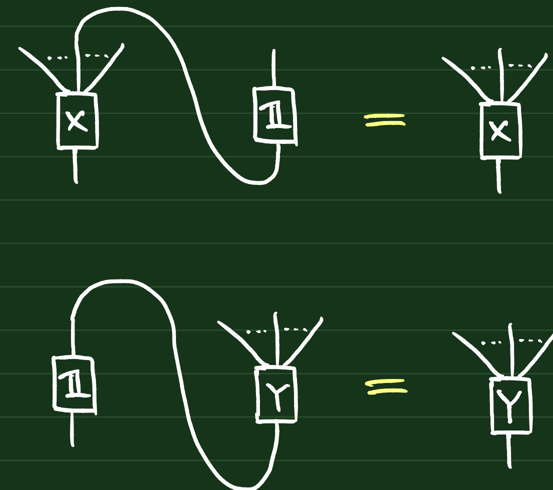
- 

Associativity

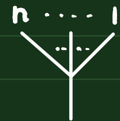


Unit

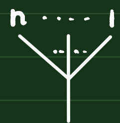
$\exists \ \boxed{\mathbb{1}} \in \mathcal{O}(1)$ such that

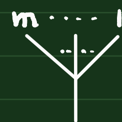


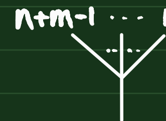
Example ① **Corolla** $= \{ \mathrm{Corolla}(n) \}_{n \geq 1}$

- $\mathrm{Corolla}(n) := \Big\{$ 

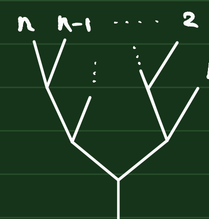
: a rooted plane tree which has no internal edge (# leaves $= n$) $\Big\}$

- 

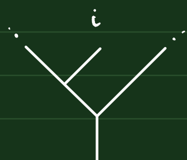
$\circ_i$ 

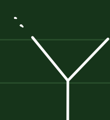
$:=$ 

② **RibTree^3** $= \{ \mathrm{RibTree}^3(n) \}_{n \geq 1}$

- $\mathrm{RibTree}^3(n) := \Big\{$ 

: a trivalent rooted plane tree (# leaves $= n$) $\Big\}$

- 

$\circ_i$  $:=$  .... (grafting)

③ **$\mathcal{W}(\mathbb{Z}_{\geq 0})$** $= \{ \mathcal{W}(\mathbb{Z}_{\geq 0})(n) \}_{n \geq 1}$

- $\mathcal{W}(\mathbb{Z}_{\geq 0})(n) := \{ f : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n \longrightarrow \mathbb{Z} \mid \text{some condition} \}$

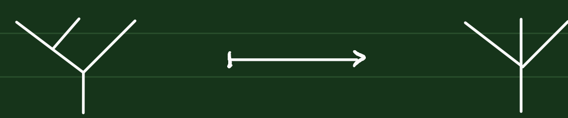
- $f \circ_i g$ can be also defined. (omit)

<u>Def.</u> A morphism between two operads $\mathcal{O}$ and $\mathcal{P}$ is a sequence

$\mathcal{F} = \{ \mathcal{F}_n : \mathcal{O}(n) \longrightarrow \mathcal{P}(n) \}_{n \geq 1}$ of maps such that

$$\begin{cases} \cdot\ \mathcal{F}_{n+m-1}(X \circ_i Y) = \mathcal{F}_n(X) \circ_i \mathcal{F}_m(Y) \quad \text{for } \forall X \in \mathcal{O}(n), \forall Y \in \mathcal{O}(m) \\ \cdot\ \mathcal{F}_1(\mathbb{1}_{\mathcal{O}}) = \mathbb{1}_{\mathcal{P}} \end{cases}$$

<u>Example</u> We define a sequence cont $= \{ \mathrm{cont}_n : \mathrm{RibTree}^3(n) \longrightarrow \mathrm{Corolla}(n) \}_{n \geq 1}$ by the contraction of internal edges of trees :



$\rightsquigarrow$ cont is a morphism of operads $\mathrm{RibTree}^3 \longrightarrow \mathrm{Corolla}$.

§ 3. The construction of morphisms $f_{\text{Käh}}$ and f_{re}

$$\left\{\begin{array}{l}
\vec{r} = (r_1, \cdots, r_n) \in \mathbb{Z}_{>0}^n, \quad T : \text{a Triangulation of } n\text{-gons} \\
\mathcal{M}(\vec{r}) = \{ (u_1, \cdots, u_n) \in S^2(r_1) \times \cdots \times S^2(r_n) \mid u_1 + \cdots + u_n = 0 \} / SO(3) \\
\mathcal{L}(\vec{r}) \longrightarrow \mathcal{M}(\vec{r}) : \text{the prequantum line bdl} \\
\pi_T^{\vec{r}} : \mathcal{M}(\vec{r}) \longrightarrow \mathbb{R}^{n-3} : \text{the bending system}
\end{array}\right.$$

$\boxed{f_{\text{Käh}}}$ V_{r_i} : the irreducible complex representation of $SO(3)$

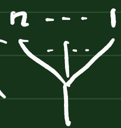
($\dim = 2r_i + 1$)

Prop. $\underline{\dim H^0(\mathcal{M}(\vec{r}), \mathcal{O}_{\mathcal{L}(\vec{r})})} = \underline{[V_{r_2} \otimes \cdots \otimes V_{r_n} : V_{r_1}]}$

α

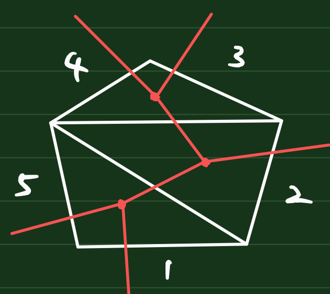
the multiplicity of V_{r_1} in $V_{r_2} \otimes \cdots \otimes V_{r_n}$

<u>Def.</u> We define $f_{\text{Kăn}} = \{ (f_{\text{Kăn}})_n : \mathrm{Corolla}(n) \to W(\mathbb{Z}_{\geq 0})(n) \}_{n \geq 1}$ by

$(f_{\text{Kăn}})_n$ () $:= \left(\begin{array}{c} d \, ; \vec{c} \\ \| \\ (c_1, \cdots, c_n) \end{array} \longmapsto [V_{c_1} \otimes \cdots \otimes V_{c_n} : V_d] \right)$

<u>Prop.</u> $f_{\text{Kăn}}$ is a morphism $\mathrm{Corolla} \longrightarrow W(\mathbb{Z}_{\geq 0})$

$\boxed{f_{re}}$



dual
$\longleftrightarrow$

a Triangulation of n-gons

a trivalent rooted plane tree
(# leaves = $n-1$)

Hereafter, we identify triangulations with trivalent rooted plane trees.

Prop. $\underbrace{\# \operatorname{Im}(\pi_T^{\vec{r}}) \cap \mathbb{Z}^{n-3}}_{\beta} = \# \underbrace{D(T, \vec{r})}_{\{\text{ some integer edge-labelings of } T \text{ relative to } \vec{r} \}}$

Def. We define $f_{re} = \{ (f_{re})_n : \mathrm{RibTree}^3(n) \to \mathcal{W}(\mathbb{Z}_{\geq 0})(n) \}_{n \geq 1}$ by

$$(f_{re})_n (T) := \left(d ; \vec{c} \longmapsto \# D(T, d ; \vec{c}) \right)$$

Prop. f_{re} is a morphism $\mathrm{RibTree}^3 \longrightarrow \mathcal{W}(\mathbb{Z}_{\geq 0})$

§ 4. The proof of (2) in the main theorem

<u>Thm.</u> (2)

$$\begin{tikzcd}
\mathrm{RibTree}^3 \arrow[r, "f_{\mathrm{re}}"] \arrow[d, "\mathrm{cont}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}(\mathbb{Z}_{\geq 0}) \\
\mathrm{Corolla} \arrow[ur, "f_{\mathrm{K\ddot{a}h}}"'] &
\end{tikzcd}$$

<u>Key lemma</u>

$\mathcal{O}$: an operad, $\mathcal{F}, \mathcal{G} : \mathrm{RibTree}^3 \longrightarrow \mathcal{O}$: morphisms of operads

$$\left.\begin{array}{l} \mathcal{F}_1 = \mathcal{G}_1 \\ \mathcal{F}_2 = \mathcal{G}_2 \end{array}\right. \Longrightarrow \quad \mathcal{F} = \mathcal{G} \quad (\forall n \geq 1 \ \ \mathcal{F}_n = \mathcal{G}_n)$$

We apply this lemma to the case where $\mathcal{F} = f_{\mathrm{K\ddot{a}h}} \circ \mathrm{cont}$, $\mathcal{G} = f_{\mathrm{re}}$.

Prop. $(\mathrm{tran} \circ \mathrm{cont})_2 = (\mathrm{tre})_2$

Note that $\mathrm{RibTree}^3(2) = \{ Y \}$

$$\big((\mathrm{tran} \circ \mathrm{cont})_2 (Y)\big)(d; c_1, c_2) = [V_{c_1} \otimes V_{c_2} : V_d]$$

the Clebsch-Gordon rule on $SO(3)$

$$= \begin{cases} 1 & \text{if } |c_1 - c_2| \leq d \leq c_1 + c_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \big((\mathrm{tre})_2 (Y)\big)(d; c_1, c_2)$$

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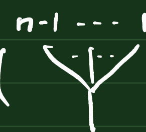
<u>Cor.</u> $\forall n \geq 4$

$\forall \vec{r} = (r_1, \cdots, r_n) \in \mathbb{Z}^n_{\geq 0}$ s.t. $\pm r_1 \pm \cdots \pm r_n \neq 0$

$\forall T$: a triangulation of n-gon (a trivalent rooted plane tree)

$\Rightarrow \dim H^0(\mathcal{M}(\vec{r}), \mathcal{O}_{\mathcal{L}(\vec{r})}) = \# \operatorname{Im}(\pi_T^{\vec{r}}) \cap \mathbb{Z}^{n-3}$

$\because$ LHS $= \Big((f_{\text{Käh}})_{n-1} \big($ 

$\big) \Big) (r_1 ; r_2, \cdots, r_n)$

$= \big((f_{\text{Käh}} \circ \text{conf})_{n-1} (T) \big) (r_1 ; r_2, \cdots, r_n)$.

RHS $= \big((f_{\text{re}})_{n-1} (T) \big) (r_1 ; r_2, \cdots, r_n)$.

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