The category of 2D rational CFT’s

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1. to briefly outline the development of the mathematical theory of 2D rational CFT's based on the representation theory of vertex operator algebras (VOA);
2. and to show how the language is refined and simplified in this process;
3. and show that it eventually lead us to the study of the category of 2D rational CFTs, which is impossible without the language simplification.

CFT was originated from the study of critical phenomena and strong interaction.

1. Gel'fand-Fuks-Virasoro algebra: Gel'fand-Fuks:1968, Virasoro:1970.
2. OPE: Polyakov (called correlation joining), Kadanoff (1969) and Wilson (1969).
3. Conformal bootstrap: Polyakov (1970), Ferrara et al. (1973b), Polyakov (1974).
4. Appearance of Virasoro algebra in CFT's: Mansouri-Nambu:1972, Ferrara-Gatto-Grillo:1972.
5. Operator algebra: Polyakov (1974)
6. Representation theory of Virasoro algebra: Feigin-Fuks:1979, Kac:1979, Feigin-Fuks:1982
7. Chiral algebra and minimal models: Belavin-Polyakov-Zamolodchikov:1984, Friedan-Qiu-Shenker:1984
8. Operator and algebraic-geometric formulations of CFT's: Polyakov, Ishibashi, Matsuo, Ooguri, Vafa, Segal, Alvarez-Gaume, Moore, Gomez, Atiyha, Friedan-Shenker, Beilinson, Schechtman, ...1984-1989
9. Chiral vertex operators and modular tensor categories: Moore-Seiberg: 1989

Four different approaches toward 2D rational CFT's:

1. Conformal nets: Longo:1994, Longo-Rehren:1995, Rehren:2000, Ocneanu, Böckenhauer-Evans-Kawahigashi:98-01, Xu, Müger, Longo-Rehren:2004,2009.
2. FRS-formalism based on 3D TQFT's: Felder-Fröhlich-Fuchs-Schweigert:2002, Fuchs-Runkel-Schweigert:2002-2007
3. Chiral algebras and chiral homology: Knizhnik, Friedan-Shenker, Beilinson-Schechtman:88, Tsuchiya-Kanie:88, Tsuchiya-Ueno-Yamada:89, Beilinson-Drinfeld:90's.
4. Vertex operator algebras (VOA):1984-2008: Frenkel-Lepowsky-Meurman:1984, Borcherds:1986, Frenkel-Lepowsky-Meurman:1988, Huang:1990, Zhu:1990, Frenkel-Zhu:92, Frenkel-Huang-Leopwsky:93, Huang-Leopwsky:90-94, Li:96, Dong-Li-Mason:97-98, Huang:05-08, K.-Huang:04-06, K.:07-08

Birth of VOA in 1980s

1. In 1978, J. McKay observed that 196884 = 196883 + 1, where

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

   and 196883 is the dimension of the smallest irreducible representation of the Monster group $\mathbf{M}$.

2. $21493760 = 1 + 196883 + 21296876$;

3. In 1979, **Monstrous Moonshine Conjecture**, J.H. Conway, S.P. Norton, "Monstrous moonshine", Bull. London Math. Soc. , 11 (1979) 308–339:

   There is an infinite-dimensional graded vector space $V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$, with the following properties: each V_k carries a finite-dimensional representation of $\mathbf{M}$; write χ_k for its character. For each $g \in \mathbf{M}$, define the Thompson–McKay series $T_g(z) = \sum_{k=1}^{\infty} \chi_k(g) q^k$ for $q = \exp(2\pi i z)$. The T_g is a generator ("Hauptmodul") of the field of modular functions for some genus-0 group $G_g \leq SL_2(\mathbb{R})$.

4. In 1982, R.L. Griess, "The friendly giant" Invent. Math. , 69 (1982) pp. 1–102

5 The construction of moonshine module $V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$:

- I. B. Frenkel, J. Lepowsky, and A. Meurman. A moonshine module for the monster. Proceedings of a Conference November 10-17, 1983, pages 231–273, Springer, New York, 1985. Publications of the Mathematical Sciences Research Institute
- I. B. Frenkel, J. Lepowsky, and A. Meurman, A natural representation of the Fischer-Griess monster with the modular function J as character, Proc. Natl. Acad. Sci. USA 81 (1984) 3256-3260..
- Relation to 2D CFT was soon noticed by audiences attending their talks.

6 Introduction of the notion of a vertex algebra based on FLM:1984: Borcherds, Vertex algebras, Kac-Moody algebras and the monster, Proc.Nat. Acad. Sci. U.S.A. 83 (1986), 3068-3071;

We will list some identities satisfied by the operators u_n and show how to construct Lie algebras from them. u, v, and w denote elements of V, and 1 is the unit of V.

For any even lattice R the operators u_n on V satisfy the following relations.

(i) $u_n(w) = 0$ for n sufficiently large (depending on u, w). This ensures convergence of the following formulae.

(ii) $1_n(w) = 0$ if $n \neq -1$, w if $n = -1$.

(iii) $u_n(1) = D^{(-n-1)}(u)$.

(iv) $u_n(v) = \sum_{i\geq 0}(-1)^{i+n+1}D^{(i)}(v_{n+i}(u))$.

(v) $(u_m(v))_n(w) = \sum_{i\geq 0}(-1)^i\binom{m}{i}(u_{m-i}(v_{n+i}(w)) - (-1)^m v_{m+n-i}(u_i(w)))$.

(The binomial coefficient $\binom{m}{i}$ is equal to $m(m-1)\ldots(m-i+1)/i!$ if $i \geq 0$ and 0 otherwise.)

7 Introduction of the notion of a vertex operator algebra (VOA):

Frenkel-Lepowsky-Meurman, "Vertex operator algebras and the Monster", Pure and Appl. Math., 134, Academic Press, New York, 1988..

(Jacobi identity) *For every* $\psi, \varphi \in \mathcal{F}$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2)$$

Definition of a VOA: Frenkel-Lepowsky-Meurman:1988, Dong's Lemma, Li:1996, Kac:1997, Li-Lepowsky:2004

1. $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, $\mathbf{1} \in V_{(0)}, \omega \in V_{(2)}$, $\dim V_{(n)} < \infty$ and $V_{(n)} = 0, n << 0$.
2. for $u, v \in V$, $u_n \in \mathrm{End}(V)$ s.t. $u_n v = 0$ for $n >> 0$.

$$Y : V \otimes V \to V((x)),$$
$$u \otimes v \mapsto Y(u, x)v = \sum_{n\in\mathbb{Z}} u_n v x^{-n-1}$$

3. $Y(\mathbf{1}, x) = \mathrm{id}_V$ and $\lim_{x\to 0} Y(u, x)\mathbf{1} = u$.
4. $\exists N \in \mathbb{N}$ for each pair of u, v such that $(x_1 - x_2)^N[Y(u, x_1), Y(v, x_2)] = 0$.
5. $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ where $L(n), n \in \mathbb{Z}$ are generators of Virasoro algebra.
   - $L(0)$ is the grading operator.
   - $[L(-1), Y(u, x)] = Y(L(-1)u, x) = \frac{d}{dx} Y(u, x)$.

Algebraic theory of VOA's and representations

1. Chiral vertex operator = Intertwining Operator Frenkel-Huang-Lepowsky:1989,1993:

Definition

Let V be a VOA. (W_i, Y_i) for $i = 1, 2, 3$ are V-modules. An intertwining operator of type $(W_1, W_2 | W_3)$ is a linear map $W_1 \otimes_{\mathbb{C}} W_2 \to W_3\{x\}$, i.e.

$$\begin{aligned} W_1 &\to (\hom(W_2, W_3))\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1} \end{aligned}$$

such that $(w_{(1)})_n w_{(2)} = 0$ for n sufficiently large and

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2) Y_2(v, x_1) w_{(2)} \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, x_2) w_{(2)}. \end{aligned}$$

2. Zhu's thesis (1990): Y.-C. Zhu, Modular invariance of vertex operator algebras, J. Amer. Math. Soc. 9 (1996) 237-302

(1) Zhu's algebra: The quotient space $A(V) = V/O(V)$, where $O(V)$ is a subspace of V spanned by $\mathrm{Res}_z(Y(a,z)z^{-2}(z+1)^{\mathrm{deg}\, a}b)$, has an associative algebraic structure defined by $a * b := \mathrm{Res}_z(Y(a,z)z^{-1}(z+1)^{\mathrm{deg}\, a}b)$.

(2) Proof of modularity of partition functions of modules over a rational VOA, i.e. $\chi_{M_i}(q) = \mathrm{Tr}_{M_i}(q^{L_0 - \frac{c}{24}})$ span a representation of $\mathrm{SL}(2,\mathbb{Z})$.

$$Tr|_{M_i} Y(e^{2\pi z_1 L_0} a_1, e^{2\pi z_1 L_0}) \cdots Y(e^{2\pi z_1 L_0} a_n, e^{2\pi z_n L_0}) q^{L_0 - \frac{c}{24}}.$$

No intertwining operators!

3. Zhu's thesis led to the classification of modules over various VOA's.

(1) I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992), 123–168.

(2) W. Wang, Rationality of Virasoro vertex operator algebras, International Mathematics Research Notices 7 (1993), 197–211.

(3) V. Kac and W. Wang, Vertex operator superalgebras and their representations, Contemp. Math. Amer. Math. Soc. 175 (1994), 161-191.

(4) W. Wang, Classification of irreducible modules of W3 algebra with $c = -2$, Comm. Math. Phys. 195 (1998), 113-128.

(5) ...

3. Tensor category theory of the modules over a rational VOA, Huang-Lepowsky:1991,1995

- Y.-Z. Huang, J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, II, Selecta Math. (N.S.) 1 (1995) 699-756, 757-786.
- Y.-Z. Huang, J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, J. Pure Appl. Alg. 100 (1995) 141-171.
- Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, J. Pure Appl. Alg. 100 (1995) 173-216.

$$(W_1 \boxtimes W_2)' \subset (W_1 \otimes_{\mathbb{C}} W_2)^*.$$

$$\{\text{Intertwining Operators} : W_1 \otimes_{\mathbb{C}} W_2 \to W_3\{x\}\} \simeq \mathrm{hom}_{\mathrm{Mod}_V}(W_1 \boxtimes W_2, W_3).$$

4. There are many important algebraic studies of VOA: Dong's Lemma, Li's local systems, generalized Zhu's algebras, regularity, twisted modules, modularity, etc.: Dong, Li, Mason, Abe, Buhl, Miyamoto, Adamović, Yamada, Tanabe, Nagatomo, Tsuchiya, ...

Since these works are not directly related to what I will talk about on 2D rational CFT's, we won't expand the discussion here.

5. Huang's proof of Verlinde formula and modular tensor category

- Y.-Z. Huang, Differential equations and intertwining operators, Commun. Contemp. Math. 7 (2005) 375-400.
- Y.-Z. Huang, Differential equations, duality and modular invariance, Commun. Contemp. Math. 7 (2005) 649-706.
- Huang, Y.-Z.: Vertex operator algebras and the verlinde conjecture. Commun. Contemp. Math. 10(1) (2008) 103-154.
- Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, Commun. Contemp. Math. 10 (2008) 871-911

Theorem (Huang's Theorem)

Let V be a simple VOA satisfying the following conditions: (1) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbb{1}$ and V' is isomorphic to V as a V-module; (2) Every $\mathbb{N}$-gradable weak V-module is completely reducible; (3) V is C_2-cofinite. Then Mod_V is a modular tensor category.

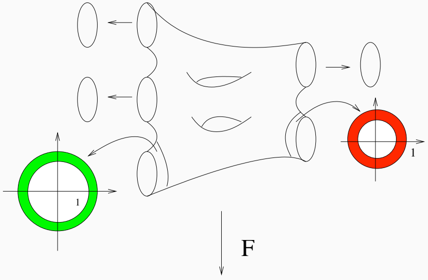
6. VOA extensions of a rational VOA as algebras in MTC: Kirillov-Ostrik:2001, Huang-Kirillov-Lepowsky:2003, Fuchs-Schweigert:2003

Theorem (Huang-Kirillov-Lepowsky:2003,2015)

For a rational VOA V, a VOA extension A of V is equivalent to a commutative algebra A in Mod_V.

Geometric theory of VOA’s and CFT’s

1. Igor Frenkel had considered the geometric meaning of VOA and had initiated a program of studying CFT based on the mathematical theory of VOA before the appearance of Segal's definition of 2D CFT.
2. Segal's definition of a 2D CFT (also Kontsevich) $\mathcal{F} : \mathrm{Bord}^2 \to \mathrm{TV}$.



$$H_{\mathrm{cl}}^{\otimes 3} \longrightarrow H_{\mathrm{cl}}^{\otimes 2}$$

Two disadvantages of Kontsevich-Segal's definition:

- The quantum fields $\phi(x), \psi(x)$ in physics usually are associated to a point in the space-time. They do not live in Kontsevich-Segal's definition in a direct way. This suggest to replace Riemann surfaces with parametrized boundaries by Riemann surfaces with parametrized punctures.

  Vafa, Conformal theories and punctured surfaces, Phys. Lett. B 199, 2 (1987) 195-202.

- Replace Hilbert spaces by graded vector spaces ($\mathrm{Vect}_{\mathbb{C}}^{gr}$). Huang's Rutgers thesis:1990.

3. Huang's thesis (1990): A VOA is equivalent to an algebra over the complex-analytic version of 2-disk operad: Huang's Rutgers thesis:1990; Huang, Geometric interpretation of vertex operator algebras. Proceedings of the National Academic Society USA, 88 :9964–9968, 1991. Y.-Z. Huang, Two-dimensional conformal geometry and vertex operator algebras, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997.

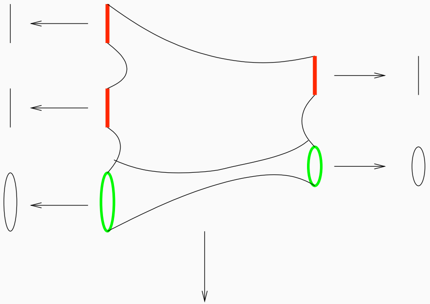
Theorem (Huang:1990,1991,1997)

A VOA is nothing but a complex-analytic E_2-algebra (2-sphere with punctures).

4. From 2-sphere with punctures to 2-sphere with holes via a completion procedure:

- Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, I, Commun. Math. Phys. 204 (1999), 61-84.
- Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, II, Commun. Math. Phys. 242 (2003), 425-444.

5. Mathematical definition of a boundary-bulk CFT (or open-closed CFT): Y.-Z. Huang, Riemann surfaces with boundaries and the theory of vertex operator algebras, Vertex operator algebras in mathematics and physics (Toronto, ON, 2000), 109-125, Fields Inst. Commun., 39, Amer. Math. Soc., Providence, RI, 2003.



$$H_{op}^{\otimes 2} \otimes H_{cl} \longrightarrow H_{op} \otimes H_{cl}$$

6. Boundary CFT and open-string VOA: Huang-K.:math/0308248.

- $Y_{op}(\phi, r) = \phi(r) = \sum_{n\in\mathbb{R}} \phi_n r^{-n-1}$.
-
$$\phi(r_1)\psi(r_2) \sim \frac{(\phi_k\psi)(r_2)}{(r_1-r_2)^{k+1}} + \frac{(\phi_{k-1}\psi)(r_2)}{(r_1-r_2)^k} + \cdots .$$
  where $k \in \mathbb{R}$ is not necessarily an integer!
- No commutativity!: $\phi(r_1)\psi(r_2) \nsim \psi(r_2)\phi(r_1)$.

Theorem (Huang-K.:math/0308248)

An open-string VOA is equivalent to a "real-analytic E_1-algebra" (2-disk with only punctures on the boundary).

7. Introduction of full field algebra for bulk CFT (or closed CFT): Huang-K.:math/0511328; K.:math/0603065

- An example of full field algebra: $V_{cl} = V_L \otimes_{\mathbb{C}} \overline{V}_R$ where V_L, V_R are VOA's;
- Full field algebra over $V_L \otimes_{\mathbb{C}} \overline{V}_R$: $V_L \otimes_{\mathbb{C}} \overline{V}_R \hookrightarrow V_{cl}$;
- $Y_{cl}(u; z, \bar{z})v = \sum_{m,n} u_{m,n} z^{-m-1} \bar{z}^{-n-1}$;
- Associativity and commutativity hold.

Theorem (K.:math/0603065)

A full field algebra is a "real-analytic E_2-algebra" (2-sphere with punctures).

8. Open-closed field algebra: K.:math/0610293.

- A full field algebra (V_{cl}, Y_{cl}), an open-string VOA (V_{op}, Y_{op}) and $Y_{cl-op}(u; z, \bar{z}) = \sum_{m,n} u_{m,n} z^{-m-1} \bar{z}^{-n-1}$.
- Associativity I & II:

$$\langle v', Y_{cl-op}(u; z, \zeta) Y_{op}(v_1, r) v_2 \rangle = \langle v', Y_{op}(Y_{cl-op}(u; z-r, \zeta - r) v_1, r) v_2 \rangle.$$

$$\langle w', Y_{cl-op}(u; z_1, \zeta_1) Y_{cl-op}(v; z_2, \zeta_2) \rangle = \langle w', Y_{cl-op}(Y_{cl}(u_1; z_1 - z_2, \zeta_1 - \zeta_2) u_2; z_2, \zeta_2) v_2 \rangle.$$

- Commutiativity I & II:
- V-symmetric boundary condition: If $V = \langle T_{\mu\nu} \rangle$, then it is called conformal-symmetric boundary condition. An open-closed field algebra satisfies a V-symmetric boundary condition is called an open-closed field algebra over V.

Theorem (K.:math/0610293)

If V is rational, an open-closed field algebra over V is an algebra over Swiss-cheese partial operad (2-disk with punctures on the boundary and in the interior).

Unification of algebraic and geometric approach: after Huang's theorem on MTC

Combining the following development:

1. Huang's Theorem: If V is a rational VOA, then Mod_V is a modular tensor category.
2. Kirillov-Ostrik:2001, Huang-Kirillov-Lepowsky:2015: For a rational VOA V, a VOA extension A of V is equivalent to a commutative algebra A in Mod_V.
3. Open-string VOA, full field algebra, open-closed field algebra, boundary-states: Huang-K.:2004-2010, K.:2007-2008,

We obtain a categorical characterization/classification of open-closed CFT over a rational V (i.e. satisfying a V-symmetric boundary condition).

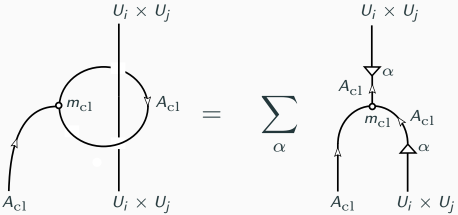
(preceded by Fuchs-Runkel-Schweigert:2002-2006: simple special Frobenius algebra A in Mod_V + state sum construction of all structure constants.)

Theorem K.:math/0612255,K.-Runkel:0807.3356: An open-closed CFT over a rational VOA V is a triple $(A_{\mathrm{op}}|A_{\mathrm{cl}}, \iota_{\mathrm{cl-op}})$, where

1. A_{cl}: a commutative symmetric Frobenius algebra in $\mathfrak{Z}_1(\mathrm{Mod}_V) = \mathrm{Mod}_V \boxtimes \overline{\mathrm{Mod}_V}$,
2. A_{op}: a symmetric Frobenius algebra in Mod_V,
3. $\iota_{\mathrm{cl-op}} : A_{\mathrm{cl}} \to Z(A_{\mathrm{op}}) \hookrightarrow \otimes^R(A_{\mathrm{op}})$: an algebra homomorphism.

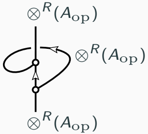
satisfying (1) modular invariance condition:

$$\frac{\dim U_i \, \dim U_j}{\dim \mathcal{C}}$$



(2). Cardy condition:

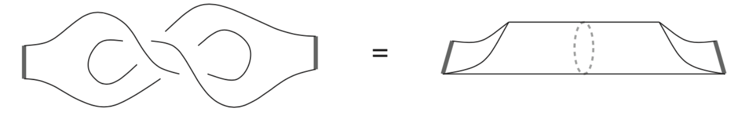
$$\iota_{\text{cl}-\text{op}} \circ \iota^*_{\text{cl}-\text{op}} =$$



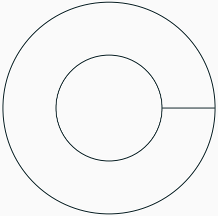
where $\iota^*_{\text{cl}-\text{op}}$ is defined using the self-duality of Frobenius algebras.

Remark: further but not significant simplification was available, e.g. replacing the modular invariant condition by the Lagrangian condition: $(\dim A_{\text{cl}})^2 = \dim \mathfrak{Z}_1(\text{Mod}_V)$. K.-Runkel:2009

Cardy condition



A weaker veresion of Cardy condition:



In physics literature, the complete Cardy condition has never been written down explicitly. Only a weaker version is commonly used in literature. The completion version can be found in K.:math/0612255

Definition 3.4. *The open-closed field algebra over V given in (2.12) and equipped with nondegenerate bilinear forms $(\cdot,\cdot)_{op}$ and $(\cdot,\cdot)_{cl}$ is said to satisfy* **Cardy condition** *if the left hand sides of the following formula, $\forall z_1, z_2 \in \mathbb{H}$, $v_1, v_2 \in V_{op}$,*

$$Tr_{V_{op}}\left(Y_{op}(\mathcal{U}(q_{s_1})v_1, q_{s_1})Y_{op}(\mathcal{U}(q_{s_2})e^{-2\pi i L(0)}v_2, q_{s_2})q_\tau^{L(0)-c/24}\right)$$
$$= \left((T_1^L \otimes T_1^R)^* \iota_{cl-op}^*(z_1, \bar{z}_1)(T_2 v_1),\ q_{-\frac{1}{\tau}}^{-c/24}(T_3^L \otimes T_3^R)^* \iota_{cl-op}^*(z_2, \bar{z}_2)(T_4 v_2)\right)_{cl} \quad (3.30)$$

converge absolutely when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$, and the right hand side of (3.30) converge absolutely for all $s_1, s_2 \in \mathbb{H}$ satisfying $Re\, s_1 = 0$, $Re\, s_2 = \frac{1}{2}$. Moreover, Eq. (3.30) holds when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$.

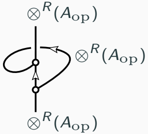
⇝ Cardy condition in terms of intertwining operators:

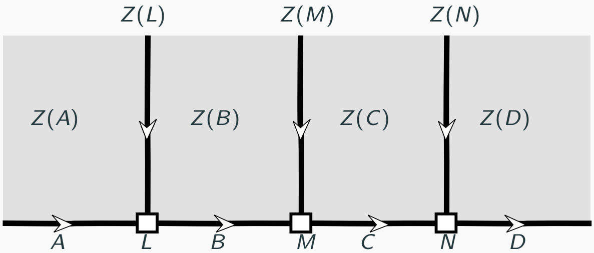
Theorem 3.10. *The Cardy condition can be rewritten as follows:*

$$\left(\theta_{W_{r_R(i)}} \circ \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}}) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_L(i)}})\right) \otimes \left(\Omega_0(\sigma_{132}(\mathcal{Y}_{\iota_{cl-op}})) \circ (\varphi_{op} \otimes \mathrm{id}_{W_{r_R(i)}})\right)$$
$$= S^{-1}\left(Y^f_{op} \otimes \left(\Omega_{-1}(Y^f_{op}) \circ (\theta_{V_{op}} \otimes \mathrm{id}_{V_{op}})\right)\right). \quad (3.73)$$

⇝ Cardy condition in categorical language:

$$\iota_{\mathrm{cl-op}} \circ \iota^*_{\mathrm{cl-op}} \quad =$$





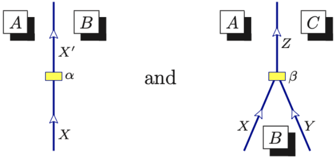
From boundary-bulk duality to the center functor (i.e. holography is functorial):

1. Bulk CFT is the center of a boundary CFT Fjelstad-Fuchs-Runkel-Schweigert:hep-th/0612306, Davydov:0908.1250;
2. A and B are Morita equivalent iff $Z(A) \simeq Z(B)$ K.-Runkel:0708.1897;
3. Center functor with a fixed chiral symmetry: Davydov-K.-Runkel:1307.5956
4. Center functor for arbitrary rational chiral symmetries is fully faithful: K.-Zheng:2018, K.-Yuan-Zheng:2021

Defects in 2D RCFT’s

Fröhlich-Fuchs-Runkel-Schweigert, Duality and defects in rational conformal field theory, Nucl. Phys. B 763 (2007) 354-430. hep-th/0607247 (based on the so-called FRS-formulation of 2D RCFT):

Defects can also be joined. The junction is labelled by an element of the relevant morphism space of bimodules. For example, when joining two A-B-defects X and X', or an A-B-defect X and a B-C-defect Y to an A-C-defect Z, according to



(2.24)

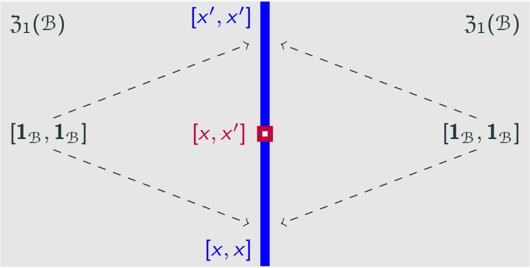
the junctions get labelled by morphisms $\alpha \in \mathrm{Hom}_{A|B}(X', X)$ and $\beta \in \mathrm{Hom}_{A|C}(Z, X \otimes_B Y)$, respectively.[6] Note also that a junction linking an A-A-defect X to the invisible defect A is

Partition function of a torus with two parallel defect line labeled by X and Y with opposite orientations and 2-cells are all labeled by $A \in \mathrm{Mod}_V$ can be determined by the formula:

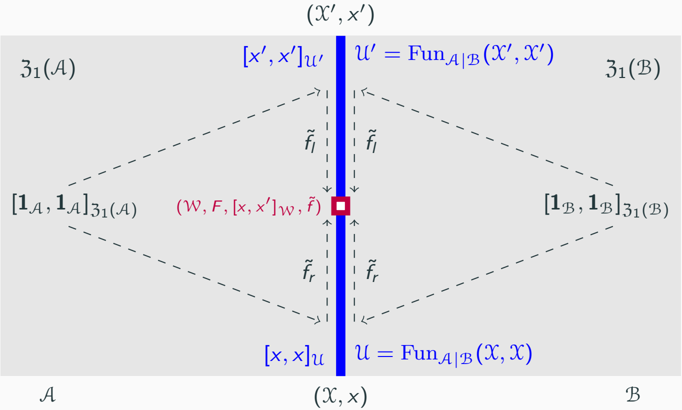
$$Z(A)_{ij}^{X|Y} = \dim \hom_{A|A}(i \otimes^{+} X \otimes^{-} j, Y).$$

The mathematical language of a RCFT was further simplified by using a powerful categorical language: internal homs. Davydov:0908.1250, Davydov-K.-Runkel:1307.5956

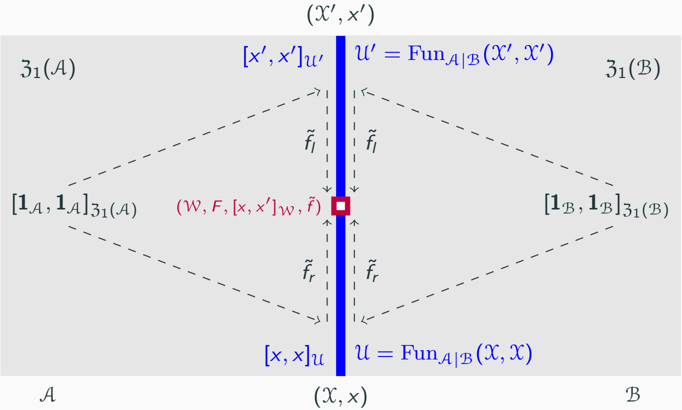
(1) $\mathcal{C}$ a fusion category, $\hom_{\mathcal{C}}(x,y) \in \mathrm{Vec}$, $\mathcal{C} = {}^{\mathrm{Vec}}\mathcal{C}$;
(2) Replacing $\hom_{\mathcal{C}}(x,y)$ by internal homs $[x,y] = y \otimes x^* \in \mathcal{C} \rightsquigarrow$, we obtain ${}^{\mathcal{C}}\mathcal{C}$.
(3) For a left $\mathcal{C}$-module category $\mathcal{M} = \mathrm{RMod}_A(\mathcal{C})$, replacing $\hom_{\mathcal{M}}(x,y)$ by internal homs $[x,y] = (x \otimes_A y^*)^*$. $\rightsquigarrow$, we obtain ${}^{\mathcal{C}}\mathcal{M}$.



Ignoring VOA, all the rest data (bulk + defects $x, x' \in \mathcal{B}$, $[x, x'] \in \mathfrak{Z}_1(\mathcal{B})$) can be reduced to $\mathcal{B}$ (or an enriched fusion category ${}^{\mathfrak{Z}_1(\mathcal{B})}\mathcal{B}$), which is called the topological skeleton of the RCFT.



$\rightsquigarrow$ The category $\mathcal{QL}^2_{\mathrm{sk}}$ of the topological skeletons of all 2D quantum liquids: objects are multi-fusion 1-categories $\mathcal{A}, \mathcal{B}$, 1-morphisms $(\mathcal{X}, x)$, ... K.-Yuan-Zheng:1912.13168



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$\rightsquigarrow$ $\mathcal{QL}^2_{\mathrm{sk}} \simeq \bullet/\Sigma^3\mathbb{C} = \bullet/3\mathrm{Vec}$. K.-Zheng:2011.02859, 2201.05726

Generalize to all quantum liquids (topological orders, SPT/SET orders, symmetry-breaking orders, CFT-like gapless phases)

1. Theory of defects: Fröhlich-Fuchs-Runkel-Schweigert:06 $\rightsquigarrow$ Kitaev-K.:2011
2. The categories of topological orders: K.-Wen:2014, K.-Wen-Zheng:2015
3. Condensation completion: Carqueville-Runkel:2012, Douglas-Reutter:2018, Gaiotto-Johnson-Freyd:2019, Johnson-Freyd:2020, K.-Lan-Wen-Zhang-Zheng:2020
4. Multi-fusion n-categories: Douglas-Reutter:2018, Johnson-Freyd:2020, K.-Zheng:2020
5. Classification theory of SPT/SET orders: Barkeshli-Bonderson-Cheng-Wang:2014, Lan-K.-Wen:2017-2018, Lan-Wen:2018, K.-Lan-Wen-Zhang-Zheng:2020, Johnson-Freyd:2020, K.-Zheng:2021
6. Rational CFT's, gapless boundaries of 2+1D topological orders and topological Wick rotation: K.-Zheng:2018,2020,2021

Combining 1-6 $\rightsquigarrow$ the category of nD quantum liquids (topological orders, SPT/SET orders, symmetry-breaking orders and CFT's). K.-Zheng:2011.02859, 2201.05726

$$\mathcal{QL}^n_{\mathrm{sk}} \simeq \bullet/\Sigma^{n+1}_*\mathbb{C}.$$

Thank you !