# Algebraic generalization of Jordan-Wigner transformation and its applications.

Kazuhiko Minami　Nagoya Univ.

- New fermionization method

  Examples（１-dim transverse Ising model,  Kitaev model,
  cluster model,   XY model,
  2-dim  Ising model ）
- Infinite number of solvable models, with c=m/2
- Jordan-Wigner transformation is a special case

- Phase diagram of 2-dim Kitaev model + Wen model
- realizations of the Onsager algebra

# Algebraic generalization of Jordan-Wigner transformation and its applications.

Kazuhiko Minami Nagoya Univ.

- New fermionization method

  Examples（１-dim transverse Ising model, Kitaev model,
  cluster model, XY model,
  2-dim Ising model ）

- Infinite number of solvable models, with c=m/2
- Jordan-Wigner transformation is a special case

- Phase diagram of 2-dim Kitaev model + Wen model
- realizations of the Onsager algebra

# Diagonalization of Hamiltonian

**partition function**

$$Z = \sum_j e^{-\beta E_j}$$

**Hamiltonian**

$$\begin{aligned} -\beta\mathcal{H} &= \sum_{j=1}^{N} (\text{ spin operators }) \\ &= \sum_p \left( c^\dagger(p) c(p) - \frac{1}{2} \right) \end{aligned}$$

Free fermion

**Jordan-Wigner transformation**

$$c_k = \exp\left[ i\pi \sum_{j=1}^{k-1} s_j^+ s_j^- \right] s_k^-, \qquad c_k^\dagger = \exp\left[ i\pi \sum_{j=1}^{k-1} s_j^+ s_j^- \right] s_k^+,$$

**c.f.**

$$\begin{aligned} e^{\pm i\pi s_j^+ s_j^-} &= 1 + \frac{(\pm i\pi)}{1!} s_j^+ s_j^- + \frac{(\pm i\pi)^2}{2!} s_j^+ s_j^- + \ldots \\ &= 1 - 2 s_j^+ s_j^- = -\sigma_j^z. \end{aligned}$$

# Fermionization formula

**Find the series of operators**

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \cdots$$

**which satisfy**

$$\eta_j \eta_k = -\eta_k \eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j \eta_k = \eta_k \eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

**Then the following Hamiltonian is solvable**

$$-\beta \mathcal{H} = \sum_{j=1}^{N} K_j \eta_j$$

**with the use of the transformation** **we have**

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j, \qquad \{\varphi_j, \varphi_k\} = \varphi_j \varphi_k + \varphi_k \varphi_j = \delta_{jk}.$$

# Transverse Ising model

$$-\beta\mathcal{H} = K\sum_{j=1}^{N}\sigma_j^z\sigma_{j+1}^z + h\sum_{j=1}^{N}\sigma_j^x$$

**that consists of the following operators**

$$\sigma_1^x \quad \sigma_1^z\sigma_2^z \quad \sigma_2^x \quad \sigma_2^z\sigma_3^z \quad \sigma_3^x \quad \dots \quad \sigma_j^z\sigma_{j+1}^z$$

**Let** $\eta_1 = \sigma_1^x$**,** $\eta_2 = \sigma_1^z\sigma_2^z$**, and generally** $\eta_{2j-1} = \sigma_j^x \quad \eta_{2j} = \sigma_j^z\sigma_{j+1}^z$**, then**

$$\eta_j\eta_k = -\eta_k\eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j\eta_k = \eta_k\eta_j \quad (|j-k| \geq 2 : \text{otherwise commute}).$$

**The transverse Ising model is, therefore, obtained from**

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \dots$$

**which satisfy the above commutation relation. The Hamiltonian is written as**

$$-\beta\mathcal{H} = K\sum_{j=even}^{N}\eta_j + h\sum_{j=odd}^{N}\eta_j$$

**Let us introduce the transformation**

$$\varphi_j = \eta_1 \eta_2 \cdots \eta_j.$$

**If** $j \neq k$**,** $\varphi_j \varphi_k = (\eta_1 \cdots \eta_j)(\eta_1 \cdots \eta_j \cdots \eta_k) = (-1)^{j-1} \eta_{j+1} \cdots \eta_k$
$\varphi_k \varphi_j = (\eta_1 \cdots \eta_j \cdots \eta_k)(\eta_1 \cdots \eta_j) = (-1)(-1)^{j-1} \eta_{j+1} \cdots \eta_k = -\varphi_j \varphi_k,$
**if** $k = j$**,** $\varphi_j \varphi_j = (\eta_1 \cdots \eta_j)(\eta_1 \cdots \eta_j) = (-1)^{j-1}.$

**Let us re-define the transformation as**

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(j-1)} \eta_1 \eta_2 \cdots \eta_j$$

**then we find the commutation relation**

$$\varphi_j \varphi_k = -\varphi_k \varphi_j, \quad \varphi_j^2 = \frac{1}{2} \quad \rightarrow \qquad \{\varphi_j, \varphi_k\} = \varphi_j \varphi_k + \varphi_k \varphi_j = \delta_{jk}.$$

**New operators $\varphi_j$ are written explicitely as**

$$\begin{aligned}
\varphi_1 &= \frac{1}{\sqrt{2}}\eta_1 = \frac{1}{\sqrt{2}}\sigma_1^x \\
\varphi_2 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}}\eta_1\eta_2 = \frac{1}{\sqrt{2}}i\sigma_1^x\sigma_1^z\sigma_2^z = \frac{1}{\sqrt{2}}\sigma_1^y\sigma_2^z \\
\varphi_3 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}2}\eta_1\eta_2\eta_3 = \frac{1}{\sqrt{2}}i^2\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x = \frac{-1}{\sqrt{2}}\sigma_1^y\sigma_2^y \\
\varphi_4 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}3}\eta_1\eta_2\eta_3\eta_4 = \frac{1}{\sqrt{2}}i^3\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x\sigma_2^z\sigma_3^z = \frac{1}{\sqrt{2}}\sigma_1^y\sigma_2^x\sigma_3^z \\
\varphi_5 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}4}\eta_1\eta_2\cdots\eta_5 = \frac{1}{\sqrt{2}}i^4\sigma_1^x\sigma_1^z\sigma_2^z\sigma_2^x\sigma_2^z\sigma_3^z\sigma_3^x = \frac{-1}{\sqrt{2}}\sigma_1^y\sigma_2^x\sigma_3^y
\end{aligned}$$

**Introducing an initial operator $\eta_0 = i\sigma_1^z$ (to avoid irregular behaviors coming from the boundary) we find**

$$\begin{aligned}
\varphi_{2j} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(2j-1)}\eta_0\eta_1\eta_2\cdots\eta_{2j} = \frac{1}{\sqrt{2}}\sigma_1^x\sigma_2^x\sigma_3^x\cdots\sigma_j^x\sigma_{j+1}^z \\
\varphi_{2j+1} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}2j}\eta_0\eta_1\eta_2\cdots\eta_{2j+1} = \frac{-1}{\sqrt{2}}\sigma_1^x\sigma_2^x\sigma_3^x\cdots\sigma_j^x\sigma_{j+1}^y
\end{aligned}$$

**Hamiltonian is written by two-body terms of** $\varphi_j$

$$-\beta\mathcal{H} = K\sum_{j=\mathrm{odd}}^{N}(-2i)\varphi_j\varphi_{j+1} + h\sum_{j=\mathrm{even}}^{N}(-2i)\varphi_j\varphi_{j+1},$$

**this is because** $\eta_{j+1}$ **is proportional to** $\varphi_j\varphi_{j+1}$

$$\begin{aligned}\varphi_j\varphi_{j+1} &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(j-1)}\eta_0\eta_1\eta_2\cdots\eta_j\frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}j}\eta_0\eta_1\eta_2\cdots\eta_j\eta_{j+1}\\ &= (\frac{1}{\sqrt{2}})^2e^{i\frac{\pi}{2}}\eta_{j+1} = \frac{i}{2}\eta_{j+1}.\end{aligned}$$

**Boundary term**

$$\begin{aligned}\varphi_{2N-1}\varphi_0 &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{2}(2N-2)}\eta_0\eta_1\cdots\eta_{2N}\frac{1}{\sqrt{2}}\eta_0\eta_1\\ &= \frac{i}{2}(-\sigma_1^x\cdots\sigma_N^x)\sigma_N^z\sigma_1^z\end{aligned}$$

**The cyclic boundary condition can be introduced**

$$\begin{aligned}\sigma_1^x\cdots\sigma_N^x = -1 \quad &\rightarrow \quad \varphi_{2N} = +\varphi_0\\ \sigma_1^x\cdots\sigma_N^x = +1 \quad &\rightarrow \quad \varphi_{2N} = -\varphi_0\end{aligned}$$

**Fourier transformation**

$$\varphi_j = \frac{1}{\sqrt{2N}} \sum_{0<q<\pi} (e^{iqj} C(q) + e^{-iqj} C^\dagger(q)),$$

**where $C(q)$ are the fermion operators**

$$\{C^\dagger(p), C(q)\} = \delta_{pq}, \quad \{C^\dagger(p), C^\dagger(q)\} = \{C(p), C(q)\} = 0.$$

**Finally, the Hamiltonian is written by two-body terms of $C(q)$ and hence diagonaizable**

$$\begin{aligned} -\beta\mathcal{H} &= K \sum_{j=1}^{N/2} (-2i) \varphi_{2j-2}\varphi_{2j-1} + h \sum_{j=1}^{N/2} (-2i) \varphi_{2j-1}\varphi_{2j} \\ &= (-i) \sum_{0<q<\pi} [(K-h) e^{-iq} C(q) C(\pi - q) + (K-h) e^{iq} C^\dagger(q) C^\dagger(\pi - q) \\ &\qquad (K+h) e^{-iq} C(q) C^\dagger(q) + (K+h) e^{iq} C^\dagger(q) C(q)] \end{aligned}$$

**Introducing the basis states $|0\,0\rangle$, $|1\,0\rangle = C^\dagger(q)|0\,0\rangle$, $|0\,1\rangle = C^\dagger(\pi - q)|0\,0\rangle$, $|1\,1\rangle = C^\dagger(\pi - q)C^\dagger(q)|0\,0\rangle$, the Hamiltonian can be expressed as**

$$-\beta\mathcal{H} = 2\sum_{0<q<\pi/2}\begin{bmatrix} (K+h)\sin q & 0 & 0 & i(K-h)\cos q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i(K-h)\cos q & 0 & 0 & -(K+h)\sin q \end{bmatrix}$$

**Eigenvalues**

$$0, \;\; 0, \;\; \pm\Lambda_q, \qquad \Lambda_q = 2\sqrt{K^2 + h^2 - 2Kh\cos 2q}$$

**Partition function**

$$Z = \prod_{0<q<\pi/2}(1 + 1 + e^{\Lambda_q} + e^{-\Lambda_q}) = \prod_{0<q<\pi/2}(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2$$

**The free energy**

$$-\beta f = \lim_{N\to\infty}\frac{\log Z}{N} = \frac{1}{\pi}\int_0^{\pi/2}\log(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})\,dq.$$

# Summary of the formula

**Find the series of operators**

$$\eta_1 \qquad \eta_2 \qquad \eta_3 \qquad \eta_4 \qquad \eta_5 \qquad \cdots$$

**which satisfy**

$$\eta_j \eta_k = -\eta_k \eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j \eta_k = \eta_k \eta_j \qquad (|j-k| \geq 2 : \text{otherwise commute}).$$

**Then the following Hamiltonian is solvable**

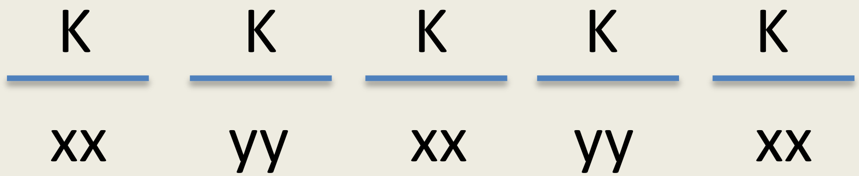
$$-\beta \mathcal{H} = \sum_{j=1}^{N} K_j \eta_j$$

**with the use of the transformation**

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2} (j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j .$$

# One-dimensional quantum systems

# Period=1




$$-\beta\mathcal{H} = K\Big(\sum_{j=\mathrm{odd}}^{N} \sigma_j^x \sigma_{j+1}^x + \sum_{j=\mathrm{even}}^{N} \sigma_j^y \sigma_{j+1}^y\Big)$$

**Series of operators**

$$\eta_1 = \sigma_1^x \sigma_2^x \quad \eta_2 = \sigma_2^y \sigma_3^y \quad \eta_3 = \sigma_3^x \sigma_4^x \quad \eta_4 = \sigma_4^y \sigma_5^y \qquad \ldots \qquad \eta_{2N} = \sigma_N^y \sigma_1^y$$

**Hamiltonian**

$$-\beta\mathcal{H} = K \sum_{j=1}^{N} \eta_j$$

**Fourier transformation**

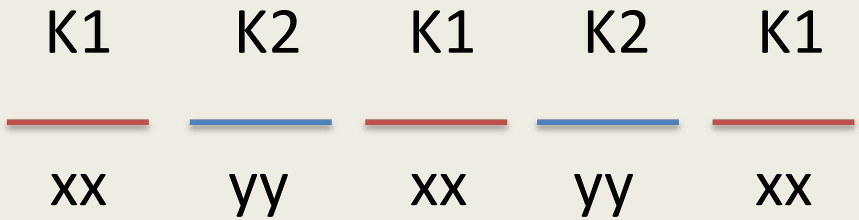
$$\varphi_j \;=\; \frac{1}{\sqrt{2N}} \sum_{0<q<\pi} \left(e^{iqj} C(q) + e^{-iqj} C^\dagger(q)\right)$$

**where** $\{C^\dagger(p), C(q)\} = \delta_{pq}, \quad \{C^\dagger(p), C^\dagger(q)\} = \{C(p), C(q)\} = 0.$

**Partition function** $\quad Z = \prod_{0<q<\pi} 2\left(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q}\right)^2 \qquad \Lambda_q = 4K \sin q$

**Free energy** $\quad -\beta f = \frac{1}{\pi} \int_0^\pi \log\left(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q}\right) dq + \frac{1}{2}\log 2$

Period=2




$$-\beta\mathcal{H} = K_1 \sum_{j=\text{odd}}^{N} \sigma_j^x \sigma_{j+1}^x + K_2 \sum_{j=\text{even}}^{N} \sigma_j^y \sigma_{j+1}^y$$

1-dim. Kitaev model

**Operators**

$$\eta_1 = \sigma_1^x \sigma_2^x \quad \eta_2 = \sigma_2^y \sigma_3^y \quad \eta_3 = \sigma_3^x \sigma_4^x \quad \eta_4 = \sigma_4^y \sigma_5^y \quad \ldots \quad \eta_{2N} = \sigma_N^y \sigma_1^y$$

**then the Hamiltonian is written as**

$$-\beta\mathcal{H} = K_1 \sum_{j=\text{odd}}^{N} \eta_j + K_2 \sum_{j=\text{even}}^{N} \eta_j$$

**Fourier transformation, let** $\varphi_1 = \varphi_1(1)$, $\varphi_2 = \varphi_2(1)$, $\varphi_3 = \varphi_1(2)$, $\varphi_4 = \varphi_2(2)$

$$\varphi_k(j) = \frac{1}{\sqrt{N}} \sum_{0<q<\pi} (e^{iqj} C_k(q) + e^{-iqj} C_k^\dagger(q)) \qquad k = 1, 2$$

**Partition function** $\quad Z = \prod_{0<q<\pi} (e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q})^2 \quad \Lambda_q = 2\sqrt{K_1^2 + K_2^2 + 2K_1K_2\cos q}$

**The free energy is identical to that of the transverse Ising model**

$$-\beta\mathcal{H} = K \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z + h \sum_{j=1}^{N} \sigma_j^x$$


# Period=3

**Series of operators**

$$\varphi_1(1),\ \ \varphi_2(1),\ \ \varphi_3(1),\ \ \varphi_1(2),\ \ \varphi_2(2),\ \ \varphi_3(2),\ \ \varphi_1(3),\ \ \ldots$$

**Hamiltonian**

$$\begin{aligned} -\beta\mathcal{H} &= K_1\sum_{j=1}^{N}\varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^{N}\varphi_2(j)\varphi_3(j) + K_3\sum_{j=1}^{N}\varphi_3(j)\varphi_1(j+1) \\ &= K_1(-2i)\sum_{0<q<\pi}[C_1^\dagger(q)C_2(q) + C_1(q)C_2^\dagger(q)] \\ &+ K_2(-2i)\sum_{0<q<\pi}[C_2^\dagger(q)C_3(q) + C_2(q)C_3^\dagger(q)] \\ &+ K_3(-2i)\sum_{0<q<\pi}[e^{iq}C_3^\dagger(q)C_1(q) + e^{-iq}C_3(q)C_1^\dagger(q)] \end{aligned}$$

**Fourier transformation**

$$\varphi_k(j) = \frac{1}{\sqrt{N}}\sum_{0<q<\pi}(e^{iqj}C_k(q) + e^{-iqj}C_k^\dagger(q)), \quad k = 1,2,3$$

**Partitiom function**

$$Z = \prod_{0<q<\pi}\sum_{l=1}^{8}e^{\lambda_l}$$

**the eigenvalues are obtained from the following equation**

$$0 = \lambda^3 - 4(K_1^2 + K_2^2 + K_3^2)\lambda \pm 16K_1K_2K_3\sin q$$

**when** $K_1 = K_2 = K_3 \ (= K)$

$$\begin{aligned} 0 &= \lambda^3 - 12K^2\lambda + 16K^3 \sin q, \qquad \sin q = -4 \sin\frac{q+\pi}{3} \sin\frac{q+2\pi}{3} \sin\frac{q+3\pi}{3} \\ &= (\lambda - 4K\sin\frac{q+\pi}{3})(\lambda - 4K\sin\frac{q+5\pi}{3})(\lambda - 4K\sin\frac{q+3\pi}{3}) \end{aligned}$$

**after all** $\lambda_l$ **are obtained as**

$$\lambda = 0,\ \ 0,\ \ 4K\sin\frac{q+n\pi}{3} \qquad n = 0,1,2,3,4,5$$

**Partition function**

$$\begin{aligned} Z &= \prod_{0<q<\pi} \left[1 + 1 + \sum_{n=0}^{5} e^{4K\sin\frac{q+n\pi}{3}}\right] \quad q = \frac{l}{N}\pi \\ &= \prod_{0<p<\pi} \left[e^{2K\sin\frac{p}{3}} + e^{-2K\sin\frac{p}{3}}\right] \qquad p = \frac{l}{3N}\pi \end{aligned}$$

# Period=4

K1 K2 K3 K4 K1

**Series of operators**

$$\varphi_1(1),\ \varphi_2(1),\ \varphi_3(1),\ \varphi_4(1),\ \varphi_1(2),\ \varphi_2(2),\ \dots$$

**Hamiltonian**

$$\begin{aligned}-\beta\mathcal{H} &= K_1\sum_{j=1}^{N}\varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^{N}\varphi_2(j)\varphi_3(j) + K_3\sum_{j=1}^{N}\varphi_3(j)\varphi_4(j) + K_4\sum_{j=1}^{N}\varphi_4(j)\varphi_1(j+1)\\ &= K_1(-2i)\sum_{0<q<\pi}[C_1^\dagger(q)C_2(q)+C_1(q)C_2^\dagger(q)] + K_2(-2i)\sum_{0<q<\pi}[C_2^\dagger(q)C_3(q)+C_2(q)C_3^\dagger(q)]\\ &+ K_3(-2i)\sum_{0<q<\pi}[C_3^\dagger(q)C_4(q)+C_3(q)C_4^\dagger(q)]\\ &+ K_4(-2i)\sum_{0<q<\pi}[e^{iq}C_4^\dagger(q)C_1(q)+e^{-iq}C_4(q)C_1^\dagger(q)]\end{aligned}$$

**Partition function**

$$Z = \prod_{0<q<\pi}\sum_{l=1}^{16} e^{\lambda_l}$$

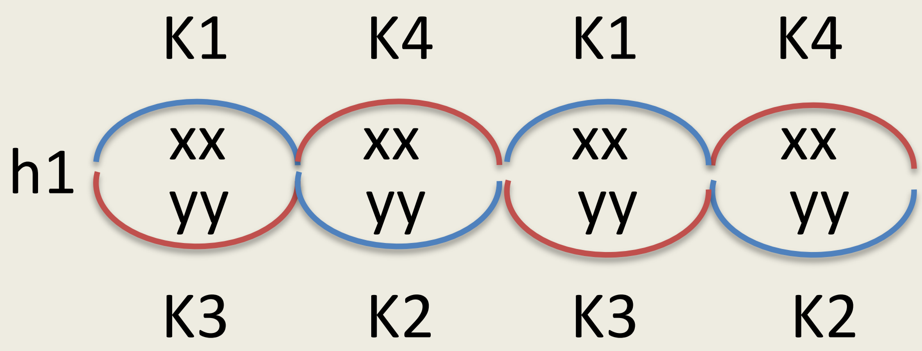
**eigenvalues** $\lambda_l$ **are** $0,0,0,0$ **and the solutions of**

$$0 = \lambda^4 - 4(K_1^2+K_2^2+K_3^2+K_4^2)\lambda^2 + 16(K_1^2K_3^2+K_2^2K_4^2-2K_1K_2K_3K_4\cos q)$$

**are** $2$**-fold degenerated eigenvalues, and the solutions of**

$$\begin{aligned}0 &= \lambda^4 - 8(K_1^2+K_2^2+K_3^2+K_4^2)\lambda^2\\ &+ 16(K_1^2+K_2^2+K_3^2+K_4^2)^2 - 64(K_1^2K_3^2+K_2^2K_4^2-2K_1K_2K_3K_4\cos q)\end{aligned}$$

# 1-dim. XY model




two series

$$\varphi_1(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-2}^z\sigma_{2j-1}^y \quad \varphi_2(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-1}^z\sigma_{2j}^x$$

$$\varphi_3(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-2}^z\sigma_{2j-1}^x \quad \varphi_4(j) = \frac{1}{\sqrt{2}}\sigma_1^z\sigma_2^z\cdots\sigma_{2j-1}^z\sigma_{2j}^y$$

**then**

external field

$$\varphi_1(j)\varphi_2(j) = \frac{i}{2}\sigma_{2j-1}^x\sigma_{2j}^x \quad \varphi_2(j)\varphi_1(j+1) = \frac{-i}{2}\sigma_{2j}^y\sigma_{2j+1}^y \quad \varphi_1(j)\varphi_3(j) = \frac{-i}{2}\sigma_{2j-1}^z$$

$$\varphi_3(j)\varphi_4 j) = \frac{-i}{2}\sigma_{2j-1}^y\sigma_{2j}^y \quad \varphi_4(j)\varphi_3(j+1) = \frac{i}{2}\sigma_{2j}^x\sigma_{2j+1}^x \quad \varphi_2(j)\varphi_4(j) = \frac{i}{2}\sigma_{2j}^z$$

**Hamiltonian**

$$\begin{aligned} -\beta\mathcal{H} &= K_1\sum_{j=1}^N \varphi_1(j)\varphi_2(j) + K_2\sum_{j=1}^N \varphi_2(j)\varphi_3(j+1) + h_1\sum_{j=1}^N \varphi_1(j)\varphi_3(j) \\ &+ K_3\sum_{j=1}^N \varphi_3(j)\varphi_4(j) + K_4\sum_{j=1}^N \varphi_4(j)\varphi_3(j+1) + h_2\sum_{j=1}^N \varphi_2(j)\varphi_4(j) \end{aligned}$$

**Partition function**

$$Z = \prod_{0<q<\pi} \sum_{l=1}^{16} e^{\lambda_l} = \prod_{0<q<\pi} (e^P + e^{-P} + e^Q + e^{-Q})^2$$

$$P = \Big(A + 2|B|\Big)^{1/2} \qquad Q = \Big(A - 2|B|\Big)^{1/2}$$

$$\begin{aligned} A &= K_1^2 + K_2^2 + K_3^2 + K_4^2 + 2K_1K_2\cos q + 2K_3K_4\cos q + h_1^2 + h_2^2 \\ B &= (K_1 + K_2e^{-iq})(K_3 + K_4e^{iq}) - h_1h_2 \end{aligned}$$

**Let** $K_1 = K_4 = K(1+\gamma)$, $K_3 = K_2 = K(1-\gamma)$, $h_1 = h_2 = h$**, then**

$$\begin{aligned} Z &= \prod_{0<p<\pi} (e^R + e^{-R})^2 \\ R &= \sqrt{(h - 2K\cos p)^2 + (2K\gamma\sin p)^2} \end{aligned}$$

**(Lieb-Schultz-Mattis 1961, Katsura 1962, Niemeijer 1967)**

# Infinite series of solvable spin chains

# 1-dim. cluster model

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + (K_2\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y + K_3\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^y)$$

**Generalized cluster model**

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + K_2\sum_{j=1}^{N}\sigma_j^x\,\mathbf{1}_{j+1}\sigma_{j+2}^x$$

**Operators $\sigma_1^x\sigma_2^z\sigma_3^x$, $\sigma_2^x\,\mathbf{1}_3\sigma_4^x$, $\sigma_3^x\sigma_4^z\sigma_5^x$, $\sigma_4^x\,\mathbf{1}_5\sigma_6^x$, $\ldots$ satisfy the condition, and hence can be solved.**

**This model is diagonalized by the transformation**

$$\begin{aligned}\varphi_{2j-1} &= \frac{1}{\sqrt{2}}\,(\,\mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\,\mathbf{1}_{2j-1})\,\sigma_{2j}^y\,\sigma_{2j+1}^x,\\ \varphi_{2j} &= \frac{1}{\sqrt{2}}\,(\,\mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\,\mathbf{1}_{2j-1}\,\sigma_{2j}^z)\sigma_{2j+1}^x\sigma_{2j+2}^x,\end{aligned}$$

**which is different from the Jordan-Wigner transformation**

Cannot be solved by the Jordan-Wigner transformation

# Solvable series of spin chains ( Minami, Nuclear Physics B 2017)

Table 1: Examples of solvable Hamiltonians $-\beta\mathcal{H}(k,n,l)$ obtained as a linear combination of $\eta_j$ and their shifted operators. Generalizations following (7) of these Hamiltonians can be found in Table 2.

| $(k,n,l)$ | $-\beta\mathcal{H}(k,n,l)$ | |
|---|---|---|
| $(1,n,l)$ | $K_1\sum_{j=1}^{N}\Big(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\Big)\Big(\prod_{\nu=1}^{l}\sigma^z_{j+n+\nu-1}\Big)\Big(\prod_{\nu=1}^{n}\sigma^x_{j+n+l+\nu-1}\Big)$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}$ |
| $(2,n,l)$ | $K_1\sum_{j=1}^{N}\Big(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\Big)\Big(\prod_{\nu=1}^{l}\sigma^z_{j+n+\nu-1}\Big)\Big(\prod_{\nu=1}^{n}\sigma^x_{j+n+l+\nu-1}\Big)$ | $+K_2\sum_{j=1}^{N}\sigma^y_j\sigma^y_{j+1}$ |
| $(3,n,l)$ | $K_1\sum_{j=1}^{N}\Big(\prod_{\nu=1}^{n}\sigma^x_{j+\nu-1}\Big)\sigma^z_{j+n}\Big(\prod_{\nu=1}^{n}\sigma^x_{j+n+\nu}\Big)$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{l}\mathbf{1}_{j+\nu}\Big)\sigma^x_{j+l+1}$ |
| $(4,n,-)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\Big)\sigma^x_{j+n+1}$ | $+K_2\sum_{j=1}^{N}\sigma^z_j\Big(\prod_{\nu=1}^{n}\mathbf{1}_{j+\nu}\Big)\sigma^z_{j+n+1}$ |
| $(5,n,-)$ | $K_1\sum_{j=1}^{N}\Big(\prod_{\nu=1}^{n}\sigma^z_{j+\nu-1}\Big)$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}$ |
| $(6,n,-)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{n}\mathbf{1}_{j+\nu}\Big)\sigma^x_{j+n+1}$ | $+K_2\sum_{j=1}^{N}\sigma^z_j$ |
| $(7,n,l)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\Big)\sigma^x_{j+n+1}$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{l}\sigma^z_{j+\nu}\Big)\sigma^x_{j+l+1}$ |
| $(8,n,l)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{n}\sigma^z_{j+\nu}\Big)\sigma^x_{j+n+1}$ | $+K_2\sum_{j=1}^{N}\sigma^y_j\Big(\prod_{\nu=1}^{l}\sigma^z_{j+\nu}\Big)\sigma^y_{j+l+1}$ |
| $(9,n,-)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\sigma^x_{j+2}\Big(\prod_{\nu=1}^{n}\sigma^z_{j+2+\nu}\Big)\sigma^x_{j+n+3}\sigma^x_{j+n+4}\sigma^x_{j+n+5}$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\Big(\prod_{\nu=1}^{n+2}\sigma^z_{j+\nu}\Big)\sigma^x_{j+n+3}$ |
| $(10,n,-)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\Big(\prod_{\nu=1}^{n}\sigma^z_{j+1+\nu}\Big)\sigma^x_{j+n+2}\sigma^x_{j+n+3}$ | $+K_2\sum_{j=1}^{N}\Big(\prod_{\nu=1}^{n+2}\sigma^z_{j+\nu-1}\Big)$ |
| $(11,-,-)$ | $K_1\sum_{j=1}^{N}\sigma^x_j\sigma^x_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$ | $+K_2\sum_{j=1}^{N}\sigma^x_j\sigma^z_{j+1}\sigma^x_{j+2}$ |

# Solvable series of spin chains

**Generally the following Hamiltonians can be diagonalized**

$$-\beta\mathcal{H} = \sum_l K_l \sum_{j=1}^{N} (-2i)\varphi_p(j)\varphi_q(j+l) \quad p,\ q = 1 \text{ or } 2$$

Two-body terms of C(q)

**For example**

$$-\beta\mathcal{H} = \sum_{j=1}^{N} [\ \cdots + K_{-1}(-2i)\varphi_2(j)\varphi_1(j-1) + K_0(-2i)\varphi_2(j)\varphi_1(j)$$
$$+K_1(-2i)\varphi_2(j)\varphi_1(j+1) + K_2(-2i)\varphi_2(j)\varphi_1(j+2) + \cdots]$$
$$= \sum_{j=1}^{N} [\ \cdots + K_{-1}\eta_{2j-3}\eta_{2j-2}\eta_{2j-1} + \underline{K_0\eta_{2j-1} + K_1\eta_{2j}} + K_2\eta_{2j}\eta_{2j+1}\eta_{2j+2} + \cdots].$$

**In the case of operators** $\eta_{2j-1} = \sigma_j^z$ **and** $\eta_{2j} = \sigma_j^x\sigma_{j+1}^x$

$$-\beta\mathcal{H} = \sum_{j=1}^{N} [\ \cdots + K_{-1}\sigma_{j-1}^y\sigma_j^y + \underline{K_0\sigma_j^z + K_1\sigma_j^x\sigma_{j+1}^x} + K_2\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + \cdots]$$

Transverse Ising

$$= \sum_{j=1}^{N} [\ \cdots + \quad YY \quad + \quad Z \quad + \quad XX \quad + \quad XZX \quad + \cdots]$$

**(Suzuki 1971)**

Table 2: Each $(k,n,l)$ provides a solvable Hamiltonian, where six interactions in (**??**) are explicitly written. One can find the operators $\{\eta_j^{(k,n,l)}\}$ in the second and third row of the first column, i.e. $\eta_{2j-1}^{(k,n,l)} = +2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j)$ and $\eta_{2j}^{(k,n,l)} = -2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$, from which a solvable series of interactions are generated. The initial operator $\eta_0^{(k,n,l)}$ and the transformation $\varphi_j^{(k,n,l)}$ that diagonalize the Hamiltonian are given in the last row. The first case $(k,n,l) = (1,0,1)$ includes the transverse Ising model, the XY model, and the cluster model, as special cases. The second case $(k,n,l) = (2,1,0)$ includes the one-dimensional Kitaev model. The operators $\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x$ are the stabilizers of the cluster state. Models except the cases $(1,0,1)$ and $(2,1,0)$ cannot be solved by the Jordan-Wigner transformation.

| $(k,n,l)$ | | | | |
|---|---|---|---|---|
| $-2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j-1)$ | | | | |
| $+2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k)}(j)$ $=\eta_{2j-1}^{(k,n,l)}$ | $-2i\varphi_1^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$ | | | |
| $-2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+1)$ $=\eta_{2j}^{(k,n,l)}$ | $-2i\varphi_2^{(k,n,l)}(j)\varphi_2^{(k,n,l)}(j+1)$ | | | |
| $+2i\varphi_2^{(k,n,l)}(j)\varphi_1^{(k,n,l)}(j+2)$ | | | | |
| $\eta_0^{(k,n,l)}$ | $\varphi_{2j}^{(k,n,l)}$ | and | $\varphi_{2j+1}^{(k,n,l)}$ | $(j=0,1,2,3,\ldots)$ |

Transv. Ising model
XY model
cluster model

| $(1,0,1)$ | | |
|---|---|---|
| $\sigma_{j-1}^y\sigma_j^y$ | | |
| $\sigma_j^z$ | $\sigma_j^y\sigma_{j+1}^x$ | |
| $\sigma_j^x\sigma_{j+1}^x$ | $\sigma_j^x\sigma_{j+1}^y$ | |
| $\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x$ | | |
| $i\sigma_1^x$ | $\varphi_{2j}^{(1,0,1)} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma_\nu^z)\sigma_{j+1}^x$ | $\varphi_{2j+1}^{(1,0,1)} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma_\nu^z)\sigma_{j+1}^y$ |

Jordan-Wigner trans.

XY model

| $(2,1,0)$ | | |
|---|---|---|
| $\sigma_{2j-3}^x\sigma_{2j-2}^z\sigma_{2j-1}^z\sigma_{2j}^x$ | | |
| $\sigma_{2j-1}^x\sigma_{2j}^x$ | $\sigma_{2j-1}^x\sigma_{2j}^z\sigma_{2j+1}^y$ | |
| $\sigma_{2j}^y\sigma_{2j+1}^y$ | $\sigma_{2j}^y\sigma_{2j+1}^z\sigma_{2j+2}^x$ | |
| $\sigma_{2j}^y\sigma_{2j+1}^z\sigma_{2j+2}^z\sigma_{2j+3}^y$ | | |
| $i\sigma_1^y$ | $\varphi_{2j}^{(2,1,0)} = \frac{(-1)^j}{\sqrt{2}}(\prod_{\nu=1}^{2j}\sigma_\nu^z)\sigma_{2j+1}^y$ | $\varphi_{2j+1}^{(2,1,0)} = \frac{(-1)^j}{\sqrt{2}}(\prod_{\nu=1}^{2j+1}\sigma_\nu^z)\sigma_{2j+2}^x$ |

Jordan-Wigner trans.

Cannot be solved by the Jordan-Wigner transformation

X Z X + X 1 X

| (3, 1, 1) | |
|---|---|
| $\sigma^x_{2j-3}\sigma^y_{2j-2}1_{2j-1}\sigma^y_{2j}\sigma^x_{2j+1}$ | |
| $\sigma^x_{2j-1}\sigma^z_{2j}\sigma^x_{2j+1}$ | $\sigma^x_{2j-1}\sigma^y_{2j}\sigma^x_{2j+1}\sigma^x_{2j+2}$ |
| $\sigma^x_{2j}1_{2j+1}\sigma^x_{2j+2}$ | $\sigma^x_{2j}\sigma^x_{2j+1}\sigma^y_{2j+2}\sigma^x_{2j+3}$ |
| $\sigma^x_{2j}\sigma^x_{2j+1}\sigma^z_{2j+2}\sigma^x_{2j+3}\sigma^x_{2j+4}$ | |
| $i\sigma^x_1\sigma^x_2$ | $\varphi^{(3,1,1)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{2\nu-1}\sigma^z_{2\nu})\sigma^x_{2j+1}\sigma^x_{2j+2}$ |
| | $\varphi^{(3,1,1)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{2\nu-1}\sigma^z_{2\nu})1_{2j+1}\sigma^y_{2j+2}\sigma^x_{2j+3}$ |

X Z X + Z 1 Z

| (3, 1, 2) | |
|---|---|
| $\sigma^x_{3j-5}\sigma^y_{3j-4}\sigma^x_{3j-3}\sigma^x_{3j-2}\sigma^y_{3j-1}\sigma^x_{3j}$ | |
| $\sigma^x_{3j-2}\sigma^z_{3j-1}\sigma^x_{3j}$ | $\sigma^x_{3j-2}\sigma^y_{3j-1}\sigma^x_{3j}1_{3j+1}\sigma^x_{3j+2}$ |
| $\sigma^x_{3j-1}1_{3j}1_{3j+1}\sigma^x_{3j+2}$ | $\sigma^x_{3j-1}1_{3j}\sigma^x_{3j+1}\sigma^y_{3j+2}\sigma^x_{3j+3}$ |
| $\sigma^x_{3j-1}1_{3j}\sigma^x_{3j+1}\sigma^z_{3j+2}\sigma^x_{3j+3}1_{3j+4}\sigma^x_{3j+5}$ | |
| $i1_1\sigma^x_2 1_3$ | $\varphi^{(3,1,2)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{3\nu-2}\sigma^z_{3\nu-1}\sigma^x_{3\nu})1_{3j+1}\sigma^x_{3j+2}$ |
| | $\varphi^{(3,1,2)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{3\nu-2}\sigma^z_{3\nu-1}\sigma^x_{3\nu})\sigma^x_{3j+1}\sigma^y_{3j+2}\sigma^x_{3j+3}$ |

| (3, 1, 3) | |
|---|---|
| $\sigma^x_{4j-7}\sigma^y_{4j-6}\sigma^x_{4j-5}1_{4j-4}\sigma^x_{4j-3}\sigma^y_{4j-2}\sigma^x_{4j-1}$ | |
| $\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$ | $\sigma^x_{4j-3}\sigma^y_{4j-2}\sigma^x_{4j-1}1_{4j}1_{4j+1}\sigma^x_{4j+2}$ |
| $\sigma^x_{4j-2}1_{4j-1}1_{4j}1_{4j+1}\sigma^x_{4j+2}$ | $\sigma^x_{4j-2}1_{4j-1}1_{4j}\sigma^x_{4j+1}\sigma^y_{4j+2}\sigma^x_{4j+3}$ |
| $\sigma^x_{4j-2}1_{4j-1}1_{4j}\sigma^x_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}1_{4j+4}1_{4j+5}\sigma^x_{4j+6}$ | |
| $i1_1\sigma^x_2$ | $\varphi^{(3,1,3)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{4\nu-3}\sigma^z_{4\nu-2}\sigma^x_{4\nu-1}1_{4\nu})1_{4j+1}\sigma^x_{4j+2}$ |
| | $\varphi^{(3,1,3)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^x_{4\nu-3}\sigma^z_{4\nu-2}\sigma^x_{4\nu-1}1_{4\nu})\sigma^x_{4j+1}\sigma^y_{4j+2}\sigma^x_{4j+3}$ |

| (4, 1, −) | |
|---|---|
| $\sigma^x_{4j-7}\sigma^z_{4j-6}\sigma^y_{4j-5}1_{4j-4}\sigma^y_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$ | |
| $\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^x_{4j-1}$ | $\sigma^x_{4j-3}\sigma^z_{4j-2}\sigma^y_{4j-1}1_{4j}\sigma^z_{4j+1}$ |
| $\sigma^z_{4j-1}1_{4j}\sigma^z_{4j+1}$ | $\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}$ |
| $\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^y_{4j+3}1_{4j+4}\sigma^z_{4j+5}$ | |
| $-i\sigma^z_1$ | $\varphi^{(4,1,-)}_{2j} = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^y_{4\nu-3}\sigma^z_{4\nu-2}\sigma^y_{4\nu-1}1_{4\nu})\sigma^z_{4j+1}$ |
| | $\varphi^{(4,1,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^y_{4\nu-3}\sigma^z_{4\nu-2}\sigma^y_{4\nu-1}1_{4\nu})\sigma^y_{4j+1}\sigma^z_{4j+2}\sigma^x_{4j+3}$ |

| $(4,2,-)$ | |
|---|---|
| $\sigma^x_{6j-11}\sigma^z_{6j-10}\sigma^z_{6j-9}\sigma^y_{6j-8}1_{6j-7}1_{6j-6}\sigma^y_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^x_{6j-2}$ | |
| $\sigma^x_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^x_{6j-2}$ | $\sigma^x_{6j-5}\sigma^z_{6j-4}\sigma^z_{6j-3}\sigma^y_{6j-2}1_{6j-1}1_{6j}\sigma^z_{6j+1}$ |
| $\sigma^z_{6j-2}\ 1_{6j-1}1_{6j}\sigma^z_{6j+1}$ | $\sigma^z_{6j-2}1_{6j-1}1_{6j}\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^x_{6j+4}$ |
| $\sigma^z_{6j-2}1_{6j-1j}1_{6j}\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^y_{6j+4}1_{6j+5}1_{6j+6}\sigma^z_{6j+7}$ | |
| $-i\sigma^z_1$ | $\varphi^{(4,2,-)}_{2j} = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{6\nu-5}\sigma^z_{6\nu-4}\sigma^z_{6\nu-3}\sigma^y_{6\nu-2}1_{6\nu-1}1_{6\nu})\sigma^z_{6j+1}$ |
| | $\varphi^{(4,2,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{6\nu-5}\sigma^z_{6\nu-4}\sigma^z_{6\nu-3}\sigma^y_{6\nu-2}1_{6\nu-1}1_{6\nu})\sigma^y_{6j+1}\sigma^z_{6j+2}\sigma^z_{6j+3}\sigma^x_{6j+4}$ |

| $(5,3,-)$ | |
|---|---|
| $\sigma^z_{3j-5}\sigma^z_{3j-4}\sigma^y_{3j-3}\sigma^y_{3j-2}\sigma^z_{3j-1}\sigma^z_{3j}$ | |
| $\sigma^z_{3j-2}\sigma^z_{3j-1}\sigma^z_{3j}$ | $\sigma^z_{3j-2}\sigma^z_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}$ |
| $\sigma^x_{3j}\sigma^x_{3j+1}$ | $\sigma^x_{3j}\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^z_{3j+3}$ |
| $\sigma^x_{3j}\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}$ | |
| $i\sigma^x_1$ | $\varphi^{(5,3,-)}_{2j} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{3\nu-2}\sigma^z_{3\nu-1}\sigma^y_{3\nu})\sigma^x_{3j+1}$ |
| | $\varphi^{(5,3,-)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^y_{3\nu-2}\sigma^z_{3\nu-1}\sigma^y_{3\nu})\sigma^y_{3j+1}\sigma^z_{3j+2}\sigma^z_{3j+3}$ |

| $(1,2,1)$ | |
|---|---|
| $\sigma^x_{j-1}1_j\sigma^z_{j+1}\sigma^z_{j+2}1_{j+3}\sigma^x_{j+4}$ | |
| $\sigma^x_j\sigma^x_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$ | $\sigma^x_j\sigma^x_{j+1}\sigma^y_{j+2}1_{j+3}\sigma^x_{j+4}$ |
| $\sigma^x_{j+2}\sigma^x_{j+3}$ | $\sigma^x_{j+2}1_{j+3}\sigma^y_{j+4}\sigma^x_{j+5}\sigma^x_{j+6}$ |
| $\sigma^x_{j+2}1_{j+3}\sigma^z_{j+4}1_{j+5}\sigma^x_{j+6}$ | |
| $i\sigma^y_1\sigma^z_2\sigma^x_3\sigma^x_4$ | $\varphi^{(1,2,1)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_\nu)\sigma^y_{j+1}\sigma^z_{j+2}\sigma^x_{j+3}\sigma^x_{j+4}$ |
| | $\varphi^{(1,2,1)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_\nu)\sigma^z_{j+1}\sigma^y_{j+2}\sigma^y_{j+3}1_{j+4}\sigma^x_{j+5}$ |

| $(2,2,1)$ | |
|---|---|
| $\sigma^x_{5j-9}\sigma^x_{5j-8}\sigma^z_{5j-7}\sigma^x_{5j-6}\sigma^z_{5j-5}\sigma^z_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^x_{5j}$ | |
| $\sigma^x_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^x_{5j}$ | $\sigma^x_{5j-4}\sigma^x_{5j-3}\sigma^z_{5j-2}\sigma^x_{5j-1}\sigma^z_{5j}\sigma^y_{5j+1}$ |
| $\sigma^y_{5j}\sigma^y_{5j+1}$ | $\sigma^y_{5j}\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^x_{5j+5}$ |
| $\sigma^y_{5j}\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^z_{5j+5}\sigma^y_{5j+6}$ | |
| $i\sigma^y_1$ | $\varphi^{(2,2,1)}_{2j} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_{5\nu-4}\sigma^x_{5\nu-3}\sigma^z_{5\nu-2}\sigma^x_{5\nu-1}\sigma^z_{5\nu})\sigma^y_{5j+1}$ |
| | $\varphi^{(2,2,1)}_{2j+1} = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{\nu=1}^{j}\sigma^z_{5\nu-4}\sigma^x_{5\nu-3}\sigma^z_{5\nu-2}\sigma^x_{5\nu-1}\sigma^z_{5\nu})\sigma^z_{5j+1}\sigma^x_{5j+2}\sigma^z_{5j+3}\sigma^x_{5j+4}\sigma^x_{5j+5}$ |

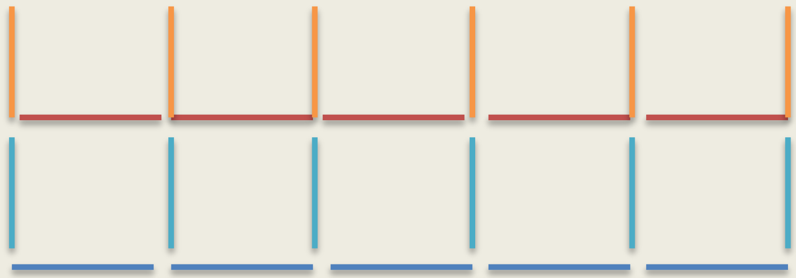
| $(4,2,2)$ | |
|---|---|
| $\sigma^x_{3j-5}\sigma^x_{3j-4}\sigma^y_{3j-3}1_{3j-2}1_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}$ | |
| $\sigma^x_{3j-2}\sigma^x_{3j-1}\sigma^z_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}$ | $\sigma^x_{3j-2}\sigma^x_{3j-1}\sigma^y_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^x_{3j+3}$ |
| $\sigma^x_{3j}1_{3j+1}1_{3j+2}\sigma^x_{3j+3}$ | $\sigma^x_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}$ |
| $\sigma^x_{3j}\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^z_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}\sigma^x_{3j+6}$ | |
| $i\sigma^x_1\sigma^x_2\sigma^x_3$ | $\varphi^{(4,2,2)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{3\nu-2}1_{3\nu-1}\sigma^z_{3\nu})\sigma^x_{3j+1}\sigma^x_{3j+2}\sigma^x_{3j+3}$ |
| | $\varphi^{(4,2,2)}_{2j+1} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} 1_{3\nu-2}1_{3\nu-1}\sigma^z_{3\nu})1_{3j+1}1_{3j+2}\sigma^y_{3j+3}\sigma^x_{3j+4}\sigma^x_{3j+5}$ |

| $(11,-,-)$ | |
|---|---|
| $\sigma^x_{4j-7}\sigma^x_{4j-6}\sigma^z_{4j-5}1_{4j-4}\sigma^z_{4j-3}1_{4j-2}\sigma^z_{4j-1}\sigma^x_{4j}\sigma^x_{4j+1}$ | |
| $\sigma^x_{4j-3}\sigma^x_{4j-2}\sigma^z_{4j-1}\sigma^x_{4j}\sigma^x_{4j+1}$ | $\sigma^x_{4j-3}\sigma^x_{4j-2}\sigma^z_{4j-1}1_{4j}\sigma^y_{4j+1}\sigma^x_{4j+2}$ |
| $\sigma^x_{4j}\sigma^z_{4j+1}\sigma^x_{4j+2}$ | $\sigma^x_{4j}\sigma^y_{4j+1}1_{4j+2}\sigma^z_{4j+3}\sigma^x_{4j+4}\sigma^x_{4j+5}$ |
| $\sigma^x_{4j}\sigma^y_{4j+1}1_{4j+2}\sigma^z_{4j+3}1_{4j+4}\sigma^y_{4j+5}\sigma^x_{4j+6}$ | |
| $i\sigma^y_1\sigma^x_2$ | $\varphi^{(11,-,-)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^z_{4\nu-3}1_{4\nu-2}\sigma^z_{4\nu-1}1_{4\nu})\sigma^y_{4j+1}\sigma^x_{4j+2}$ |
| | $\varphi^{(11,-,-)}_{2j} = \frac{1}{\sqrt{2}}(\prod_{\nu=1}^{j} \sigma^z_{4\nu-3}1_{4\nu-2}\sigma^z_{4\nu-1}1_{4\nu})\sigma^z_{4j+1}1_{4j+2}\sigma^z_{4j+3}\sigma^x_{4j+4}\sigma^x_{4j+5}$ |

K.M. , Nucl. Phys.B 2017

# Two-dimensional classical systems

# 2-dim. Ising model（period m）



$$-\beta\mathcal{H} = \sum_{i=1}^{M}\sum_{j=1}^{N}(K_{1i}\sigma_{ij}^z\sigma_{i+1j}^z + K_{2i}\sigma_{ij}^z\sigma_{ij+1}^z)$$

**Partition function**

$$Z = \mathrm{tr}\Big(\prod_{i=1}^{m} V_{1i}V_{2i}\Big)^{M/m}$$

**Transfer matrix**

$$V_{1i} = \Big(\frac{e^{K_{1i}}}{\cosh K_{1i}^*}\Big)^N \exp(K_{1i}^*\sum_{j=1}^{N}\sigma_j^x) \qquad V_{2i} = \exp(K_{2i}\sum_{j=1}^{N}\sigma_j^z\sigma_{j+1}^z)$$

$$= \Big(\frac{e^{K_{1i}}}{\cosh K_{1i}^*}\Big)^N \prod_{0<q<\pi} \exp(K_{1i}^*(-2i)C_{12}(q)) \qquad = \prod_{0<q<\pi} \exp(K_{2i}(-2i)\tilde{C}_{21}(q))$$

**where**

$$C_{12}(q) = C_1^\dagger(q)C_2(q) + C_1(q)C_2^\dagger(q)$$

$$\tilde{C}_{21}(q) = e^{iq}C_2^\dagger(q)C_1(q) + e^{-iq}C_2(q)C_1^\dagger(q) \qquad C^3 = -C$$

**Transformation**

$$\varphi_1(j) = \frac{1}{\sqrt{2}}(\prod_{k=1}^{j}\sigma_k^x)\,\sigma_{j+1}^z \qquad \varphi_2(j) = \frac{-1}{\sqrt{2}}(\prod_{k=1}^{j}\sigma_k^x)\,\sigma_{j+1}^y$$

**Transfer matrix** $V = \prod_{i=1}^{m} V_{1i} V_{2i}$
**(represented by** $|0\,0\rangle$**,** $|1\,0\rangle = C_1^\dagger(q)|0\,0\rangle$**,** $|0\,1\rangle = C_2^\dagger(q)|0\,0\rangle$ $|1\,1\rangle = C_2^\dagger(q) C_1^\dagger(q)|0\,0\rangle$**)**

$$V = \prod_{i=1}^{m} \Big( \frac{e^{K_{1i}}}{\cosh K_{1i}^*} \Big)^N \prod_{0<q<\pi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A_i(q) & \bar{B}_i(q) & 0 \\ 0 & B_i(q) & \bar{A}_i(q) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

**where**

$$\begin{aligned} A_i(q) &= \cosh 2K_{1i}^* \cosh 2K_{2i} - e^{iq} \sinh 2K_{1i}^* \sinh 2K_{2i} \\ B_i(q) &= i(\sinh 2K_{1i}^* \cosh 2K_{2i} - e^{iq} \cosh 2K_{1i}^* \sinh 2K_{2i}) \end{aligned}$$

**maximum eigenvalue is obtained from the block-element**

$$v(q) = \prod_{i=1}^{m} \begin{pmatrix} A_i(q) & \bar{B}_i(q) \\ B_i(q) & \bar{A}_i(q) \end{pmatrix}, \qquad 0 = \lambda^2 - (\mathrm{tr}\, v(q))\lambda + \det v(q).$$

$$\cosh m\epsilon_q = \frac{1}{2} \mathrm{tr}\, v(q) \qquad \lambda = e^{m\epsilon_q}, e^{-m\epsilon_q} \qquad \epsilon_q > 0$$

**The free energy is**

$$
\begin{aligned}
-\beta f &= \lim_{(N,M)\to(\infty,\infty)} \frac{1}{NM} \log Z \\
&= \lim_{N\to\infty} \frac{1}{N}\left[\frac{1}{m} \log \prod_{i=1}^{m} \left(\frac{e^{K_{1i}}}{\cosh K_{1i}^*}\right)^N + \sum_{0<q<\pi} \epsilon_q\right] \\
&= \frac{1}{m} \log \prod_{i=1}^{m} \left(\frac{e^{K_{1i}}}{\cosh K_{1i}^*}\right) + \frac{1}{2\pi} \int_0^\pi \epsilon_q \, dq.
\end{aligned}
$$

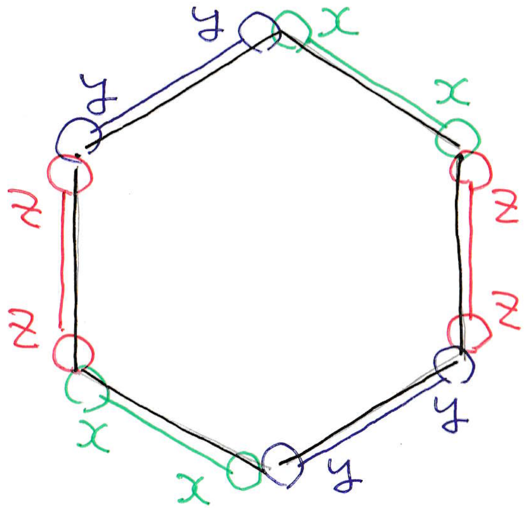
**When m=1,** $K_{1i} = K_1 \ K_{2i} = K_2$**, then**

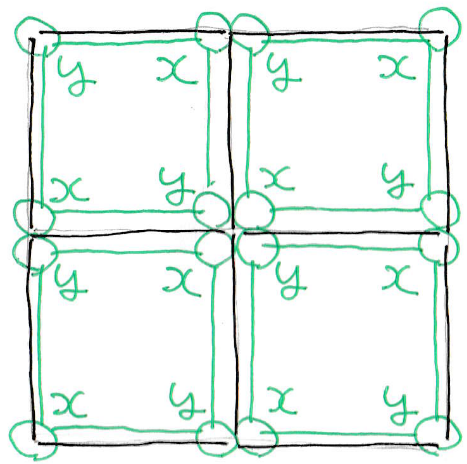
$$
\begin{aligned}
-\beta f &= \log \left(\frac{e^{K_1}}{\cosh K_1^*}\right) + \frac{1}{2\pi} \int_0^\pi \epsilon_q \, dq, \\
\cosh \epsilon_q &= \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos q.
\end{aligned}
$$

**(Onsager 1944)**

two-dimensional quantum systems

$$x = \sigma_j^x$$
$$y = \sigma_j^y$$
$$z = \sigma_j^z$$




**Kitaev model**

$$-\beta\mathcal{H} = \sum [K_1\sigma_i^x\sigma_j^x + K_2\sigma_k^y\sigma_l^y + K_3\sigma_m^z\sigma_n^z]$$

**Wen model**

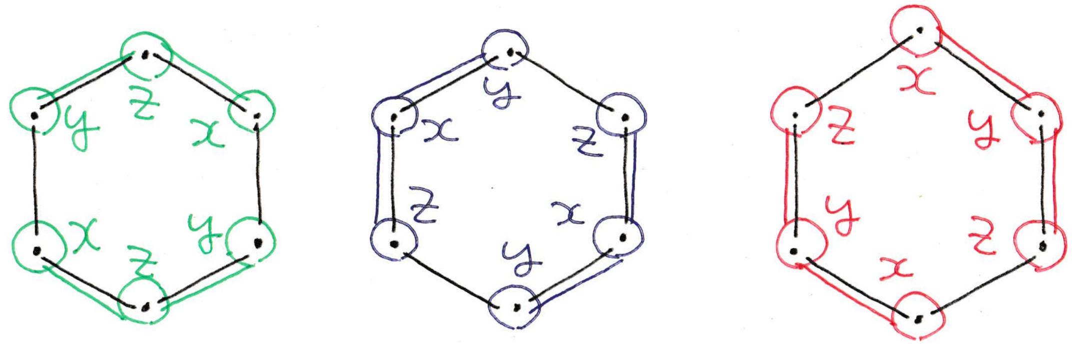
$$-\beta\mathcal{H} = K\sum_{ij}\sigma_{ij}^x\sigma_{ij+1}^y\sigma_{i+1j}^x\sigma_{i+1j+1}^y$$

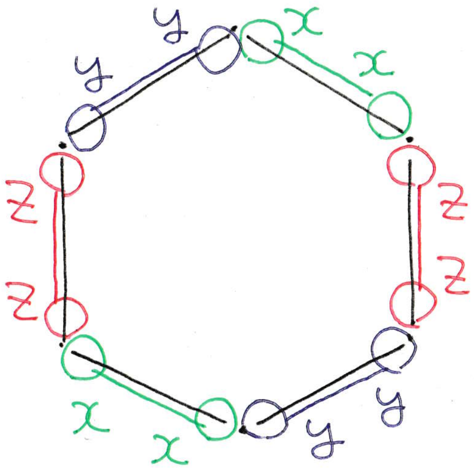
Lee et el. 2007

Si et al. 2009

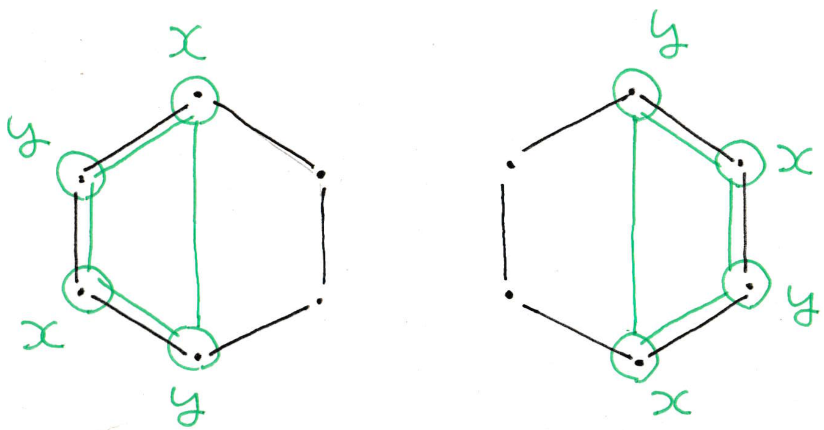
Yu 2008

Yu Wang 2008

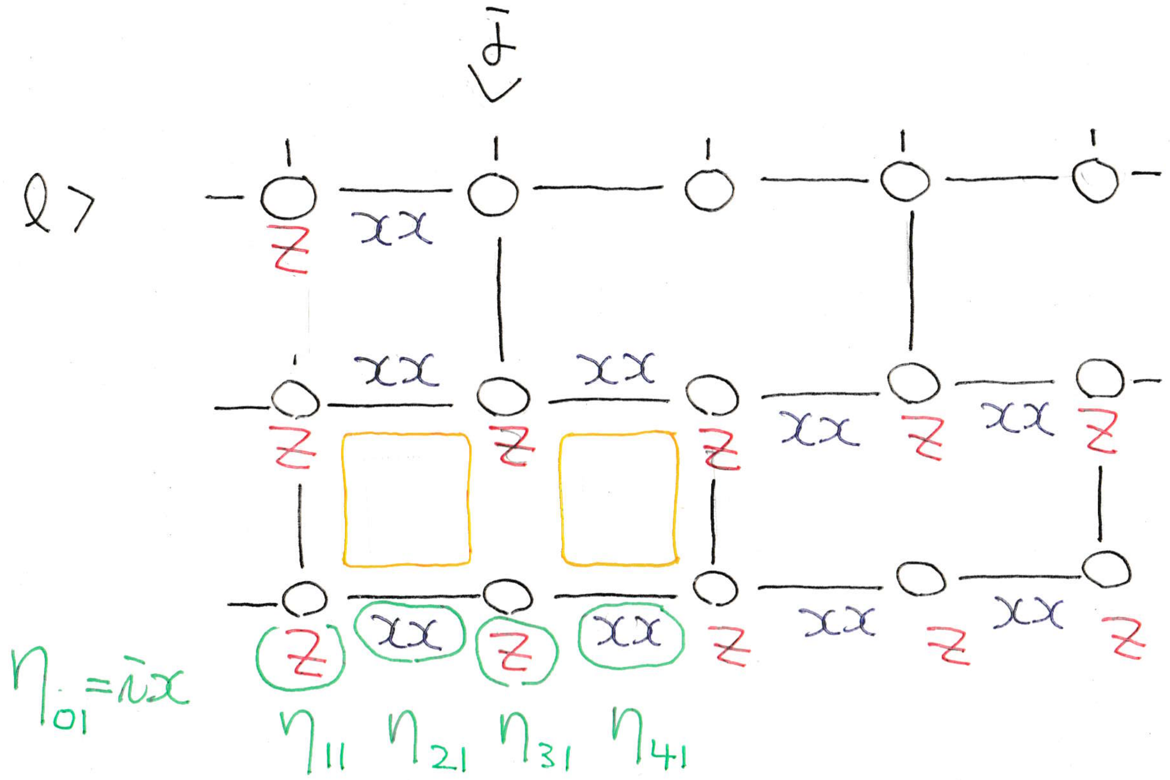




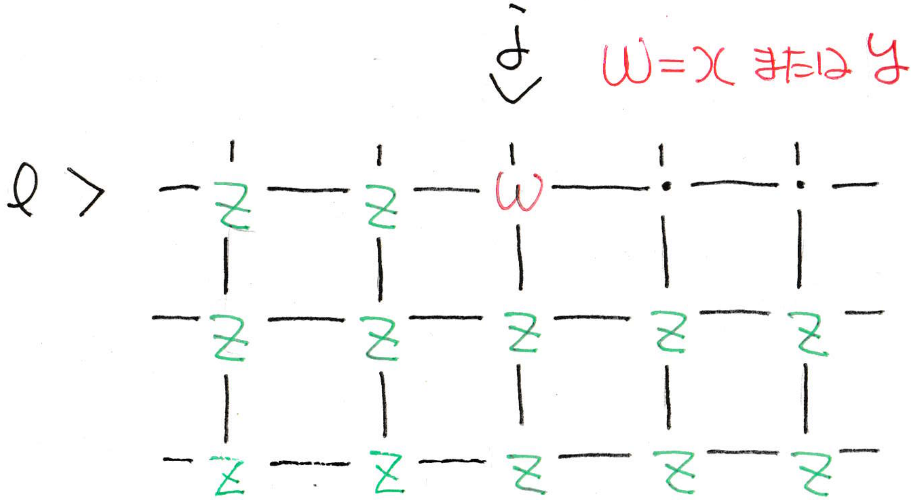
Kitaev 2006

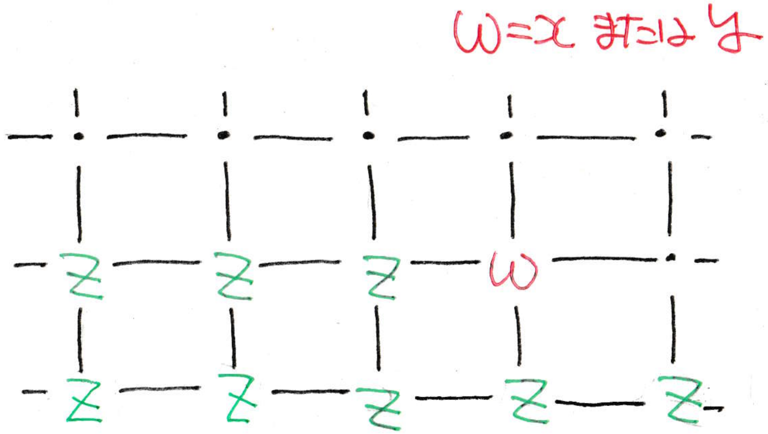


4-body terms (Wen model)




Series of operators in two-dimension



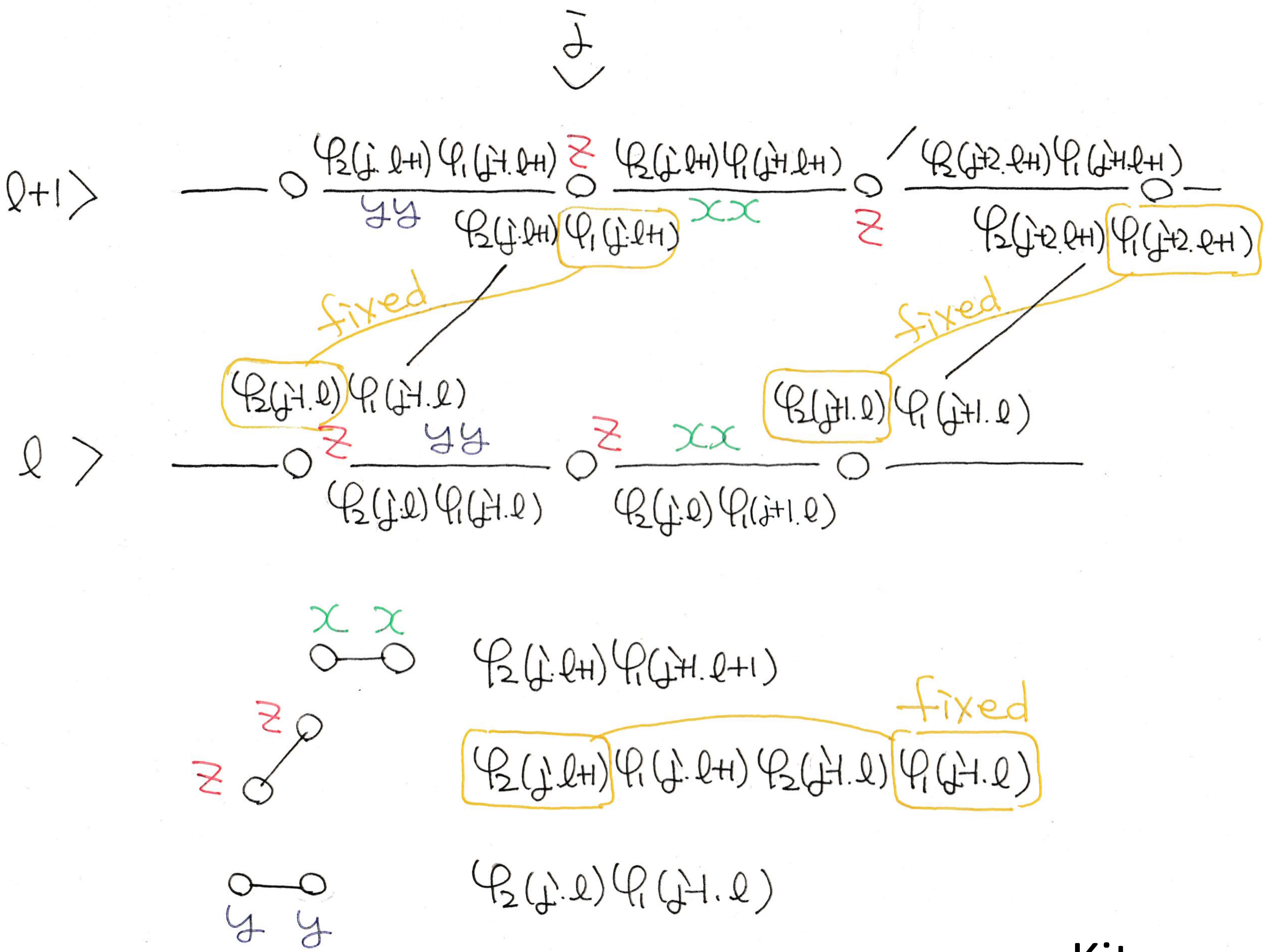

$\varphi_K(j,l) =$

$\varphi_{K'}(j',l') =$

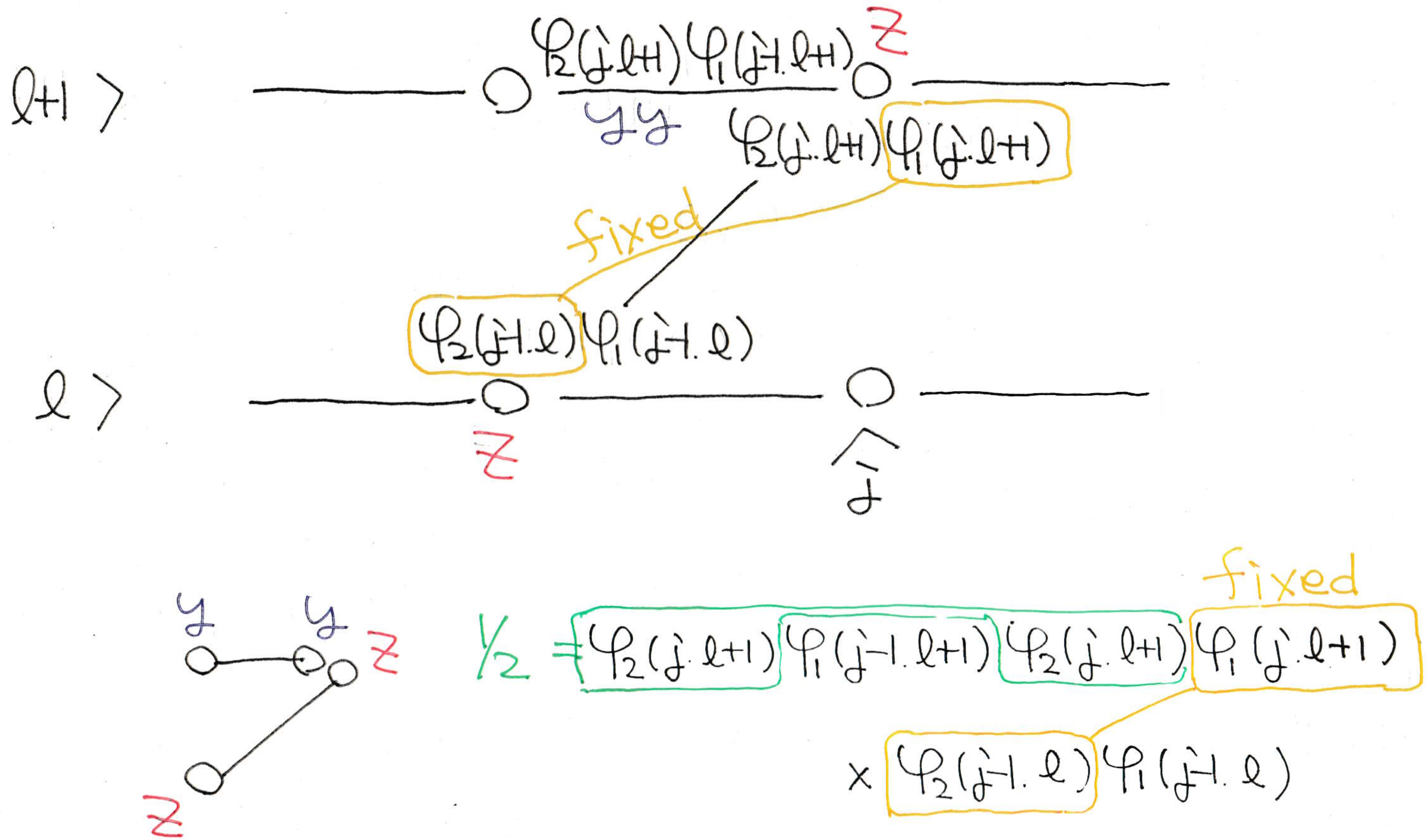
$$\{\varphi_K(j,l), \varphi_{K'}(j',l')\} = \delta_{jj'}\delta_{ll'}$$

Two-dimesnional
Jordan-Wigner transformation

Feng et al. 2007

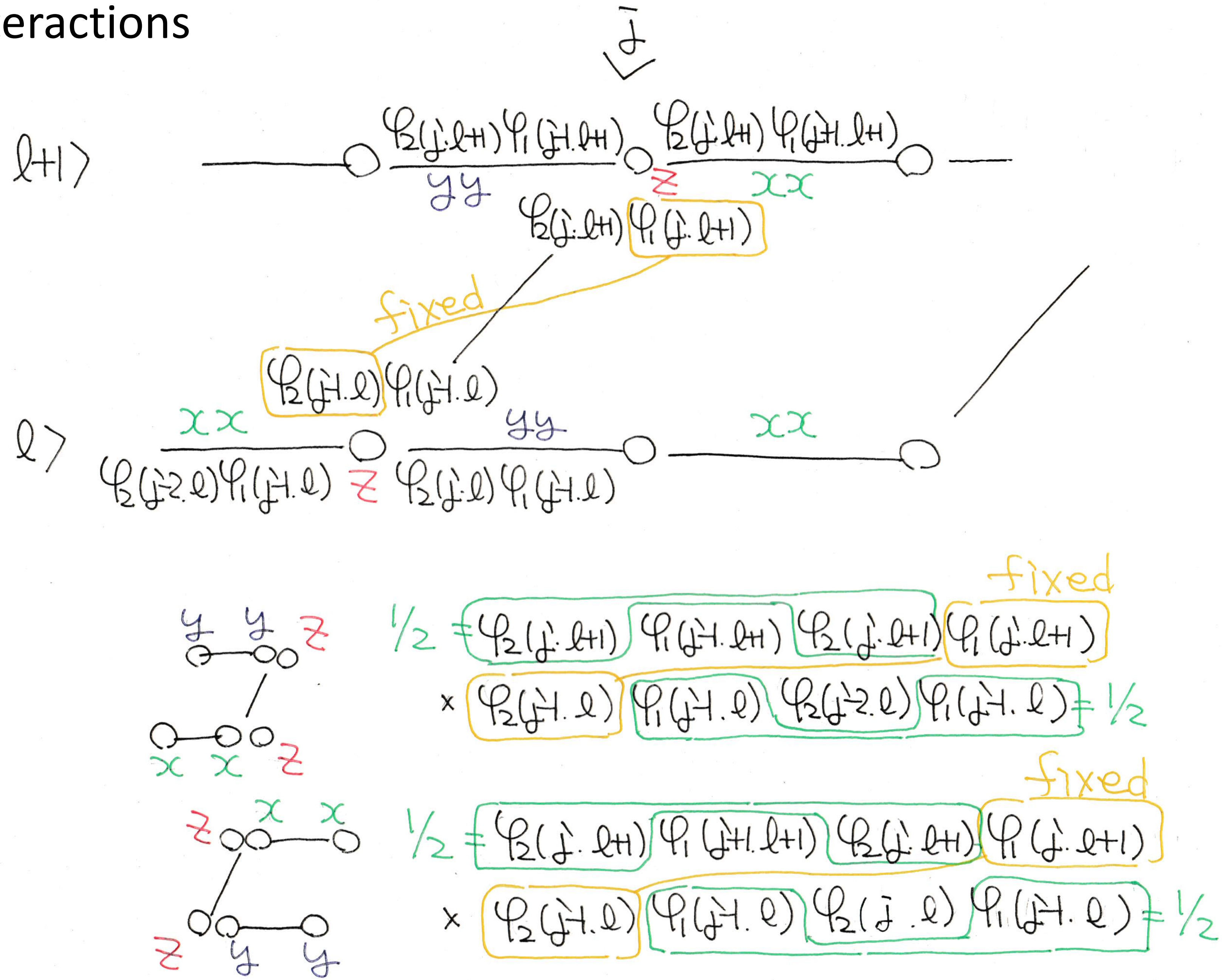


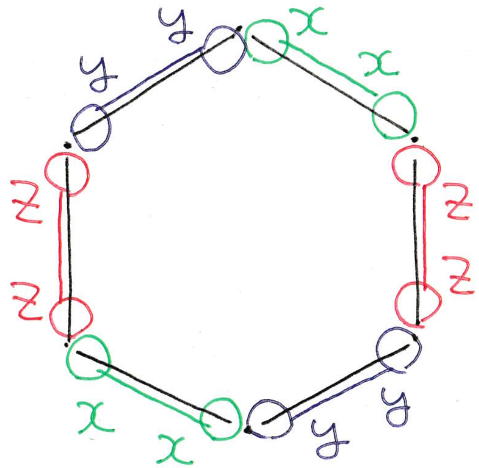

Kitaev model




Six kind of three-body

# Wen interactions

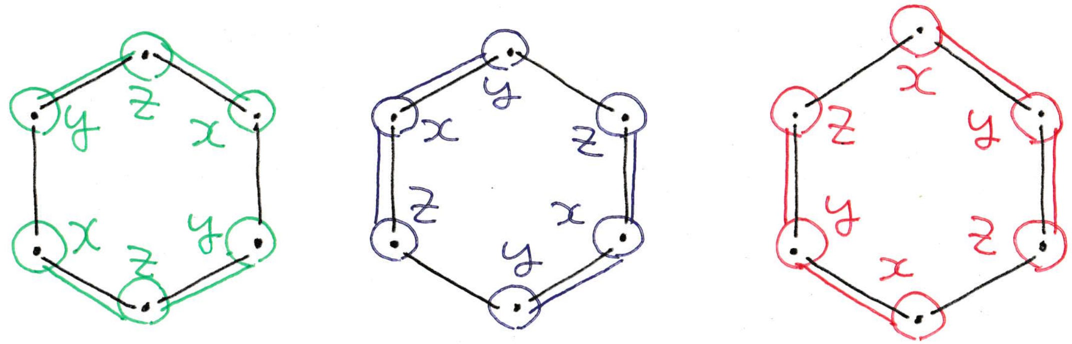


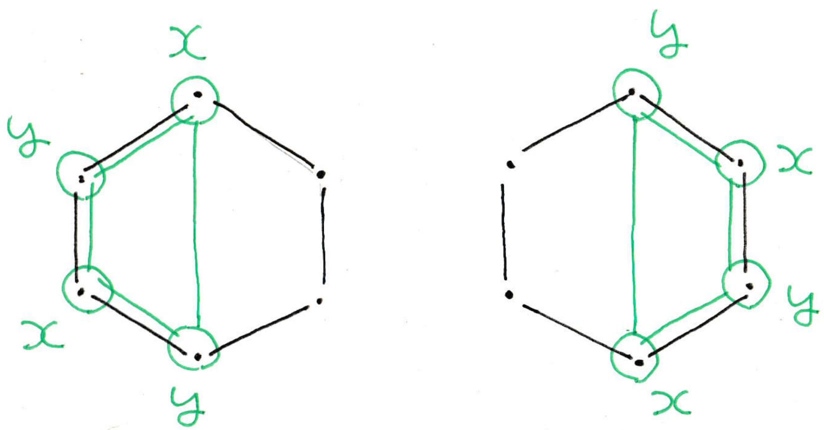


Kitaev 2006

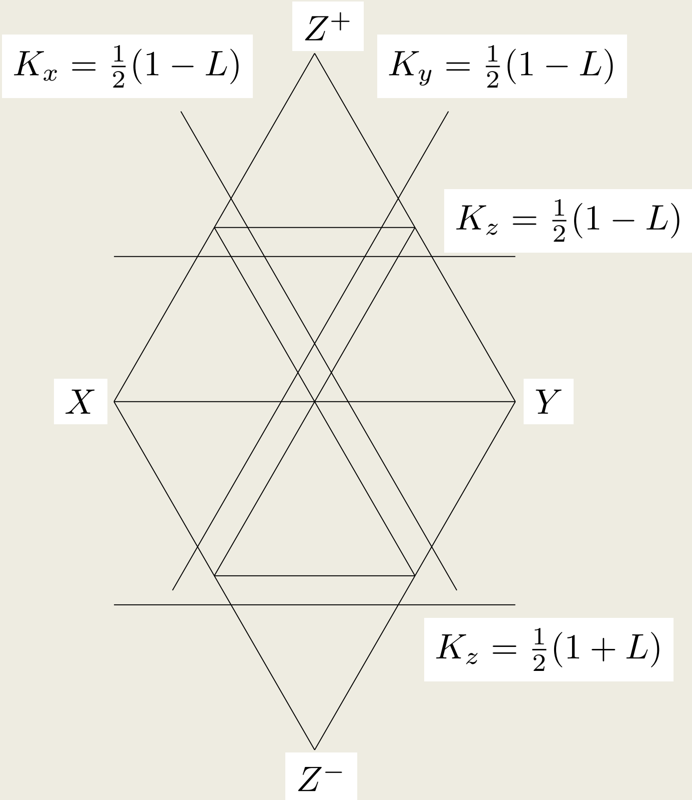
Lee et el. 2007
Si et al. 2009

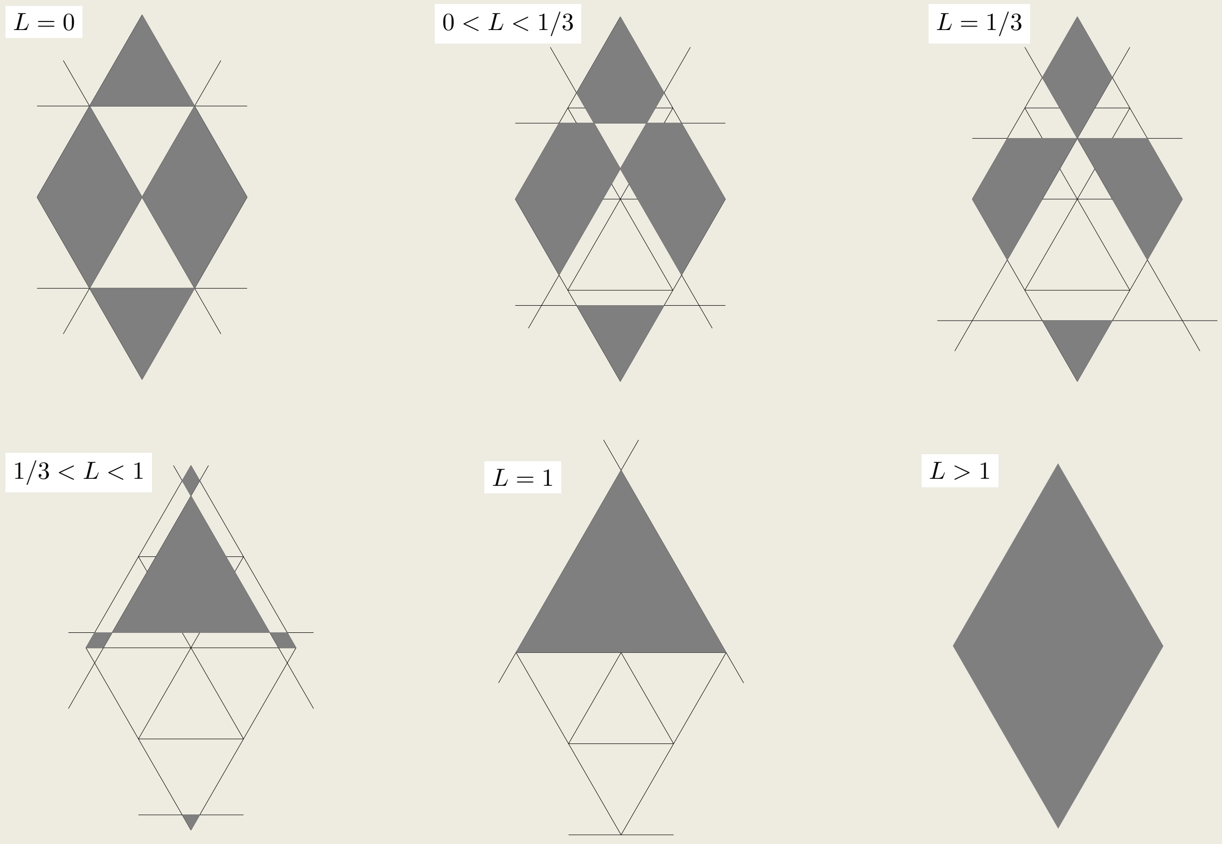
Yu 2008
Yu Wang 2008





4-body terms (Wen model)




| | | |
|---|---|---|
| | | $(K_x, K_y, K_z)$ |
| $X$ | : | $(1, 0, 0)$ |
| $Y$ | : | $(0, 1, 0)$ |
| $Z^+$ | : | $(0, 0, 1)$ |
| $Z^-$ | : | $(0, 0, -1)$ |

$$
\begin{aligned}
&(a)\ \ (K_x, K_y) \mapsto (-K_x, -K_y) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (-\sigma^x_{jl}, -\sigma^y_{jl}, \sigma^z_{jl}) \ \text{ if } \ j = \text{odd},\\
&(b)\ \ (L, K_z) \mapsto (-L, -K_z) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (\sigma^x_{jl}, -\sigma^y_{jl}, -\sigma^z_{jl}) \ \text{ if } \ l = \text{odd},\\
&(c)\ \ (K_x, K_z) \mapsto (-K_x, -K_z) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (-\sigma^x_{jl}, \sigma^y_{jl}, -\sigma^z_{jl}) \ \text{ if } \ j = \text{odd},\\
&(d)\ \ (K_y, K_z) \mapsto (-K_y, -K_z) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (\sigma^x_{jl}, -\sigma^y_{jl}, -\sigma^z_{jl}) \ \text{ if } \ j = \text{odd},\\
&(e)\ \ (L, K_x) \mapsto (-L, -K_x) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (-\sigma^x_{jl}, \sigma^y_{jl}, -\sigma^z_{jl})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ (j, l) = (\text{odd}, \text{even}) \ \text{ or } \ (\text{even}, \text{odd}),\\
&(f)\ \ (L, K_y) \mapsto (-L, -K_y) \quad (\sigma^x_{jl}, \sigma^y_{jl}, \sigma^z_{jl}) \mapsto (\sigma^x_{jl}, -\sigma^y_{jl}, -\sigma^z_{jl})\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ (j, l) = (\text{odd}, \text{even}) \ \text{ or } \ (\text{even}, \text{odd}),
\end{aligned}
$$

$\mathcal{H}(\cdots, K_i, K_j, \cdots)$ **and** $\mathcal{H}(\cdots, -K_i, -K_j, \cdots)$ **are equivalent.**

## Symmetry of the system

$$
\begin{aligned}
&(q_1, q_2) \mapsto (-q_1, -q_2), \qquad \text{and}\\
&(K_x, K_y) \mapsto (K_y, K_x), \quad (K_z, L) \mapsto (L, K_z),\\
&\tilde{c}^\dagger_1(q_1, q_2) = c^\dagger_1(-q_1, -q_2) \quad (= c_1(q_1, q_2)),\\
&\tilde{c}_2(q_1, q_2) = c_2(-q_1, -q_2) \quad (= c^\dagger_2(q_1, q_2)).
\end{aligned}
$$

$\mathcal{H}(K_x, K_y, K_z, L)$ **and** $\mathcal{H}(K_y, K_x, L, K_z)$ **are identical as an operator.**

$$
\begin{aligned}
&(q_1, q_2) \mapsto (q_1, q) \qquad \text{where } \ q = q_1 + q_2,\\
&(K_x, K_y) \mapsto (K_z, L), \quad (K_z, L) \mapsto (K_x, K_y),\\
&c^\dagger_1(q_1, q - q_1) = \tilde{c}^\dagger_1(q_1, q),\\
&c_2(q_1, q - q_1) = \tilde{c}_2(q_1, q).
\end{aligned}
$$

$\mathcal{H}(K_x, K_y, K_z, L)$ **and** $\mathcal{H}(K_z, L, K_x, K_y)$ **are identical as an operator.**

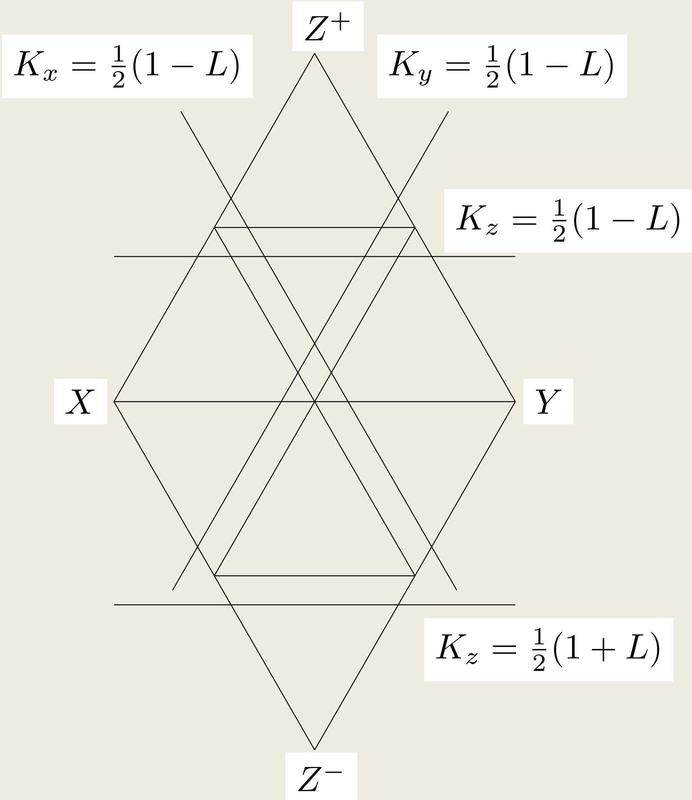
# Anyon generator changes its location

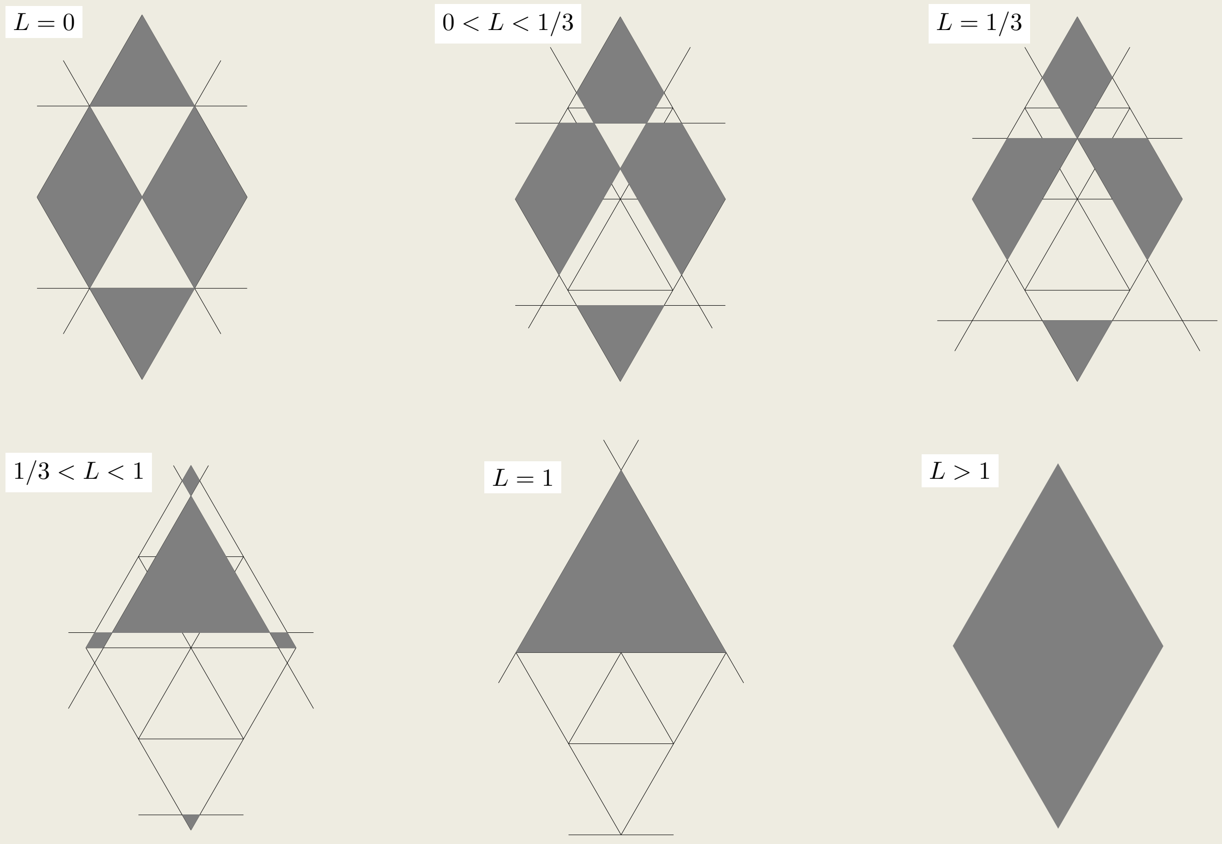
**The spin operators can be expressed by $\varphi_\alpha(j,l)$ as**

$$\begin{aligned}
\sigma^z_{jl} &= \eta_{2j-1\,l} = (+2i)\varphi_2(j,l)\varphi_1(j,l), \\
\sigma^x_{jl} &= \sqrt{2}\Big(\prod_{r=1}^{l-1}\prod_{k=1}^{N}\eta_{2k-1\,r}\Big)\Big(\prod_{k=1}^{j-1}\eta_{2k-1\,l}\Big)\varphi_1(j,l), \\
\sigma^y_{jl} &= \sqrt{2}\Big(\prod_{r=1}^{l-1}\prod_{k=1}^{N}\eta_{2k-1\,r}\Big)\Big(\prod_{k=1}^{j-1}\eta_{2k-1\,l}\Big)\varphi_2(j,l),
\end{aligned}$$

**where $\eta_{2k-1\,r}$ are also written by $\varphi_\alpha(j,l)$. The external field and string operators are, therefore, transformed together with $\varphi_\alpha(j,l)$.**

**We also find that the anyon excitations appear in all of the regions shown in the phase diagram, and they can be transformed each other.**




$$
\begin{array}{rcl}
 & & (K_x, K_y, K_z) \\
X & : & (1, 0, 0) \\
Y & : & (0, 1, 0) \\
Z^+ & : & (0, 0, 1) \\
Z^- & : & (0, 0, -1)
\end{array}
$$

# Relations with other methods

# The Jordan-Wigner transformation is a special case

**The Jordan-Wigner transformation is a special case of our formula.**
**Let**

$$\Psi_j = u_3\varphi_3(j) + u_1\varphi_1(j), \qquad \Psi_j^\dagger = u_3^*\varphi_3(j) + u_1^*\varphi_1(j),$$

**where** $\varphi_3(j)$ **and** $\varphi_1(j)$ **were introduced in the case of the XY model.**
**Canonical condition**

$$\begin{aligned} \delta_{jk} &= \{\Psi_j^\dagger, \Psi_k\} = (u_3^* u_3 + u_1^* u_1)\delta_{jk}, & u_3 &= \frac{1}{\sqrt{2}}e^{i\theta}, \\ 0 &= \{\Psi_j, \Psi_k\} = (u_3 u_3 + u_1 u_1)\delta_{jk}, & u_1 &= \frac{\pm i}{\sqrt{2}}e^{i\theta}. \end{aligned}$$

**Let** $\theta = (j-1)\pi$ **and choose negative sign, then**

$$\begin{aligned} \Psi_j &= \frac{(-1)^{j-1}}{\sqrt{2}}\Big(\varphi_3(j) - i\varphi_1(j)\Big), & \Psi_j^\dagger &= \frac{(-1)^{j-1}}{\sqrt{2}}\Big(\varphi_3(j) + i\varphi_1(j)\Big). \\ &= \exp\Big[i\pi\sum_{k=1}^{2j-2} s_k^+ s_k^-\Big] s_{2j-1}^-, & &= \exp\Big[-i\pi\sum_{k=1}^{2j-2} s_k^+ s_k^-\Big] s_{2j-1}^+. \end{aligned}$$

**These are the operators** $c_{2j-1}$**and** $c_{2j-1}^\dagger$ **in the Jordan-Wigner transformation.**
**Similarly we can obtain** $c_{2j}$ **and** $c_{2j}^\dagger$ **from** $\varphi_4(j)$ **and** $\varphi_2(j)$**.**

# Transformation by Nambu

**Nambu solved the equare lattice Ising model (Nambu 1950)**
**The inverse of his transformation is our transformation for the XY chain**

$$\varphi_1(j) \;=\; \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-2}\sigma_k^z)\sigma_{2j-1}^y, \quad \varphi_2(j) = \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-1}\sigma_k^z)\sigma_{2j}^x,$$

$$\varphi_3(j) \;=\; \frac{(-1)^{j-1}}{\sqrt{2}}(\prod_{k=1}^{2j-2}\sigma_k^z)\sigma_{2j-1}^x, \quad \varphi_4(j) = \frac{(-1)^{j}}{\sqrt{2}}(\prod_{k=1}^{2j-1}\sigma_k^z)\sigma_{2j}^y,$$

**i) with the initial operators** $\eta_0 = -ix_2$ **and** $\zeta_0 = -ix_1$
**ii) the sign of the additional phase is opposite** $\exp(-i\frac{\pi}{2}(j-1))$

# Algebraic structure

# Onsager Algebra

**In his solution of the 2-dimensional Ising model, Onsager introduced**

$$A_n = \sum_{j=1}^{N} \sigma_j^x \Big( \prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^x,$$

$$G_n = \frac{1}{2}\mathrm{i} \sum_{j=1}^{N} \Big[ \sigma_j^x \Big( \prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^y + \sigma_j^y \Big( \prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^x \Big].$$

**We assume periodicity** $\sigma_{j\pm N}^\alpha = \sigma_j^\alpha$**,** $\alpha = x, y, z$**, and** $(\sigma_k^z)^2 = 1$**, we have**

$$\prod_{k=j+1}^{j} \sigma_k^z = P, \qquad \prod_{k=j+1}^{j-m} \sigma_k^z = P \prod_{k=j-m+1}^{j} \sigma_k^z,$$

**so that, using this and the Pauli matrix product rules, we find**

$$A_0 = -\sum_{j=1}^{N} \sigma_j^z, \quad A_{-n} = \sum_{j=1}^{N} \sigma_j^y \Big( \prod_{k=j+1}^{j+n-1} \sigma_k^z \Big) \sigma_{j+n}^y, \quad P \equiv \prod_{k=1}^{N} \sigma_k^z,$$

$$A_{n\pm N} = -PA_n = -A_n P, \quad A_{n\pm 2N} = A_n,$$

$$G_0 = 0, \quad G_{-n} = -G_n, \quad G_{n\pm N} = -PG_n = -G_n P, \quad G_{n\pm 2N} = G_n.$$

**Onsager derived the following commutation rules:**

$$[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.$$

**From these we also have**

$$[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k], \quad [A_j, [A_j, G_k]] = 16G_k,$$

# 演算子列からの構成

$$A_1 = \sum_{j=1}^{N} \eta_{2j} \qquad A_2 = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\eta_{2j+2} \qquad A_3 = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4}$$

$$A_0 = \sum_{j=1}^{N} \eta_{2j-1} \qquad A_{-1} = \sum_{j=1}^{N} \eta_{2j-3}\eta_{2j-2}\eta_{2j-1} \qquad A_{-2} = \sum_{j=1}^{N} \eta_{2j-5}\eta_{2j-4}\eta_{2j-3}\eta_{2j-2}\eta_{2j-1}$$

$$G_0 = 0 \qquad G_1 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1} - \eta_{2j-1}\eta_{2j})$$

$$G_2 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\eta_{2j+2})$$

$$G_3 = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4}\eta_{2j+5} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\eta_{2j+2}\eta_{2j+3}\eta_{2j+4})$$

**generally**

$$A_l = \sum_{j=1}^{N} \eta_{2j}\eta_{2j+1}\cdots\eta_{2j+2l-2},$$

$$A_{-l} = \sum_{j=1}^{N} \eta_{2j-2l-1}\eta_{2j-2l}\cdots\eta_{2j-1},$$

$$G_l = \frac{1}{2}\sum_{j=1}^{N}(\eta_{2j}\eta_{2j+1}\eta_{2j+2}\cdots\eta_{2j+2l-1} - \eta_{2j-1}\eta_{2j}\eta_{2j+1}\cdots\eta_{2j+2l-2})$$

Onsager algebra

$$[A_j, A_k] = 4G_{j-k}$$
$$[G_m, A_l] = 2A_{l+m} - 2A_{l-m}$$
$$[G_j, G_k] = 0$$

we also have

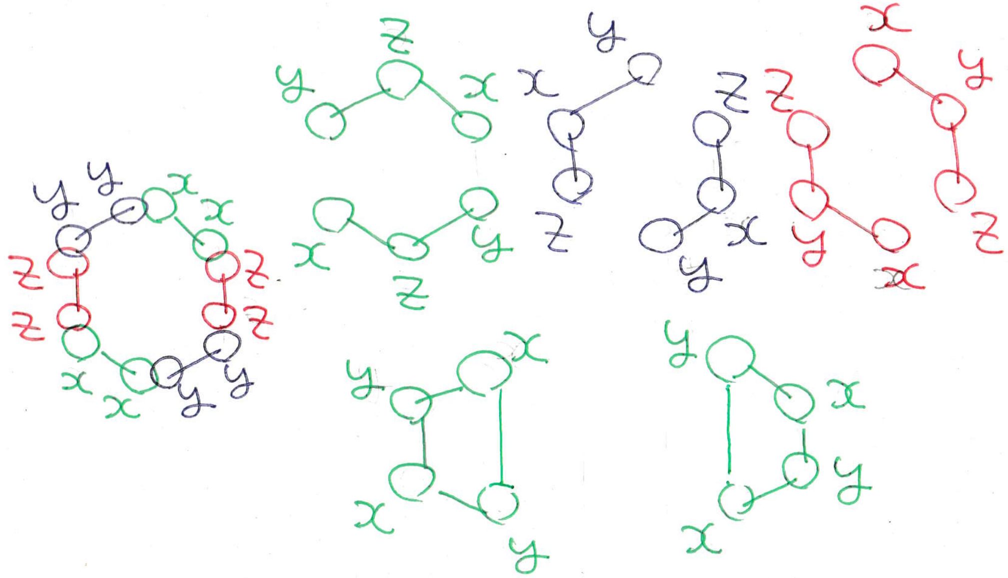
$$[A_j, [A_j, [A_j, A_k]]] = 16[A_j, A_k]$$
$$[A_j, [A_j, G_k]] = 16G_k$$

## Summary

- New fermionization method
  Examples（1-dim transv. Ising model, Kitaev model,cluster model,
  2-dim Ising model XY model）
- Infinite number of solvable models, with c=m/2
- Jordan-Wigner transformation is a special case

- Phase diagram of the 2-dim
  Kitaev model + Wen model
- excitations, Anyon

- realizations of the
  Onsager algebra



# Summary of the formula

**Find the series of operators**

$$\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4 \quad \eta_5 \quad \cdots$$

**which satisfy**

$$\eta_j \eta_k = -\eta_k \eta_j \quad (k = j \pm 1 : \text{when adjacent anticommute}), \quad \eta_j^2 = 1 \quad (k = j),$$
$$\eta_j \eta_k = \eta_k \eta_j \quad (|j - k| \geq 2 : \text{otherwise commute}).$$

**Then the following Hamiltonian is solvable**

$$-\beta \mathcal{H} = \sum_{j=1}^{N} K_j \eta_j$$

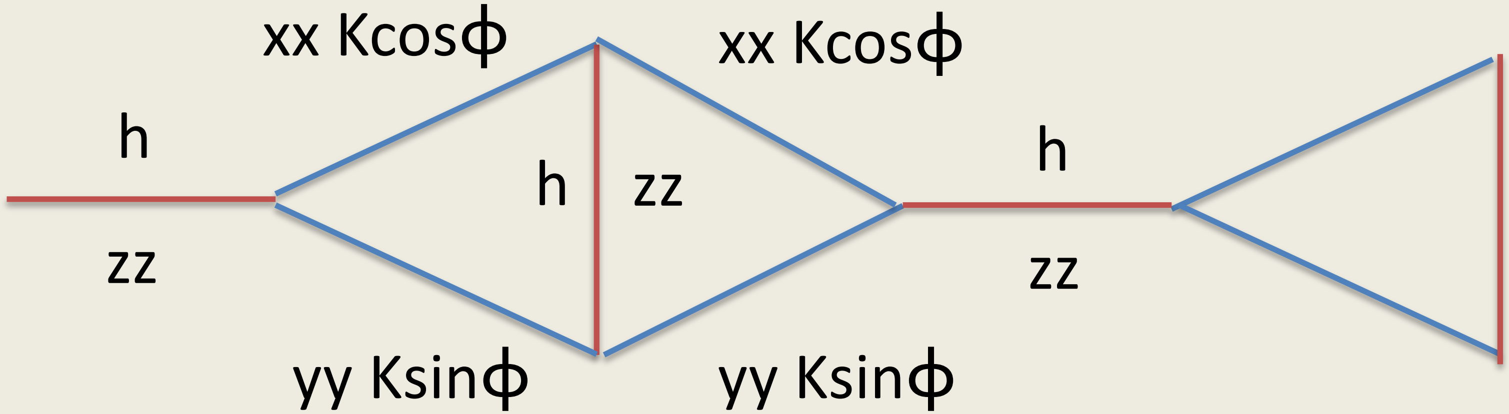
**with the use of the transformation**

$$\varphi_j = \frac{1}{\sqrt{2}} e^{i \frac{\pi}{2}(j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j .$$

## References

[1] K. Minami, J. Phys. Soc. Jpn. 85, 024003 (2016).
[2] K. Minami, Nuclear Physics B 925, 144 (2017).
[3] K. Minami, Nuclear Physics B 939, 465 (2019).
[4] K. Minami, Nuclear Physics B 973, 115599 (2021).
[5] http://www.math.nagoya-u.ac.jp/~minami/

# Other systems




**Series of operators**

$$\begin{aligned}
\eta_1 &= \sigma^z_{j-3}\sigma^z_{j-2} \quad \eta_2 = \sigma^x_{j-2}\tilde{\sigma}^x_{j-1}\cos\phi + \sigma^y_{j-2}\sigma^y_{j-1}\sin\phi \\
\eta_3 &= \tilde{\sigma}^z_{j-1}\sigma^z_{j-1} \quad \eta_4 = \tilde{\sigma}^x_{j-1}\sigma^x_{j}\cos\phi + \sigma^y_{j-1}\sigma^y_{j}\sin\phi \\
\eta_5 &= \sigma^z_{j}\sigma^z_{j+1} \quad \eta_6 = \sigma^x_{j+1}\tilde{\sigma}^x_{j+2}\cos\phi + \sigma^y_{j+1}\sigma^y_{j+2}\sin\phi \quad \ldots
\end{aligned}$$

**Commutation relationst**

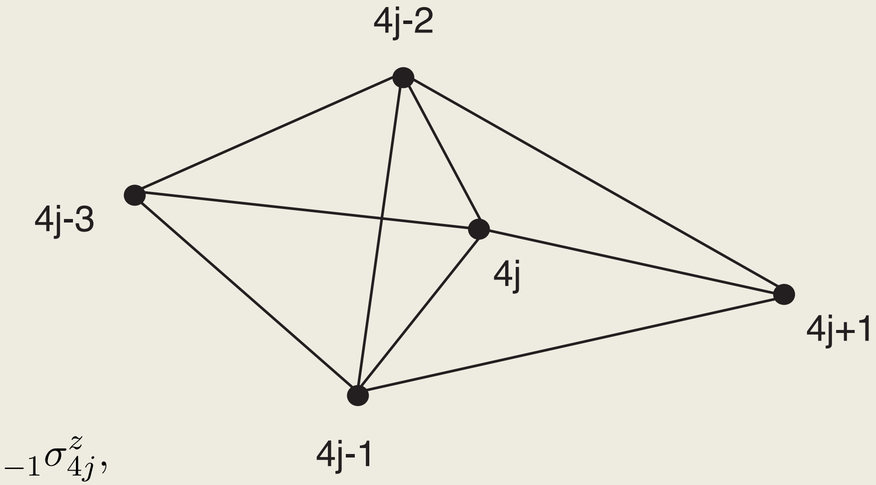
$$\eta_j\eta_k = -\eta_k\eta_j \quad (k = j \pm 1), \quad \eta_j\eta_k = \eta_k\eta_j \quad (|j-k| \geq 2), \quad \eta_j^2 = 1$$

**Hamiltonian**

$$-\beta\mathcal{H} = K\sum_{j=odd}^{N}\eta_j + h\sum_{j=even}^{N}\eta_j$$

**Partition function**

$$Z = \prod_{0<q<\pi}\left(e^{\frac{1}{2}\Lambda_q} + e^{-\frac{1}{2}\Lambda_q}\right)^2 \quad \Lambda_q = 2\sqrt{K^2 + h^2 + 2Kh\cos q}$$




**$n$-body products of the Pauli spin operators**

$$\begin{aligned} \eta_{4j-3} &= \sigma^x_{4j-3}, \\ \eta_{4j-2} &= \sigma^z_{4j-3}\sigma^z_{4j-2}\sigma^z_{4j-1}\sigma^z_{4j}, \\ \eta_{4j-1} &= \sigma^x_{4j-2}\sigma^x_{4j-1}\sigma^x_{4j}, \\ \eta_{4j} &= \sigma^z_{4j-2}\sigma^z_{4j-1}\sigma^z_{4j}\sigma^z_{4j+1}. \end{aligned}$$

**These operators satisfy the condition.**

**Hamiltonian**

$$-\beta\mathcal{H} = h\sum_{j=\mathrm{odd}}\eta_j + K\sum_{j=\mathrm{even}}\eta_j.$$

**generally when we consider**

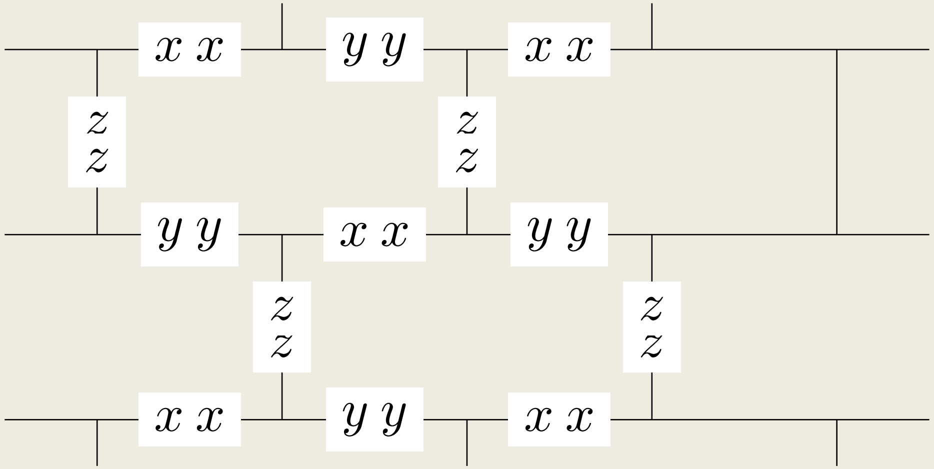
$$\eta_j = \mathcal{O}^L_j\mathcal{O}_j\mathcal{O}^R_j,$$

**where $\mathcal{O}^L_j$, $\mathcal{O}_j$, $\mathcal{O}^R_j$ are products of Pauli operators which satisfy**

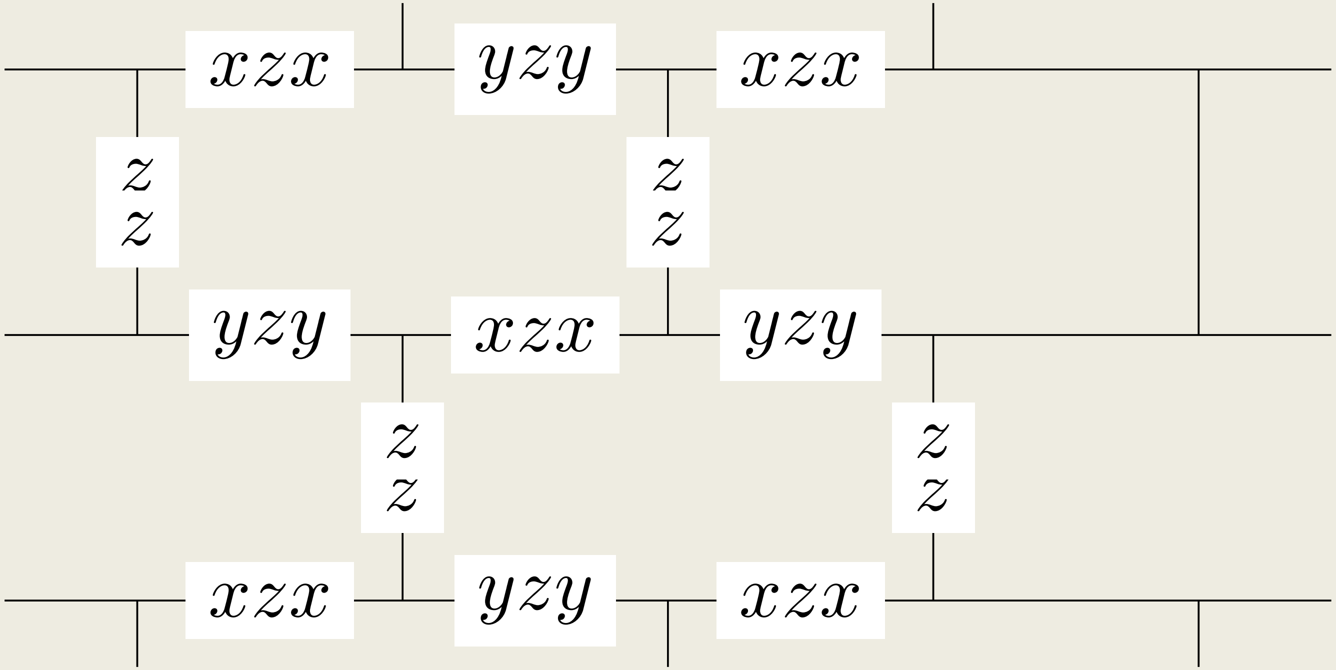
$$\{\mathcal{O}^R_{j-1}, \mathcal{O}^L_j\} = 0,$$

**then $\eta_j$ satisfy the condition.**

Kitaev model

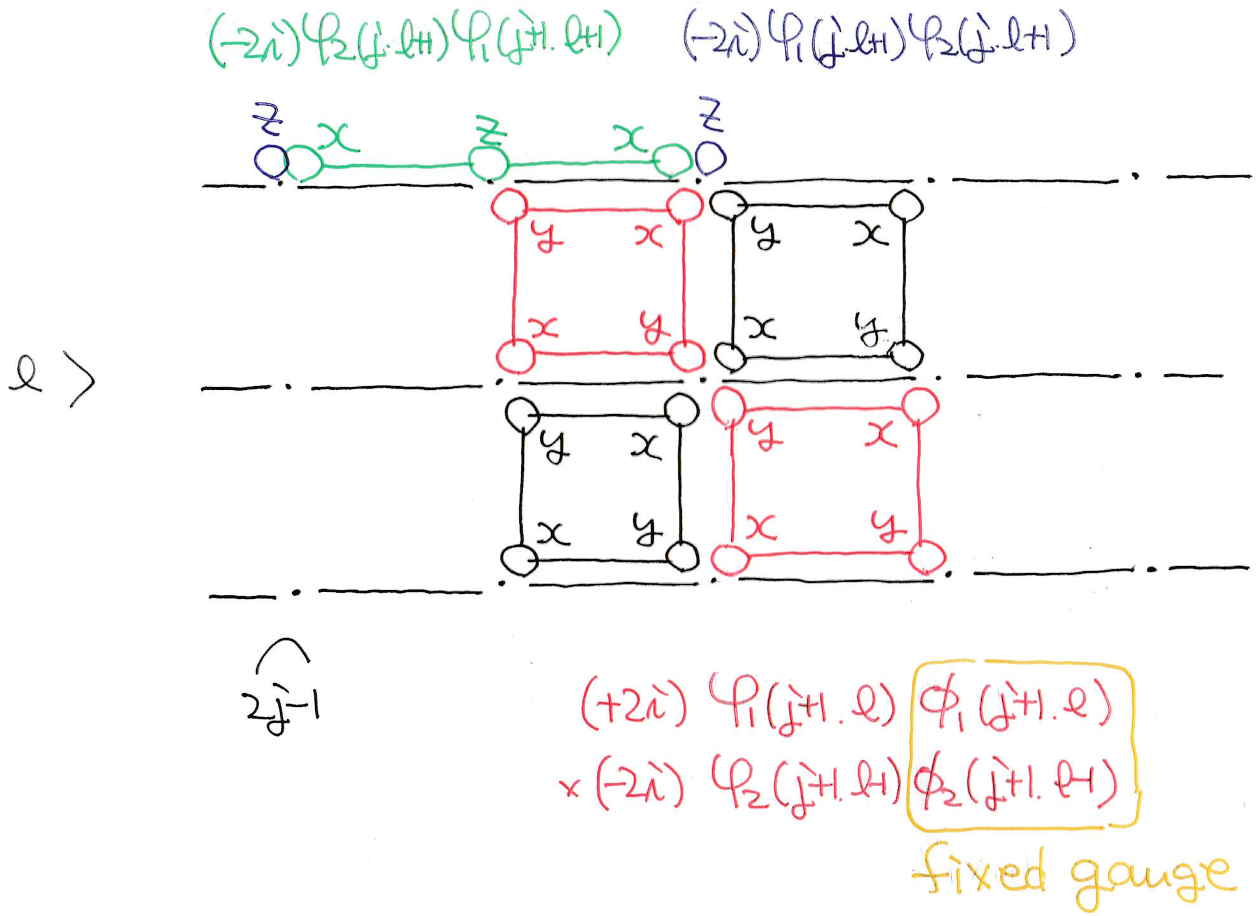



# Equivalent to the Honeycomb-lattice Kitaev model



A two-dimensional model

hat cannnot be solved by the Jordan-Wigner transformation



# 拡張されたcluster模型

**cluster 模型**

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + (K_2\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y + K_3\sum_{j=1}^{N}\sigma_j^y\sigma_{j+1}^y)$$

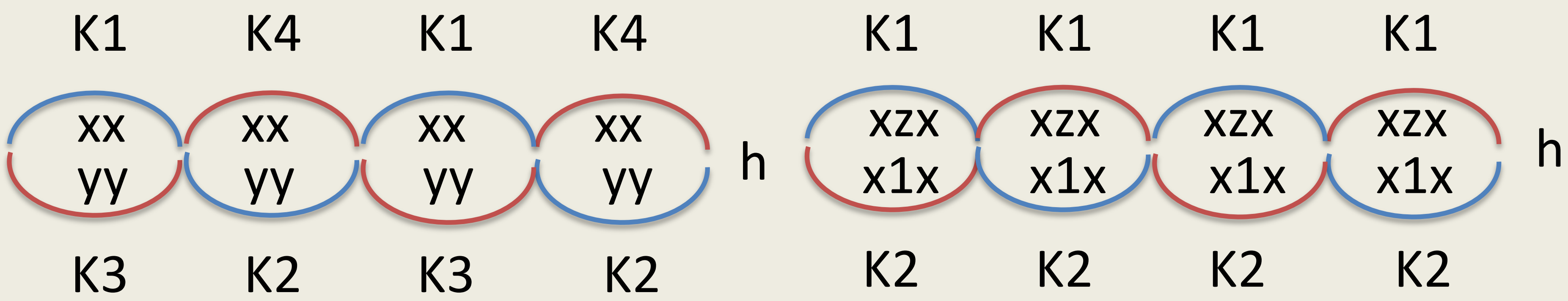
一般化された **cluster** 模型 **1**

$$-\beta\mathcal{H} = K_1\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + K_2\sum_{j=1}^{N}\sigma_j^x\,\mathbf{1}_{j+1}^z\sigma_{j+2}^x$$

変換

$$\varphi_1(j) = \begin{cases} \frac{1}{\sqrt{2}}\ \mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\mathbf{1}_j\,\sigma_{j+1}^y\,\sigma_{j+2}^x & j = \text{odd} \\ \frac{1}{\sqrt{2}}\ \mathbf{1}_1\,\sigma_2^z\,\mathbf{1}_3\,\sigma_4^z\,\mathbf{1}_5\,\sigma_6^z\cdots\cdots\sigma_j^z\,\sigma_{j+1}^x\,\sigma_{j+2}^x & j = \text{even} \end{cases}$$

$$\varphi_2(j) = \begin{cases} \frac{1}{\sqrt{2}}\ \sigma_1^z\,\mathbf{1}_2\,\sigma_3^z\,\mathbf{1}_4\,\sigma_5^z\,\mathbf{1}_6\cdots\cdots\sigma_j^z\,\sigma_{j+1}^x\,\sigma_{j+2}^x & j = \text{odd} \\ \frac{1}{\sqrt{2}}\ \sigma_1^z\,\mathbf{1}_2\,\sigma_3^z\,\mathbf{1}_4\,\sigma_5^z\,\mathbf{1}_6\cdots\cdots\mathbf{1}_j\,\sigma_{j+1}^y\,\sigma_{j+2}^x & j = \text{even} \end{cases}$$



**cluster 模型**

$$-\beta\mathcal{H} = K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^x + (K_2 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^z \sigma_{j+2}^y + K_3 \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y)$$

**一般化された cluster 模型 1**

$$-\beta\mathcal{H} = K_1 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^x + K_2 \sum_{j=1}^{N} \sigma_j^x \, \mathbf{1}_{j+1}^z \sigma_{j+2}^x$$

**結合した transverse Ising 模型**

$$\begin{aligned} -\beta\mathcal{H} &= (K_1 \sum_{j=odd} \sigma_j^x \, \mathbf{1}_{j+1}^z \, \sigma_{j+l+1}^x + K_2 \sum_{j=odd} \sigma_j^z) \\ &+ (K_1 \sum_{j=even} \sigma_j^x \, \mathbf{1}_{j+1}^z \, \sigma_{j+l+1}^x + K_2 \sum_{j=even} \sigma_j^z) \\ &+ K_3 \sum_{j=1}^{N} \sigma_1^z \sigma_2^z \sigma_3^z \cdots \sigma_j^z \end{aligned}$$

**Hamiltonian**

$$-\beta\mathcal{H} = K_{-m}\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa-m) + K_0\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa) + K_m\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa+m).$$

**In this case**

$$\begin{aligned}
f(p) &= \frac{L^\dagger}{\sqrt{LL^\dagger}}e^{ip\kappa}, \qquad L = \alpha_{-m}e^{ip(\kappa-m)} + e^{ip\kappa} + \alpha_m e^{ip(\kappa+m)} \\
\alpha_m &= K_m/K_0, \;\; \alpha_{-m} = K_{-m}/K_0, \;\; a_m = \alpha_m t > a_{-m} = \alpha_{-m}t \\
I_\infty &= \lim_{n\to\infty}\frac{\det M_n}{\mu^n} = \exp(\sum_{n=1}^{\infty} n g_n g_{-n}) \\
&= \begin{cases} \pm[(1-a_m^2)(1-a_{-m}^2)/(1-a_m a_{-m})]^{|m|/4} & |a_m| < 1 \\ 0 & |a_m| > 1 \end{cases}
\end{aligned}$$

Exponent=m/4

**Ex. transverse Ising model**

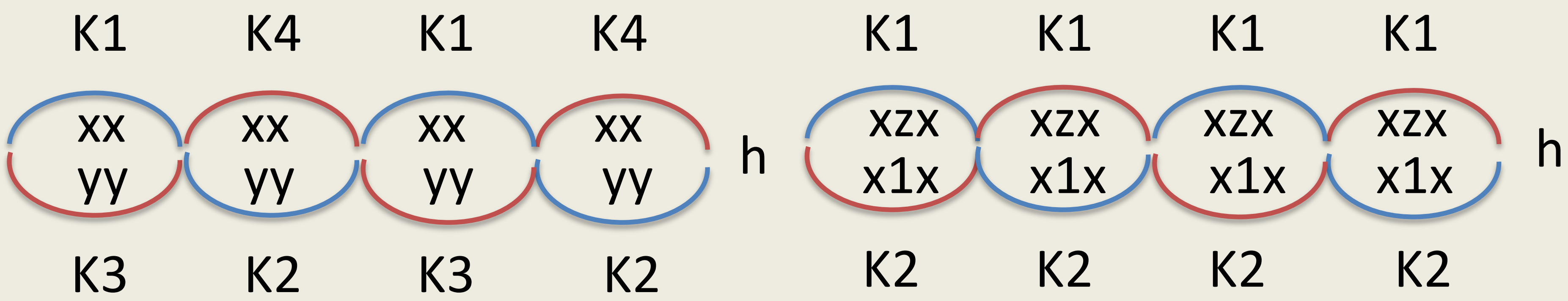
$$\langle\sigma_j^x\rangle_0^2 = \lim_{n\to\infty}\langle\sigma_j^x\sigma_{j+n}^x\rangle_0 \simeq \pm[(1-K_m/K_0)(1+K_m/K_0)]^{1/4} \qquad \beta = 1/8.$$

相関関数

$$\begin{aligned} \langle \sigma_j^z \rangle &= \langle \sigma_1^z \cdots \sigma_{j-1}^z \cdot \sigma_1^z \cdots \sigma_j^z \rangle = \rho_{j-1j} \\ \langle \sigma_2^z \cdots \sigma_{n+1}^z \rangle &= \langle \sigma_1^z \cdot \sigma_1^z \cdots \sigma_{n+1}^z \rangle = \rho_{1n+1} \end{aligned}$$

ただし

$$\begin{aligned} \rho_{lm} &= \frac{1}{4} \begin{vmatrix} G_{ll} & G_{lm} \\ G_{ml} & G_{mm} \end{vmatrix} \\ G_{lm}(\beta) &= -\frac{1}{2} \frac{K_1}{\sqrt{K_1^2 + K_2^2}} L_{r+1} + \frac{1}{2} \frac{K_2}{\sqrt{K_1^2 + K_2^2}} L_{r-1}, \qquad r = |l - m| \\ L_r &= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\Lambda_k} \cos k(r-1) \tanh \frac{1}{2} \beta \Lambda_k \, dk \end{aligned}$$



**Hamiltonian (generalized XY-chain)**

$$-\beta\mathcal{H} = K_{-m}\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa-m) + K_0\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa) + K_m\sum_{j=1}^{N}\varphi_2(j)\varphi_1(j+\kappa+m).$$

**Ground state correlation function**

$$\langle(-2i)\varphi_2(j)\varphi_1(j+\kappa)\rangle_0 = \frac{1}{2\pi}\int_0^{\pi}\frac{Le^{iq\kappa}+L^{\dagger}e^{-iq\kappa}}{\sqrt{LL^{\dagger}}}\,dq, \qquad L=\sum_l K_l e^{iql}$$

**String correlation function**

$$I_n = \langle\prod_{j=j_0}^{j_0+n-1}(-2i)\varphi_2(j)\varphi_1(j+\kappa)\rangle_0 = (-2i)^n \det M_n,$$

Wick theorem

$n\to\infty$ **limit and exponent**

Szego theorem

$$\lim_{n\to\infty}\frac{\det M_n}{\mu^n} = \exp(\sum_{n=1}^{\infty} n g_n g_{-n}) = \begin{cases} \pm[(1-a_m^2)(1-a_{-m}^2)/(1-a_m a_{-m})]^{|m|/4} & |a_m|<1 \\ 0 & |a_m|>1 \end{cases}$$

**Central charge** $\qquad c = m/2$

exponent |m|/4

**Ex. transverse Ising model**

$$\langle\sigma_j^z\rangle_0^2 = \lim_{n\to\infty}\langle\sigma_j^z\sigma_{j+n}^z\rangle_0 \simeq \pm[(1-K_m/K_0)(1+K_m/K_0)]^{1/4} \qquad \beta = 1/8.$$

That of 2-dim Ising model

# Central charge

**Estimation of the conformal charge from finite-size correction**
**Dispersion**

$$\Lambda(q) = 2\sqrt{LL^\dagger} \simeq 2|K|m\gamma\,|q| \quad (\frac{l}{N}\pi = q \simeq 0) \tag{1}$$

**Then conformal invariant normalization is**

$$2|K|m\gamma = 1 \tag{2}$$

**Consider finite size correction of the ground state energy**

$$E_0 \quad = \quad -\sum_{0<q<\pi} 2|K|2\gamma|\sin\frac{ml\pi}{2N}|[1 - \frac{\gamma^2-1}{\gamma^2}\sin^2\frac{ml\pi}{2N}]^{1/2} \tag{3}$$

**The term proportional to $1/N$ is obtained as**

$$-2|K|\gamma m\,\frac{1}{6}\frac{m\pi}{2N} = -\frac{m\pi}{12N} \quad (= -\frac{c\pi}{6N}) \tag{4}$$

**Conformal charge is**

$$c = m/2 \qquad (m = 1, 2, 3, \ldots) \tag{5}$$