MacMahon KZ equation for quantum toroidal $\mathfrak{gl}_1$ algebra

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Overview

1. Introduction and Motivation

2. Quantum toroidal $\mathfrak{gl}_1$ algebra, its representations, and intertwiner

3. Derivation of the MacMahon KZ equations

Introduction and Motivation

Introduction and Motivation

AGT correspondence

- $4d$ AGT correspondence [Alday-Gaiotto-Tachikawa, 0906.3219]

Nekrasov factor of SUSY gauge theory on $\mathbb{R}^4$ with U(N) gauge group $=$ Conformal block of W_N algebra

- $5d$ AGT correspondence [Awata-Yamada, 0910.4431]

Nekrasov factor of SUSY gauge theory on $\mathbb{R}^4 \times S^1$ with U(N) gauge group $=$ Conformal block of qW_N algebra

It was shown in [Feigin-Shiraishi-et al., 1002.2485] that all of the q-deformed W_N algebra can be generated from an algebra called quantum toroidal $\mathfrak{gl}_1$ algebra. This motivates us to regard quantum toroidal $\mathfrak{gl}_1$ algebra $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ as the algebra underlying the $5d$ AGT correspondence. One way to justify this statement is to consider Fock q-KZ equation for quantum toroidal $\mathfrak{gl}_1$ [Awata-Kanno-et al., 1703.06084].

Introduction and Motivation

Fock q-KZ equation for quantum toroidal $\mathfrak{gl}_1$

q-KZ equation (of quantum toroidal $\mathfrak{gl}_1$) corresponding to representation ρ

Let ρ be a representation of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$. Then the q-KZ equation is defined to be the q-difference equation of the form

$$q^{z_i \partial/\partial z_i} \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \left[\prod_{i,j} R_{ij} \left(\frac{z_i}{z_j} \right) \right] \langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle .$$

where $\Phi_i(z_i)$ are the intertwiners of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ with respect to representation ρ, and $R_{ij}(z_i/z_j)$ are the R-matrix of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ with respect to the representation ρ.

- **Why q-KZ ?** $\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle = \prod_{i<j} \langle \Phi_i(z_i) \Phi_j(z_j) \rangle \overset{?}{=} \prod N_{\lambda\mu}$
- **Why Fock ?** Fock intertwiner $\Phi_\lambda(z)$ is labeled by a partition.

Introduction and Motivation

Goal

Goal

Our goal is to derive the MacMahon[1] KZ equation and solve it. Then, we express the solution of the KZ equation as the generalized Nekrasov factor.

- **Why MacMahon ?**
  - From the perspective of affine Yangian $\mathfrak{gl}_1$, the rational limit of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$, the MacMahon representation (plane partition representation) is more natural than the Fock representation.
  - MacMahon representation preserves the S_3 symmetry of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$, while Fock representation breaks it.

[1] MacMahon representation can be considered as the generalization of Fock representation

Quantum toroidal $\mathfrak{gl}_1$ algebra, its representations, and intertwiner

The goal of the part 2 is to introduce the notion of intertwiner of quantum toroidal algebra. However, in order to do this, we have to first discuss about the quantum toroidal algebra and its representations.

Definition of quantum toroidal $\mathfrak{gl}_1$ algebra

- Let $q_1, q_2, q_3 \in \mathbb{C}$ such that $q_1 q_2 q_3 = 1$ and if $q_1^a q_2^b q_3^c = 1$, then $a = b = c$.
- Quantum toroidal $\mathfrak{gl}_1$ algebra $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ (a.k.a. Ding-Iohara-Miki algebra) is the associative unital algebra with the generators $E_k, F_k, K_0^{\pm}, H_r$ $(k \in \mathbb{Z}, r \in \mathbb{Z}\backslash\{0\})$ and central element C satisfying the following defining relations:

$$g(z,w)\,K^{\pm}(C^{(1\mp 1)/2}z)E(w) + g(w,z)\,E(w)K^{\pm}(C^{(1\mp 1)/2}z) = 0,$$
$$g(w,z)\,K^{\pm}(C^{(1\pm 1)/2}z)F(w) + g(z,w)\,F(w)K^{\pm}(C^{(1\pm 1)/2}z) = 0,$$
$$\vdots$$

where $g(z,w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)$, and

$$E(z) = \sum_{k\in\mathbb{Z}} E_k z^{-k}, \quad F(z) = \sum_{k\in\mathbb{Z}} F_k z^{-k}, \quad K^{\pm}(z) = K_0^{\pm} \exp\left(\pm \sum_{r=1}^{\infty} H_{\pm r} z^{\mp r}\right).$$

- $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ has the Hopf algebra structure. Particularly, it has a coproduct $\Delta : U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \to U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \otimes U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$.

Representations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$

Vertical Fock representation

- Let $\mathcal{F}(v)$ be the free vector space which has the set of (2d) partitions as a basis. We define the map $\rho_v^{\mathcal{F}} : U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}(\mathcal{F}(v))$ by

$$\rho_v^{\mathcal{F}}(K^{\pm}(z))|\lambda\rangle = \mathfrak{q}^{-1}\prod_{i=1}^{\infty}\frac{(1-q_1^{\lambda_i}q_2^{i}v/z)(1-q_1^{\lambda_i-1}q_2^{i-2}v/z)}{(1-q_1^{\lambda_i}q_2^{i-1}v/z)(1-q_1^{\lambda_i-1}q_2^{i-1}v/z)}|\lambda\rangle,$$

$$\rho_v^{\mathcal{F}}(E(z))|\lambda\rangle = \sum_{k=1}^{\infty}\frac{1}{(1-q_1)}\delta\left(q_1^{\lambda_k}q_2^{k-1}\frac{v}{z}\right)\left[\prod_{i=1}^{k-1}\frac{(1-q_1^{\lambda_k-\lambda_i}q_2^{k-i-1})(1-q_1^{\lambda_k-\lambda_i+1}q_2^{k-i+1})}{(1-q_1^{\lambda_k-\lambda_i}q_2^{k-i})(1-q_1^{\lambda_k-\lambda_i+1}q_2^{k-i})}\right]|\lambda+1_k\rangle,$$

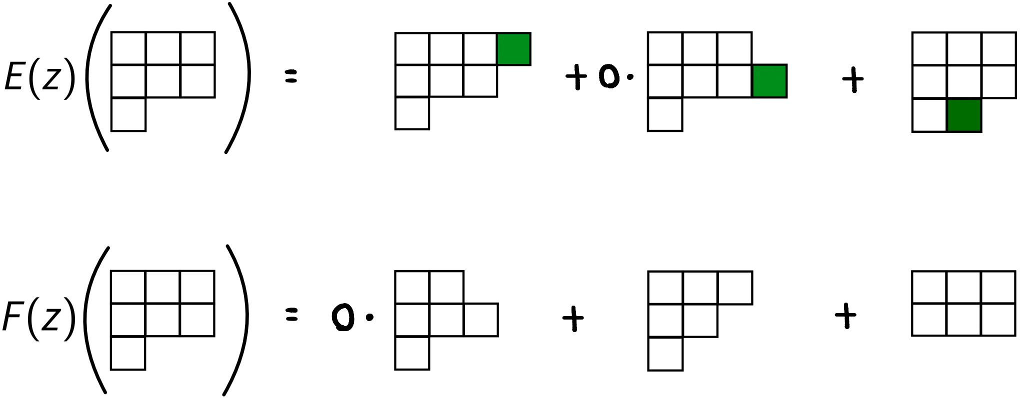
$$\rho^{\mathcal{F}}(F(z))|\lambda\rangle = \mathfrak{q}^{-1}\sum_{k=1}^{\infty}\frac{1}{(1-q_1^{-1})}\delta\left(q_1^{\lambda_k-1}q_2^{k-1}\frac{v}{z}\right)\left[\prod_{i=k+1}^{\infty}\frac{(1-q_1^{\lambda_i-\lambda_k+1}q_2^{i-k+1})(1-q_1^{\lambda_i-\lambda_k}q_2^{i-k-1})}{(1-q_1^{\lambda_i-\lambda_k+1}q_2^{i-k})(1-q_1^{\lambda_i-\lambda_k}q_2^{i-k})}\right]|\lambda-1_k\rangle.$$

  We can directly check that these actions satisfy all of the defining relations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$. So $\rho_v^{\mathcal{F}}$ is an algebra homomorphism and it is called the ***vertical Fock representation with spectral parameter*** v. We can interpret vertical Fock representation by using diagrams. In terms of the diagrams, the action of $E(z)$ (resp. $F(z)$) adds (resp. removes) a box into the diagrams.

Representations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$

Vertical Fock representation

- **Example** Consider the partition $\lambda = (3, 3, 1)$



Representations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$

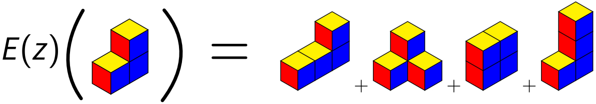
MacMahon representation

- Generalization of vertical Fock representation
- Let $\mathcal{M}(K;v)$ be the vector space which has the set of plane partitions as a basis. We call the algebra homomorphism $\rho^{\mathcal{M}}_{K,v} : U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}\left(\mathcal{M}(K;v)\right)$ defined by

$$\rho^{\mathcal{M}}_{K,v}\left(K^{\pm}(z)\right)|\Lambda\rangle = (\cdots)\,|\Lambda\rangle, \qquad \rho^{\mathcal{M}}_{K,v}\left(E(z)\right)|\Lambda\rangle = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(\cdots)|\Lambda + 1^{(k)}_m\rangle,$$

$$\rho^{\mathcal{M}}_{K,v}\left(F(z)\right)|\Lambda\rangle = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}(\cdots)|\Lambda - 1^{(k)}_m\rangle,$$

  the ***MacMahon representation with spectral parameter*** v. The MacMahon representation is first constructed in the paper [Feigin-Jimbo-Miwa-Mukhin, 2011].
- **Example :** Consider the plane partition $\Lambda = (\,(2),(1)\,)$



Representations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$

Horizontal Fock representation

$$\kappa_r = (q_1^r - 1)(q_2^r - 1)(q_3^r - 1)$$
$$\mathfrak{q} = q_3^{1/2}$$

- Let $\{a_r\}_{r\in\mathbb{Z}\setminus\{0\}}$ be the operators satisfying the commutation relation

$$[a_r, a_s] = \delta_{r+s,0}\frac{r}{\kappa_r}(\mathfrak{q}^r - \mathfrak{q}^{-r}).$$

- Define $\mathcal{H} = \mathrm{span}\,\{a_{-\lambda_1}a_{-\lambda_2}\cdots a_{-\lambda_n}|0\rangle \mid \lambda_1 \geq \cdots \geq \lambda_n\}$ where $|0\rangle$ is the vacuum state satisfying the annihilation condition $a_n|0\rangle = 0$
- $\rho_H^{(\mathfrak{q},1)} : U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}\,\mathcal{H}$ defined by

$$\rho_H^{(\mathfrak{q},1)}(E(z)) = \frac{u}{(1-q_1)(1-q_2)}\exp\left(\sum_{n=1}^{\infty}\frac{\kappa_n}{n}\frac{\mathfrak{q}^{-n/2}}{(\mathfrak{q}^n-\mathfrak{q}^{-n})}a_{-n}z^n\right)\exp\left(-\sum_{n=1}^{\infty}\frac{\kappa_n}{n}\frac{\mathfrak{q}^{-n/2}}{\mathfrak{q}^n-\mathfrak{q}^{-n}}a_nz^{-n}\right),$$

$$\rho_H^{(\mathfrak{q},1)}(F(z)) = \frac{u^{-1}}{(1-q_1^{-1})(1-q_2^{-1})}\exp\left(-\sum_{n=1}^{\infty}\frac{\kappa_n}{n}\frac{\mathfrak{q}^{n/2}}{\mathfrak{q}^n-\mathfrak{q}^{-n}}a_{-n}z^n\right)\cdot\exp\left(\sum_{n=1}^{\infty}\frac{\kappa_n}{n}\frac{\mathfrak{q}^{n/2}}{\mathfrak{q}^n-\mathfrak{q}^{-n}}a_nz^{-n}\right)$$

$$\rho_H^{(\mathfrak{q},1)}(K^{\pm}(z)) = \exp\left(\pm\sum_{r=1}^{\infty}\frac{\kappa_r}{r}\mathfrak{q}^{\pm r/2}a_{\pm r}z^{\mp r}\right), \qquad \rho_H^{(\mathfrak{q},1)}(C) = \mathfrak{q}.$$

is an algebra homomorphism. We call it the ***horizontal Fock representation of level*** 0.

Representations of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$

Horizontal Fock representation

- The map $\rho_H^{(\mathfrak{q},\mathfrak{q}^N)} : U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}\left(\mathcal{H}\right)$ defined by

$$\rho_H^{(\mathfrak{q},\mathfrak{q}^N)}\big(E(z)\big) = \rho_H^{(\mathfrak{q},1)}\big(E(z)\big) \cdot \left(\frac{\mathfrak{q}}{z}\right)^N,$$

$$\rho_H^{(\mathfrak{q},\mathfrak{q}^N)}\big(F(z)\big) = \rho_H^{(\mathfrak{q},1)}\big(F(z)\big) \cdot \left(\frac{\mathfrak{q}}{z}\right)^{-N},$$

$$\rho_H^{(\mathfrak{q},\mathfrak{q}^N)}\big(K^{\pm}(\mathfrak{q}^{1/2}z)\big) = \rho_H^{(\mathfrak{q},1)}\big(K^{\pm}(\mathfrak{q}^{1/2}z)\big)\mathfrak{q}^{\mp N}$$

is an algebra homomorphism. In other word, it forms a representation of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$. We call it the ***horizontal Fock representation of level*** N.

Intertwiner

Definition

Definition of intertwiner

Let $\mathcal{V}$ be a Fock (resp. MacMahon) representation, and $\mathcal{H}$ and $\mathcal{H}'$ be horizontal Fock representations. A Fock (resp. MacMahon) intertwiner is a map $\Psi : \mathcal{V} \otimes \mathcal{H} \to \mathcal{H}'$ defined by the intertwining relation

$$a\Psi = \Psi\Delta(a) \qquad \forall a \in U_{q_1,q_2}(\hat{\hat{\mathfrak{gl}}}_1).$$

- Since in the Fock (and MacMahon) representation, $[H_r, H_s] = 0$, there exists a simultaneous eigenbasis of $\{H_r\}_{r\in\mathbb{Z}\setminus\{0\}}$, say $\{\alpha\}$. For each α in this basis, we define the α-component intertwiner $\Psi_\alpha(\bullet)$ by

$$\Psi_\alpha(\bullet) = \Psi(\alpha \otimes \bullet) : \mathcal{H} \to \mathcal{H}'.$$

Intertwiner

Explicit expression

- Fock intertwiner

$$\Phi_\lambda(v) \sim \begin{aligned}&\exp\left(\sum_{r=1}^{\infty}\frac{1}{\mathfrak{q}^r-\mathfrak{q}^{-r}}\frac{\kappa_r}{r}a_{-r}\mathfrak{q}^{-r/2}v^r\left(\sum_{(i,j)\in\lambda}q_1^{r(j-1)}q_2^{r(i-1)}+\frac{1-q_3^r}{\kappa_r}\right)\right)\\&\times\exp\left(-\sum_{r=1}^{\infty}\frac{1}{\mathfrak{q}^r-\mathfrak{q}^{-r}}\frac{\kappa_r}{r}a_r\mathfrak{q}^{-r/2}v^{-r}\left(\sum_{(i,j)\in\lambda}q_1^{-r(j-1)}q_2^{-r(i-1)}-\frac{1-q_3^{-r}}{\kappa_r}\right)\right)\end{aligned}$$

- MacMahon intertwiner

$$\Xi_\Lambda(K;v) \sim \begin{aligned}&\exp\left(\sum_{r=1}^{\infty}\frac{1}{\mathfrak{q}^r-\mathfrak{q}^{-r}}\frac{\kappa_r}{r}\mathfrak{q}^{-r/2}a_{-r}v^r\left(\sum_{(i,j,k)\in\Lambda}q_1^{r(j-1)}q_2^{r(i-1)}q_3^{r(k-1)}+\frac{1-K^r}{\kappa_r}\right)\right)\\&\cdot\exp\left(-\sum_{r=1}^{\infty}\frac{1}{\mathfrak{q}^r-\mathfrak{q}^{-r}}\frac{\kappa_r}{r}\mathfrak{q}^{-r/2}a_r v^{-r}\left(\sum_{(i,j,k)\in\Lambda}q_1^{-r(j-1)}q_2^{-r(i-1)}q_3^{-r(k-1)}-\frac{1-K^{-r}}{\kappa_r}\right)\right).\end{aligned}$$

Remark

Note that if we take the 2*d partition limit* $\Lambda=\lambda,\ K=q_3$, the MacMahon intertwiner will reduce to the Fock intertwiner.

Derivation of the MacMahon KZ equations

Derivation of the MacMahon KZ equations

MacMahon KZ equation

Now we have already introduced all of the ingredients needed to derive the MacMahon KZ equation. Let's get started!

- First recall that a difference equation will be called MacMahon KZ equation if it takes the following form

$$q^{z_i \partial/\partial z_i} \langle 0| \underbrace{\Xi_1(z_1)\cdots\Xi_n(z_n)}_{\text{MacMahon intertwiners}} |0\rangle = \underbrace{\left[\prod_{i,j} R_{ij}\left(\frac{z_i}{z_j}\right)\right]}_{\text{Products of MacMahon } R\text{-matrix}} \langle 0|\Xi_1(z_1)\cdots\Xi_n(z_n)|0\rangle .$$

  for certain parameter q.
- The goal of this part is to find the equation which takes the above form.

Derivation of the MacMahon KZ equations

MacMahon KZ equation

- If we act on the MacMahon intertwiner $\Xi_\Lambda(K;v)$ by the operator $(\mathfrak{q}^{-2})^{v\frac{\partial}{\partial v}}$, the result will be the MacMahon intertwiner whose spectral parameter is scaled by $\mathfrak{q}^{-2}$. We can always write the scaled MacMahon intertwiner as a composition of $\mathcal{T}_\Lambda^{\pm}$ and the original MacMahon intertwiner.

$$(\mathfrak{q}^{-2})^{v\frac{\partial}{\partial v}}\Xi_\Lambda(K;v) = \Xi_\Lambda(K;\mathfrak{q}^{-2}v) = \mathcal{T}_\Lambda^{-} \cdot \Xi_\Lambda(K;v) \cdot \mathcal{T}_\Lambda^{+}$$

and

$$\mathcal{T}_\Lambda^{+}(v,K)\Xi_{\Lambda'}(K',v') = R_{\Lambda,\Lambda'}^{K,K'}\left(\frac{v}{\mathfrak{q}^{-1/2}v'}\right) \cdot \Xi_{\Lambda'}(K',v')\mathcal{T}_\Lambda^{+}(v,K),$$

$$\mathcal{T}_\Lambda^{-}(v,K)\Xi_{\Lambda'}(K',v') = R_{\Lambda',\Lambda}^{K',K}\left(\frac{v'}{\mathfrak{q}^{-1/2}v}\right) \cdot \Xi_{\Lambda'}(K',v')\mathcal{T}_\Lambda^{-}(v,K),$$

- If one uses other scaling parameter rather than $\mathfrak{q}^{\pm 2}$, the $\mathcal{T}_\Lambda^{\pm}(v,K)$ operators will no longer produce the MacMahon R-matrix.
- The MacMahon R-matrix $R_{\Lambda_1,\Lambda_2}^{K_1,K_2}(z)$ is computed in [Awata-Kanno-et al, 1810.07676] using the universal R-matrix of $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$ [Feigin-Jimbo-Miwa-Mukhin, 1603.02765]

Derivation of the MacMahon KZ equations

MacMahon KZ equation

- Define $G_{\Lambda^1\ldots\Lambda^n}\binom{K_1\ldots,K_n}{v_1,\ldots,v_n} := \langle 0|\Xi_{\Lambda^1}(K_1;v_1)\cdots\Xi_{\Lambda^n}(K_n;v_n)|0\rangle$
- If we act on $G_{\Lambda^1\ldots\Lambda^n}\binom{K_1\ldots,K_n}{v_1,\ldots,v_n}$ by the operator $(\mathfrak{q}^{-2})^{v_i\frac{\partial}{\partial v_i}}$, we then get that

$$
\begin{aligned}
&(\mathfrak{q}^{-2})^{v_i\frac{\partial}{\partial v_i}} G_{\Lambda^1\ldots\Lambda^n}\binom{K_1\ldots,K_n}{v_1,\ldots,v_n}\\
&= \langle 0|\Xi_{\Lambda^1}(K_1;v_1)\cdots\Xi_{\Lambda^i}(K_i;\mathfrak{q}^{-2}v_i)\cdots\Xi_{\Lambda^n}(K_n;v_n)|0\rangle\\
&= \langle 0|\Xi_{\Lambda^1}(K_1;v_1)\cdots\mathcal{T}^{-}_{\Lambda^i}\Xi_{\Lambda^i}(K_i;v_i)\mathcal{T}^{+}_{\Lambda^i}\cdots\Xi_{\Lambda^n}(K_n;v_n)|0\rangle\\
&= \left(\prod_{j=1}^{i-1} R^{K_j,K_i}_{\Lambda^j,\Lambda^i}\left(\mathfrak{q}^{1/2}\frac{v_j}{v_i}\right)\right)^{-1}\left(\prod_{j=i+1}^{n} R^{K_i,K_j}_{\Lambda^i,\Lambda^j}\left(\mathfrak{q}^{1/2}\frac{v_i}{v_j}\right)\right) G_{\Lambda^1\ldots\Lambda^n}\binom{K_1\ldots,K_n}{v_1,\ldots,v_n}
\end{aligned}
$$

This is the ***MacMahon KZ equation of*** $U_{q_1,q_2}(\widehat{\widehat{\mathfrak{gl}}}_1)$.

Derivation of the MacMahon KZ equations

Generalized Nekrasov factor

- (*Solution*) Define $A^{K_1,K_2}_{\Lambda_1,\Lambda_2}(v,w) = \langle 0|\Xi_{\Lambda^1}(K_1;v)\Xi_{\Lambda_2}(K_2;w)|0\rangle$. Then,

$$G_{\Lambda^1\dots\Lambda^n}\begin{pmatrix} K_1,\dots,K_n \\ v_1,\dots,v_n \end{pmatrix} = \prod_{\substack{j,l=1 \\ j<l}}^{n} A^{K_j,K_l}_{\Lambda^j,\Lambda^l}(v_j,v_l).$$

- We regard the quantity $A^{K_1K_2}_{\Lambda\Lambda'}(v,w)^{-1}$ as the ***generalized Nekrasov factor***. This is because when we take the 2*d partition limit* $\Lambda_1 = \lambda_1, \quad \Lambda_2 = \lambda_2, \quad K_1 = K_2 = q_3$, we then see that

$$A^{K_1K_2}_{\Lambda\Lambda'}(v,w)^{-1} \to A^{q_3,q_3}_{\lambda\lambda'}(v,w)^{-1} = (\text{proportional factors}) \cdot N_{\lambda\lambda'}(w,v).$$

- The MacMahon KZ equations and the generalized Nekrasov factor should be related to various objects in field and string theory, though currently many of these links remain to be discovered.