Holomorphic differential operators via Fedosov quantization

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Motivation

The main motivation is to quantize a subclass of smooth functions on a Kähler manifold to differential operators on certain holomorphic vector bundles. This has the following advantage comparing to the Berezin-Toeplitz quantization:

- This gives rise to a non-formal deformation of the classical pointwise multiplication of functions.
- This construction is not limited to the prequantum line bundle.
- Functions are quantized to operators which are local instead of asymptotic locality.

Background

Deformation quantization is mathematical description of observables in a quantum mechanical system. More precisely, the Poisson algebra of classical observables is deformed to an associative algebra of quantum observables.

Definition

Let (M, ω) be a symplectic manifold, then a deformation quantization of M is an associative product $*$ on $C^\infty(M)[[\hbar]]$ such that

$$f * g = f \cdot g + \sum_{i \geq 1} \hbar^i C_i(f, g),$$

where C_i's are bi-differential operators, with $C_1(f, g) - C_1(g, f) = \{f, g\}$.

Fedosov's construction of deformation quantization

Fedosov gives a simple geometric construction of deformation quantization on a general symplectic manifold M: he considers the Weyl bundle $\mathcal{W}_{M,\mathbb{R}} := \widehat{\mathrm{Sym}}(T^\vee_{M,\mathbb{R}})$, with the associative Moyal product. And he shows the following theorem:

Theorem (Fedosov)

- *There exists a flat connection D on the Weyl bundle, which is compatible with the fiberwise (quantum) Moyal product;*
- *There is the following isomorphism between formal smooth functions and flat sections of the Weyl bundle:*

$$\Gamma^{flat}(M, \mathcal{W}_{M,\mathbb{R}}) \cong C^\infty(M)[[\hbar]]$$

In particular, the Moyal product on the flat sections induces a deformation quantization on $C^\infty(M)[[\hbar]]$.

Berezin-Toeplitz quantization

Toeplitz operators are defined on a prequantizable Kähler manifold M, in the following way:

- Pick a prequantum line bundle L on M.
- For every positive integer $k > 0$. Given any smooth function f, we define the Toeplitz operator on holomorphic sections of $L^{\otimes k}$ associated to f as

$$T_f := \Pi \circ m_f,$$

  where m_f denotes multiplication by f, and Π denotes the orthogonal projection to holomorphic sections.
- The asymptotic formula of the composition of two Toeplitz operators induces a deformation quantization.

Example: Fock spaces

Definition

The Fock spaces are the holomorphic function spaces

$$\mathcal{H}L^2(\mathbb{C}^n, \mu_\hbar),$$

where the volume $\mu_\hbar$ is given by

$$\mu_\hbar(z) = (\pi\hbar)^{-n} e^{-|z|^2/\hbar}.$$

Here $\hbar$ is a positive number.

The Bargmann-Fock representation of the Wick algebra is the algebraic formulation of Toeplitz operators on $\mathcal{H}L^2(\mathbb{C}^n, \mu_\hbar)$.

Example: Fock spaces

The Berezin-Toeplitz operators on the Fock spaces are operators of the form "multiply then project", i.e., multiplication by a smooth function f which is in general non-holomorphic and then project back to the holomorphic subspaces. In particular, for $n = 1$, there are the following:

$$
\begin{aligned}
T_z &= m_z, \\
T_{\bar{z}} &= \hbar \frac{d}{dz}, \\
T_{\bar{z}^m z^n} &= \left(\hbar \frac{d}{dz} \right)^m \circ m_{z^n},
\end{aligned}
$$

where m_{z^n} denotes the multiplication by z^n.

Wick algebra

The Toeplitz operators on $\mathbb{C}^n$ gives rise to the following Wick product.

Definition

The Wick product on the space $\mathcal{W}_{\mathbb{C}^n} := \mathbb{C}[[z^1, \bar{z}^1, \cdots, z^n, \bar{z}^n]][[\hbar]]$ is defined by

$$f * g := \exp\left(-\hbar \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{w}^i}\right) (f(z, \bar{z}) g(w, \bar{w}))|_{z=w}$$

And there is also the representation of $\mathcal{W}_{\mathbb{C}^n}$ on the Bargmann-Fock space which consists of only holomorphic functions:

$$\mathcal{F}_{\mathbb{C}^n} := \mathbb{C}[[z^1, \cdots, z^n]][[\hbar]].$$

There are several drawbacks of these two quantization schemes:

1. In both Fedosov and Toeplitz quantization, we only get formal deformation of smooth functions.
2. There is no Hilbert space in the Fedosov quantization scheme.
3. In Toeplitz quantization, although we have Hilbert spaces $H^0(X, L^{\otimes k})$, this is only a representation of the deformation quantization in an asymptotic way as $k \to \infty$.

Non-formal quantization

A simple observation of the above formula is that if we restrict to polynomials on $\mathbb{C}^n$, then we can evaluate $\hbar$ at any complex number and get a non-formal quantum algebra.

A natural question is if there are appropriate generalizations to Kähler manifolds, so that we get some non-formal quantization of functions?

L_∞ structure on Kähler manifolds

The L_∞-structure on a Kähler manifold X is equivalent to a flat connection on the holomorphic Weyl bundle

$$\mathcal{W}_X := \widehat{\mathrm{Sym}}(T_X^\vee),$$

of the following form:

$$D_K = \nabla - \delta + \sum_{n\geq 2} \tilde{R}_n^*,$$

The $\tilde{R}_n^* \in \mathcal{A}_X^1 \otimes (\mathrm{Sym}^n(T_X^\vee) \otimes T_X)$'s are defined as partial transpose of the covariant derivatives of the curvature tensor.

Fedosov quantization on Kähler manifolds

Definition

Let X be a Kähler manifold, the complexified Weyl bundle on X is defined as follows:

$$\mathcal{W}_{X,\mathbb{C}} = \widehat{\mathrm{Sym}}\, T_X^\vee \otimes \widehat{\mathrm{Sym}}\, \bar{T}_X^\vee.$$

Here $T_X^\vee$ and $\bar{T}_X^\vee$ denote the holomorphic and anti-holomorphic cotangent bundle respectively. It is clear that $\mathcal{W}_{X,\mathbb{C}}$ is the complexification of $\mathcal{W}_{X,\mathbb{R}}$, and there is a fiberwise Wick product on $\mathcal{W}_{X,\mathbb{C}}$ induced by the Kähler form.

It is clear that the Levi-Civita connection naturally extends to a connection on $\mathcal{W}_{X,\mathbb{C}}$, which we denote by ∇.

Fedosov quantization on Kähler manifolds

With respect to a local holomorphic coordinate system $\{z^1, \cdots, z^n\}$, we let y^i and $\bar{y}^j$'s denote their corresponding local sections of $T_X^\vee$ and $\bar{T}_X^\vee$ respectively. A local section of $\mathcal{W}_{X,\mathbb{C}}$ is of the following explicit form:

$$\sum_{k\geq 0} a_{k,i_1,\cdots,i_m,j_1,\cdots,j_n} \hbar^k y^{i_1} \cdots y^{i_m} \bar{y}^{j_1} \cdots \bar{y}^{j_n}.$$

There is the following fiberwise de Rham differential operator δ on $\mathcal{A}_X \otimes \mathcal{W}_{X,\mathbb{C}}$, which is a derivation:

$$\delta(y^i) = dz^i,$$
$$\delta(\bar{y}^j) = d\bar{z}^j.$$

Quantization of L_∞ structure

We can define the following natural operator

$$L : \mathcal{A}_X^* \otimes (\widehat{\mathrm{Sym}}(T_X^\vee) \otimes T_X) \to \mathcal{A}_X^* \otimes (\widehat{\mathrm{Sym}}(T_X^\vee) \otimes \bar{T}_X^\vee)$$

by using the symplectic form to "lift the last subscript". In particular, we can define

$$I_n := L(\tilde{R}_n) = R_{i_1\cdots i_n}^j \omega_{j\bar{k}} y^{i_1} \otimes \cdots \otimes y^{i_n} \otimes \bar{y}^k \in \mathcal{A}_X^{0,1} \otimes \mathcal{W}_{X,\mathbb{C}}.$$

There are the following three simple observations:

$$\tilde{R}_n^* = \frac{1}{\hbar}[I_n, -]_\star|_{\mathcal{W}_X},$$

$$\nabla \circ L = L \circ \nabla,$$

$$L([A, B]) = [L(A), L(B)]_\star.$$

Quantization of L_∞ structure

Theorem (Chan-Leung-L)

There exists a Fedosov flat connection:

$$D_F := \nabla - \delta + \sum_{n\geq 2} \frac{1}{\hbar}[I_n, -]_\star.$$

which is a quantum extension of L_∞ structure in the sense that

$$D_F|_{\mathcal{W}_X} = D_K.$$

From the general theory of Fedosov quantization, we know that there is a canonical one-to-one correspondence between smooth functions and flat sections of Weyl bundle under Fedosov connection:

$$C^\infty(X)[[\hbar]] \cong \Gamma^{\text{flat}}(X, \mathcal{W}_{X,\mathbb{C}}).$$

Quantizable functions

The Fedosov connection in the Theorem satisfies the property that we can evaluate the variable $\hbar = 1/k$ for any non-zero complex number and to obtain a flat connection which we call D_k. We can define a subclass of smooth functions:

Definition

A smooth function $f \in C^\infty(X)$ is called a quantizable function of level k if there exists a section ξ of the Weyl bundle such that the following conditions are satisfied:

1. The symbol of ξ is the function f, i.e. $\sigma(\xi) = f$.
2. $D_k(\xi) = 0$.
3. ξ has a uniform bound in the anti-holomorphic degree of Weyl bundle. Precisely, there exists $N > 0$, such that ξ is a section of $\mathcal{W}_X \otimes (\overline{\mathcal{W}_X})_{\leq N}$.

Quantizable functions

By explicit computation, we can show that the follow are examples of quantizable functions:

1. Holomorphic derivatives of Kähler potentials
2. Quantum moment maps of hamiltonian symmetries which preserves the complex structure

To show that the subclass of smooth functions can be quantized to differential operators, we define the Bargmann-Fock sheaf.

First of all, the Kähler form on X enables us to define the fiberwise Bargmann-Fock action, making the holomorphic Weyl bundle $\mathcal{W}_X$ a sheaf of $\mathcal{W}_{X,\mathbb{C}}$-modules. Explicitly, a monomial in $\mathcal{W}_{X,\mathbb{C}}$ acts as an operator on $\mathcal{W}_X$ as follows:

$$y^{i_1}\cdots y^{i_k}\bar{y}^{j_1}\cdots\bar{y}^{j_l}\mapsto \left(-\frac{\sqrt{-1}}{2}\hbar\right)^l\omega^{p_1\bar{j}_1}\cdots\omega^{p_l\bar{j}_l}\frac{\partial}{\partial y^{p_1}}\circ\cdots\frac{\partial}{\partial y^{p_l}}\circ m_{y^{i_1}\cdots y^{i_k}}$$

Bargmann-Fock sheaf

The Fedosov connection can be defined on $\mathcal{W}_X$ in a compatible way. However, this connection is not flat. For this we twist it by tensor powers of the prequantum line bundle L.

Definition

For every $k > 0$, we define the level k Bargmann-Fock sheaf by twisting $\mathcal{W}_X$ with tensor powers of the prequantum line bundle L:

$$\mathcal{F}_{L^{\otimes k}} := \mathcal{W}_X \otimes_{\mathcal{O}_X} L^{\otimes k}.$$

Theorem (L)

There exists a connection D_k on the level k Bargmann-Fock sheaf $\mathcal{F}_{L^{\otimes k}}$ which is compatible with the Bargmann-Fock action.

Bargmann-Fock sheaves

Similar to the standard result in Fedosov quantization that flat sections of the Weyl bundle corresponds to smooth functions, there is the following twisted version:

Theorem (L)

For any open set $U \subset X$, the space of flat sections of $\mathcal{F}_{L^{\otimes k}}$ under the connection $D_{\alpha,k}$ is canonically isomorphic to holomorphic sections of $L^{\otimes k}$.

The compatibility between the Fedosov connections and the Bargmann-Fock action implies that quantizable functions acts on the space of holomorphic sections $H^0(X, L^{\otimes k})$. From the explicit construction, it is easy to see that functions acts as differential operators.

Quantizable functions as differential operators

Theorem (L)

Suppose that X is prequantizable Kähler manifold, then for any integer $k \in \mathbb{Z}$, there is a natural isomorphism

$$\varphi : \mathcal{C}_k^\infty \to \mathcal{D}(L^{\otimes k}),$$

from the sheaf of algebra of level k almost holomorphic functions to the sheaf of holomorphic differential operators on $L^{\otimes k}$. In particular, this isomorphism is compatible with the filtration on almost holomorphic functions and that on differential operators by orders.

Thank You!