Categorical descriptions of 1-dimensional gapped phases with abelian onsite symmetries

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There is a famous saying by Eugene Wigner which states that:

Mathematics is unreasonably effective in the natural sciences.

What I want is to give you a feeling that, among the fields of mathematics,

Category theory is (unreasonably) effective in the realm of many body physics.

Our paper is indeed inherited from this idea and will be used as an example to convince you later on. Before that, we may begin by revisiting the core concept in many body systems, the "phase".

I don't know the background of some of you guys, but I think you all have some senses about a "phase". We all encounter the phase transition about ice, water, and vapor everyday, right? And people would accept that the phase or phase transition is a "macroscopic phenomenon" and should have a direct mathematical description for it. By "direct" we mean **to collect all observables in the long wave length limit (LWLL) in a physical system and see what mathematical structure they form**.

However, for a long period of time, the pursuing of this structure has been put aside, especially for phases within Landau's paradigm. And it was partially because we did not have a clear language to approach it. So for the rest of the slides, I will show you how to find the observables in a lattice model and how to use categorical language to describe them.

We start from 1d (space dimension). In a recent work Kong-Wen-Zheng:2021 arXiv:2108.08835, Kong, Wen and Zheng use Ising chain to show that the macroscopic observables in a 1d gapped quantum phase form an enriched fusion category. (In my opinion, this is a very important result, it tells us that even the most basic phases are not completely understood. Our paper is actually a generalization of their results. So if you have some physical background and are interested in this topic, I recommend you to read this paper.)

This paper reveals that, for a given 1d lattice model with a total Hilbert space $\mathcal{H}_{tot} = \otimes_{i\in\mathbb{Z}}\mathcal{H}_i$ and for a Hamiltonian with local interactions, there are two kinds of observables in LWLL : **topological sectors of operators and topological sectors of states**.

In the LWLL, many degrees of freedom are not "observables". Because there are always a lot of microscopic degrees of freedom (local operators) surrounding a physical item. Observables should be things that are non-local and invariant under the action of these microscopic degrees of freedom so as to "stand out". Therefore, the observables are the so called "topological sectors of operators or states" instead of bare operators or states. Specifically speaking:

- The topological sectors of operators are the spaces of non-local unconfined operators that are invariant under the action of local operators. The sector consisting of only local operators is denoted by $\mathbb{1}$. A morphism between two sectors of operators is an operator that intertwines the action of local operators. Then the topological sectors of operators form a category, denoted by $\mathcal{B}$. Moreover, these operators can be composed and braided in $1+1$D spacetime. In consequence, **the topological sectors of operators form a braided fusion category $\mathcal{B}$**.

- The topological sectors of states are the topological defect lines (also known as topological excitations) in 1+1D, which are the subspaces of $\mathcal{H}_{tot}$ that are invariant under the action of local operators. The sector generated by the ground state is called the vacuum sector. The topological defect lines can be mapped to each other or fused together, which indicates that **the topological sectors of states form a fusion category denoted by $\mathcal{S}$**.

Certainly the topological sectors of operators act on those of states, i.e. $\mathcal{B}$ should act on $\mathcal{S}$. We denote the space of non-local operators which maps a sector of states a to another sector of states b by $\mathrm{Hom}(a, b)$, then $\mathrm{Hom}(a, b)$ should be an object in the topological sectors of operators $\mathcal{B}$. This shows that the set of topological sectors of states, along with the spaces of morphism $\mathrm{Hom}(a, b)$, form a $\mathcal{B}$-enriched category.

Furthermore, the fusion behavior of non-local operators should be compatible with the fusion of topological sectors of states. **All together they form a $\mathcal{B}$-enriched fusion category $^{\mathcal{B}}\mathcal{S}$** defined by a braided equivalence $\mathcal{B} \to \mathfrak{Z}_1(\mathcal{S})$, where $\mathfrak{Z}_1(\mathcal{S})$ is the monoidal center of $\mathcal{S}$. $^{\mathcal{B}}\mathcal{S}$ is also called the *topological skeleton*, which is just what we want in describing the given phase of a lattice model.

Moreover, a topological skeleton can be associated to different phases depending on what symmetry we assign. **In the presence of an onsite symmetry, we only consider the sectors that are invariant under the action of symmetric local operators**.

As an example, for a 1d symmetry protected topological (SPT) order with a finite onsite symmetry G, the topological excitations are symmetry charges, and they form a fusion category $\mathcal{S} = \mathrm{Rep}(G)$ in the bosonic case Kong-Lan-Wen-Zhang-Zheng:2020, arXiv:2003.08898. Therefore, the topological skeleton of a 1d SPT order with onsite symmetry G should be given by the enriched fusion category ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$ for $\mathcal{S} = \mathrm{Rep}(G)$ obtained from a braided equivalence $\phi\colon \mathfrak{Z}_1(\mathrm{Rep}(G)) \to \mathfrak{Z}_1(\mathcal{S})$.

For the next few slides, I will use a 1d lattice model in our paper as a physical example to explicitly show that **the macroscopic observables of the trivial** (i.e. braided equivalence $\phi\colon \mathfrak{Z}_1(\mathrm{Rep}(G)) \to \mathfrak{Z}_1(\mathcal{S})$ is just identity) **SPT order indeed form** ${}^{\boldsymbol{\mathfrak{Z}_1(\mathrm{Rep}(G))}}\mathbf{Rep}(\boldsymbol{G})$.

The abelian lattice model

In order to check the observables in a 1d SPT order, we construct a lattice model with abelian onsite G-symmetry. So consider the 1d lattice with the total Hilbert space $\mathcal{H}_{tot} = \otimes_{i\in\mathbb{Z}}\mathcal{H}_i$, where each local Hilbert space $\mathcal{H}_i$ is spanned by an orthonormal basis $\{|g\rangle_i \mid g \in G\}$. For each site i and $g \in G$, we define an operator L_g^i acting on $\mathcal{H}_i$ as follows:

$$L_g^i|h\rangle_i := |gh\rangle_i, \forall h \in G.$$

The global G-symmetry is defined by

$$U(g) := \otimes_i L_g^i, \forall g \in G.$$

The abelian lattice model

Denote the set of equivalence class of irreducible representations of G by $\widehat{G}$. For each site i and $\rho \in \widehat{G}$, we define another operator Z_ρ^i acting on $\mathcal{H}_i$ by

$$Z_\rho^i |g\rangle_i := \rho(g) \cdot |g\rangle_i.$$

Let $H \subseteq G$ be a subgroup of G. Note that $\widehat{G/H}$ can be naturally viewed as a subgroup of $\hat{G}$. For each site i, define two Hermitian operators:

$$X_H^i := \frac{1}{|H|} \sum_{h \in H} L_h^i, \qquad Z_H^{i,i+1} := \frac{|H|}{|G|} \sum_{\rho \in \widehat{G/H}} (Z_\rho^i)^\dagger Z_\rho^{i+1},$$

It is easy to verify that these operators mutually commute and are G-symmetric.

The abelian lattice model

Then we can define our Hamiltonian as follows:

$$\mathcal{H} := \sum_i (1 - X_H^i) + \sum_i (1 - Z_H^{i,i+1}) \tag{1}$$

This lattice model spontaneously breaks the G-symmetry to the subgroup H.

Remark

When $G = \mathbb{Z}_2$, this Hamiltonian recovers the well-known 1d Ising model.

By taking different subgroups $H \subseteq G$ in (1), we can realize different symmetry breaking phases. Here we just consider the trivial SPT phase, that is $H = G$, the Hamiltonian is

$$\mathcal{H} = \sum_i (1 - X_G^i)$$

topological sector of operators

Now let us analyse the topological sector of operators within this model. We can find two kinds of G-symmetric non-local operators:

$$M_g^i := \prod_{j \leq i} L_g^j, \forall g \in G, \qquad E_\rho^i = \prod_{j \geq i} Z_\rho^j (Z_\rho^{j+1})^\dagger, \forall \rho \in \hat{G}.$$

Apparently, the product $M_g^i E_\rho^j$ is also a symmetric non-local operator. **We use $\mathcal{O}_{(g,\rho)}$ to denote the topological sector of operators generated by $M_g^i E_\rho^j$.** Now we are about to show $\mathcal{O}_{(g,\rho)}$ is an object in braided fusion category $\mathfrak{Z}_1(\mathrm{Rep}(G))$.

Topological sectors of operators

We can calculate that

$$M_g^i E_\rho^j = \begin{cases} E_\rho^j M_g^i, & i < j, \\ \rho(g)^{-1} E_\rho^j M_g^i, & i \geq j. \end{cases} \quad (2)$$

Thus, $M_g^i E_\rho^j M_h^k E_\sigma^l$ and $M_g^i M_h^k E_\rho^j E_\sigma^l$ differ by a coefficient at most. It follows that the fusion products of these topological sectors of operators are given by

$$\mathcal{O}_{(g,\rho)} \otimes \mathcal{O}_{(h,\sigma)} = \mathcal{O}_{(gh,\rho\sigma)}.$$

This recovers the fusion structure in $\mathfrak{Z}_1(\mathrm{Rep}(G))$.

Moreover, we can double braid the topological sector of operators by the following process: For $i < k < j$, we can first create a pair of $M_g E_\rho$ at sites i and j, then wind another $M_h E_\sigma$ around one of the sector, after this process, we annihilate the pair of $M_g E_\rho$ and obtain a phase $\rho(h)\sigma(g)$.

$$M_g^i E_\rho^i E_{\rho^{-1}}^j M_{g^{-1}}^j M_h^k E_\sigma^k M_g^j E_\rho^j E_{\rho^{-1}}^i M_{g^{-1}}^i = \rho(h)\sigma(g) M_h^k E_\sigma^k,$$

this equation gives the double braiding

$$\mathcal{O}_{(h,\sigma)} \otimes \mathcal{O}_{(g,\rho)} \xrightarrow{\sigma(g)\rho(h)} \mathcal{O}_{(h,\sigma)} \otimes \mathcal{O}_{(g,\rho)}. \tag{3}$$

Thus we show that **the topological sectors of operators $\{\mathcal{O}_{(g,\rho)}\}_{g\in G,\rho\in\hat{G}}$ indeed form a braided fusion category equivalent to $\mathfrak{Z}_1(\mathrm{Rep}(G))$**.

Topological sector of states (SPT)

Now we show that the topological sectors of states of the SPT phase from fusion category $\mathrm{Rep}(G)$.

For Hamiltonian $\mathcal{H} = \sum_i (1 - X_G^i)$, the unique ground state is a product state.

$$|\Omega_G\rangle := \bigotimes_i \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle_i.$$

M_g^i has only trivial action on the ground state: $M_g^i |\Omega_G\rangle = |\Omega_G\rangle$ for all sites i and $g \in G$. We use $\mathcal{S}_1$ to denote the trivial topological sector of states generated by the ground state.

For each $\rho \in \hat{G}$ and site i, we define $|\rho, i\rangle := E_\rho^i |\Omega_G\rangle$.

We use $\mathcal{S}_\rho$ to denote the topological sector of states generated by $|\rho, i\rangle$.

Topological sector of states (SPT)

Now consider the state

$$|\rho, i; \sigma, j\rangle := E^i_\rho E^j_\sigma |\Omega_G\rangle.$$

Intuitively, $|\rho, i; \sigma, j\rangle$ should generate the topological sector of states $\mathcal{S}_\rho \otimes \mathcal{S}_\sigma$ (i.e. the fusion of $\mathcal{S}_\rho$ and $\mathcal{S}_\sigma$). On the other hand, by acting a local operator $Z^j_\rho (Z^i_\rho)^\dagger$ on this state we find that $|\rho, i; \sigma, j\rangle$ generates the same topological sector of states as $|\rho\sigma, j\rangle$:

$$Z^j_\rho (Z^i_\rho)^\dagger |\rho, i; \sigma, j\rangle = E^j_\rho E^j_\sigma |\Omega_G\rangle = E^j_{\rho\sigma} |\Omega_G\rangle = |\rho\sigma, j\rangle.$$

It follows that the fusion rules of topological sectors of states are given by

$$\mathcal{S}_\rho \otimes \mathcal{S}_\sigma = \mathcal{S}_{\rho\sigma}.$$

Therefore, **topological sector of states form a fusion category equivalent to $\mathbf{Rep}(G)$**.

Topological skeleton (SPT)

Since M_g^i only has trivial actions on the ground state, $M_g^i E^j_{\sigma\rho^{-1}}$ maps the topological sector $\mathcal{S}_\rho$ to $\mathcal{S}_\sigma$ for all $g \in G$. Thus the hom spaces between topological sectors of states are given by

$$\mathrm{Hom}(\mathcal{S}_\rho, \mathcal{S}_\sigma) = \bigoplus_{g \in G} \mathcal{O}_{(g, \sigma\rho^{-1})}. \tag{4}$$

Hence by carefully analyzing the observables in a lattice model, we prove that

Theorem[ph]

The topological skeleton of the 1d trivial SPT order with the bosonic onsite abelian symmetry G is the enriched fusion category ${}^{\mathfrak{Z}_1(\mathrm{Rep}(G))}\mathrm{Rep}(G)$.

(You may check our paper for more examples of the symmetry-breaking phases to see the unifying power of the enriched-categorical approach.)