Analytic indices in lattice gauge theory and their continuous limits

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Joint work with

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§1. Main theorem

<u>Thm</u> (FFMOYY)

Fix a smooth connection A on a torus $\mathbb{T}^d$ for $d \geq 0$.
For any sufficiently fine lattice approximation of A and $\mathbb{T}^d$, we have

the spectral flow of the Wilson-Dirac op on the lattice

$\parallel$

the analytic index of D_A on $\mathbb{T}^d$.

Moreover, the same is true for $\mathbb{Z}/2$-indices or family case (Clifford indices)

§2. Motivation

My motivation comes from 4-dimensional topology.

Fundamental problem in topology:

Classify "all the manifold" in given dimension $d \geq 0$
"
spaces

The basic strategy to attack this problem is

the construction of invariants:

$$\text{Manifolds} \rightsquigarrow \text{Numbers (group, } \cdots \text{)}$$

$$X \longmapsto I(X)$$

Fundamental problem in topology:

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$\underbrace{\text{manifolds}}_{\text{spaces}}$

The basic strategy to attack this problem is the construction of invariants:

$$\text{Manifolds} \rightsquigarrow \text{Numbers (group, } \dots)$$

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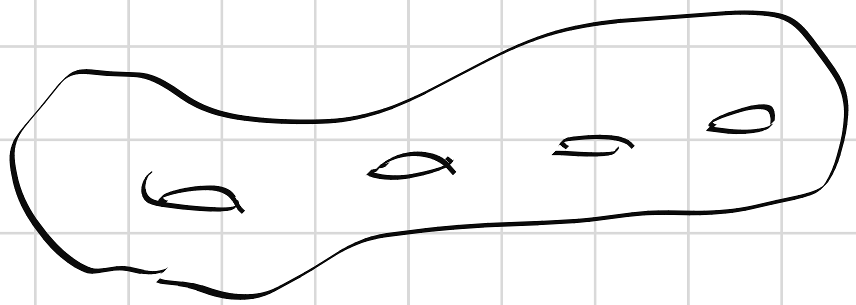
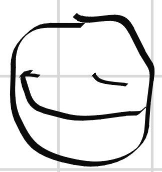
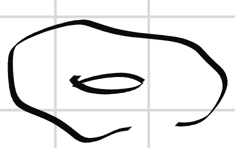
Eg. Euler numbers

$$\text{a surface } \Sigma \longmapsto \chi(\Sigma) \in \mathbb{Z}$$



If $\chi(\Sigma) \neq \chi(\Sigma')$, then $\Sigma \not\cong \Sigma'$.

$\chi($  $) = \chi(S^2) = 2$

$\chi($  $) = \chi(T^2) = 0$.

$\rightsquigarrow$ Classification of all the surfaces.

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$$X \longmapsto I(X)$$

In dimension 4 $(d = 4)$, we have the Seiberg-Witten invariants

$$X \longmapsto SW(X) \in \mathbb{Z}$$

4-dim. mfd (with spin^c str)

The construction of $SW(X)$ is based on geometric analysis of the Seiberg-Witten eq.

$$\begin{cases} D_A \Phi = 0 \\ iF_A^+ = \sigma(\Phi) \end{cases} \leftarrow \text{PDE of geometric origin.}$$

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A basic problem in SW theory :

Give a combinatorial description of SW invariants (or more refined invariants)

e.g. Bauer-Furuta invariants

In dimension 4 $(d=4)$, we have the Seiberg-Witten invariants

$$\underset{\substack{\text{4-dim. mfd} \\ \text{(with spin}^c\text{ str)}}}{X} \longmapsto \underset{\mathbb{Z}}{SW(X)}$$

$$SW(X) \in \mathbb{Z}$$

The construction of $SW(X)$ is based on geometric analysis of the Seiberg-Witten eq.

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$$\begin{cases} \boxed{D_A \Phi = 0} \\ i F_A^+ = \sigma(\Phi) \end{cases}$$

Dirac equation.

A basic problem in SW theory:

Give a combinatorial description of SW invariants (or more refined invariants) e.g. Bauer-Furuta invariants

My goal is to give a combinatorial description of Bauer-Furuta inv. by discrete approximation of the SW eq.

As a first step, we are considering discrete Dirac eq.

$\rightsquigarrow$ Today's main thm.

§3. lattice Dirac operators

⊙ (continuous) Dirac op on $\mathbb{T}^d$

$$D^{\text{cont}} := \sum_{j=1}^{d} \gamma^j \frac{\partial}{\partial x_j}$$

$\{\gamma^1, \dots, \gamma^d\}$: γ-matrices (generators of Clifford alg)

$$\begin{cases} (\gamma_j)^* = -\gamma_j \\ \gamma_j^2 = -\mathrm{id} \\ \{\gamma_i, \gamma_j\} := \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \quad \text{if } i \neq j \end{cases}$$

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<u>e.g.</u> • $d = 3$ (Pauli matrices)

$\gamma_1 = \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\gamma_2 = \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\gamma_3\ \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

• $d = 4$

$\gamma_1 = \begin{pmatrix} 0 & -\sigma_1^* \\ \sigma_1 & 0 \end{pmatrix}$ $\gamma_2 = \begin{pmatrix} 0 & -\sigma_2^* \\ \sigma_2 & 0 \end{pmatrix}$ $\gamma_3 = \begin{pmatrix} 0 & -\sigma_3^* \\ \sigma_3 & 0 \end{pmatrix}$ $\gamma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\epsilon := \gamma_1 \cdots \gamma_4$ $(=: \gamma_5)$ $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

⊙ (continuous) Dirac op on $\mathbb{T}^d$

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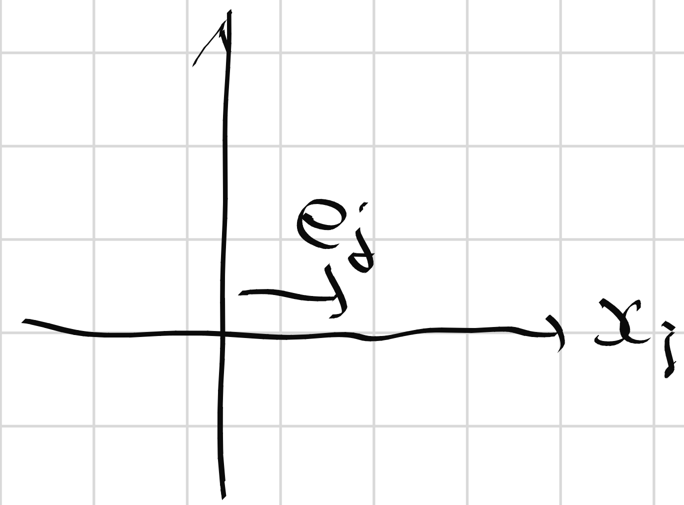
$$\epsilon = \gamma_1 \cdots \gamma_d$$

$$\rightsquigarrow \quad \{D^{\mathrm{cont}}, \epsilon\} = D^{\mathrm{cont}}\epsilon + \epsilon D^{\mathrm{cont}} = 0$$

$\rightsquigarrow$ <u>Def</u> $\mathrm{ind}(D^{\mathrm{cont}}) := \mathrm{tr}(\epsilon|_{\ker D})$

◎ lattice Dirac op on a regular lattice

Fix $a > 0$, $(a\mathbb{Z})^d \subset \mathbb{R}^d$.



For $\phi : (a\mathbb{Z})^d \to \mathbb{C}^N$,

$$\begin{cases} \nabla_j^f \phi(x) := \dfrac{\phi(x + a e_j) - \phi(x)}{a} \\[2ex] \nabla_j^b \phi(x) := \dfrac{\phi(x) - \phi(x - a e_j)}{a} \end{cases}$$

$$(\nabla_j^f)^* = - \nabla_j^b .$$

$$D^{naive} := \sum \gamma^j \left(\frac{\nabla_j^f + \nabla_j^b}{2} \right).$$

$\rightsquigarrow$ $D^{naive} \varepsilon + \varepsilon D^{naive} = 0$.

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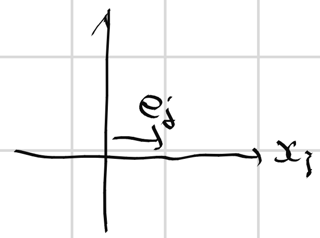

For $\phi : (a\mathbb{Z})^d \to \mathbb{C}^N$,

$$\begin{cases} \nabla_j^f \phi(x) := \dfrac{\phi(x+ae_j) - \phi(x)}{a} \\ \nabla_j^b \phi(x) := \dfrac{\phi(x) - \phi(x-ae_j)}{a} \end{cases}$$

$(\nabla_j^f)^* = -\nabla_j^b$.

$$D := \sum \gamma^j \left(\frac{\nabla_j^f + \nabla_j^b}{2} \right).$$

$\rightsquigarrow D\epsilon + \epsilon D = 0.$

← This is not a good op.

Nielsen–Ninomiya thm

$\updownarrow$

Poincaré–Hopf thm.

e.g. $\operatorname{ind}(D^{\text{naive}}) = 0$.

Fundamental problem
for odd order PD op.

◎ Wilson-Dirac op

$$D^{\text{Wilson}} := D^{\text{naive}} + \epsilon W \qquad \left(\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

Wilson term $\quad W = \sum \left(\nabla_j^f - \nabla_j^b\right)/2$

D^{Wilson} is self-adjoint, but not $D^{\text{Wilson}} \epsilon + \epsilon D^{\text{Wilson}} \neq 0$.

<u>Remark</u> Relations to top insulators.

$$\begin{cases} d = 1 & \longleftrightarrow \quad \text{SSH model} \\ d = 2 & \longleftrightarrow \quad \text{Chern insulators (class A)} \\ d = 3 & \longleftrightarrow \quad \text{class CII.} \end{cases}$$

§ Main thm again

For a lattice approximation, we consider Wilson-Dirac op

<u>Def.</u> $\mathrm{index}(D^{\mathrm{Wils.}}) :=$ spectral flow of $\{ D^{\mathrm{Wilson}} + \underbrace{m\varepsilon}_{\text{mass term}} \}_{m \in [-1,1]}$

<u>Remarks</u>

- $\mathrm{index}(D^{\mathrm{Wils.}}) \in KO^1(I, \partial I) = \mathbb{Z}$

  $I = [-1,1]$

- $\mathrm{index}\, D^{\mathrm{Wilson}}$ = ind of overlap Dirac op

  Karoubi K-theory.

Thm (FFMOYY)

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the analytic index of D_A on $\mathbb{T}^d$.

Moreover, the same is true for $\mathbb{Z}/2$-indices or family case
(Clifford indices)

The proof is based on direct computations in K-theory.
(We don't use Fujikawa's method)