Non-Kerov deformation of the Macdonald polynomials

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The talk is mostly a review.

But partly based on works w/ H. Awata, A. Mironov, A. Morozov

arXiv 1912.12897, 2002.12746, 2005.10563 //

1. Introduction

Macdonald polynomials $P_\lambda(x;q,t)$ are very interesting symmetric polynomials.

Two features of $P_\lambda(x;q,t)$

1. They are orthogonal polynomials.

   a deformation of the inner product $\longrightarrow$ Kerov deformation

2. They are simultaneous eigenfunctions of mutually commuting Hamiltonians.

   $\longrightarrow$ possibility of non-Kerov deformation

- $P_\lambda(x;q,t)$ are simultaneous eigenfunctions of mutually commuting Hamiltonians.

We find a bispectral nature of eigenvalue problem.

Duality (exchange of "coordinates" and "momenta")

$\Longrightarrow$ $\exists\, M(x,y;q,t)$ a mother function (母関数)

s.t. $M(x,y;q,t) = M(y,x;q,t)$

$M(x,y;q,t)$ allows a formal power series expansion

a "systematic" deformation of the coefficinents (non-Kerov)

$\underset{\text{partition}}{P_{\lambda}}(x;\underset{\text{two parameters}}{q,t})$ $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > \lambda_{\ell+1} = 0)$

$\ell(\lambda)$ length of λ

(more parameters after deformation)

2 Symmetric polynomials and Kerov deformation

$$\Lambda_N = \mathbb{Z}[x_1, x_2, \cdots, x_N]^{S_N} \ni f, \quad f(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(N)}) = f(x_1, x_2, \cdots, x_N) \quad (\sigma \in S_N)$$

the ring of symmetric polynomials

λ : partition $\quad \ell(\lambda) \leq N \quad \longrightarrow \quad x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$

monomial symmetric polynomials $\quad m_\lambda(x) = \mathrm{Sym}(x^\lambda)$

symmetrization

FACT $\quad P_N = \{\lambda \mid \ell(\lambda) \leq N\}, \quad \{m_\lambda\}_{\lambda \in P_N}$ is a basis of Λ_N

example $(N=3)$

$m_{(1)}(x) = x_1 + x_2 + x_3, \quad m_{(2)}(x) = x_1^2 + x_2^2 + x_3^2$

$m_{(1,1)}(x) = x_1 x_2 + x_2 x_3 + x_3 x_1, \quad m_{(3)}(x) = x_1^3 + x_2^3 + x_3^3$

$m_{(2,1)}(x) = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1, \quad m_{(1,1,1)}(x) = x_1 x_2 x_3, \cdots$

power sum symmetric polynomials $\quad p_n(x) = \sum_{i=1}^{N} x_i^{n}$

λ : partition $\longrightarrow \quad p_\lambda(x) = \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(x)$

<u>FACT</u> $\quad \{p_\lambda(x)\}_{\lambda \in P_N}$ is a basis of $\Lambda_{N,\mathbb{Q}} := \Lambda_N \otimes_{\mathbb{Z}} \mathbb{Q}$

$$m_{(1)} = p_1, \qquad m_{(2)} = p_2, \qquad m_{(1,1)} = \frac{1}{2}p_1^2 - \frac{1}{2}p_2$$

$$m_{(3)} = p_3 \qquad m_{(2,1)} = p_1 p_2 - p_3 \qquad m_{(1,1,1)} = \frac{1}{3}p_3 - \frac{1}{2}p_2 p_1 + \frac{1}{6}p_1^3$$

......

The standard inner product on $\Lambda_{N,\mathbb{Q}}$ is

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \qquad z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$$

$$m_i(\lambda) = \#\{j \,;\, \lambda_j = i\}$$

Deform it with $\{g_n\}$ as

$$\langle p_\lambda, p_\mu \rangle^{(g)} = \delta_{\lambda\mu} z_\lambda \prod_{j=1}^{\ell(\lambda)} g_{\lambda_j}$$

$\leftarrow$ depends on some parameters

(extending $\mathbb{Q} \Rightarrow F$ appropriately s.t. $g_n \in F$)

Kerov "functions" (Unfortunately "Kerov polynomials" is used for different objects)

Kerov (1991)

$\{K_\lambda^{(g)}[p]\}$

(1) $K_\lambda^{(g)}[p] = m_\lambda[p] + \sum_{\lambda > \mu} \mathcal{K}_{\lambda,\mu}^{(g)} m_\mu[p]$

triangular transformation from $\{m_\lambda\}$ ← semi-ordering

(2) $\langle K_\lambda^{(g)} | K_\mu^{(g)} \rangle^{(g)} = 0 \quad (\lambda \neq \mu)$

$$\Longrightarrow K_{\square}^{(g)} = p_1, \quad K_{\substack{\square\\\square}}^{(g)} = \frac{1}{2}p_1^2 - \frac{1}{2}p_2 \qquad \lambda = (1^m) : \text{minimal}$$

$$K_{\square\square}^{(g)} = \frac{1}{g_2 + g_1^2}\left(g_1^2 p_2 + g_2 p_1^2\right), \cdots\cdots \qquad //$$

Example

$$g_n = \prod_{\alpha=1}^{k} \frac{q_\alpha^{n/2} - q_\alpha^{-n/2}}{t_\alpha^{n/2} - t_\alpha^{-n/2}}$$

$k = 0$ Schur polynomials $s_\lambda(x)$

$k = 1$ Macdonald polynomials $P_\lambda(x; q, t)$

$k = 2$ $K_\lambda^{(g)}(x; q_1, q_2, t_1, t_2)$

$$g_n = \frac{q^{n/2} - q^{-n/2}}{t^{n/2} - t^{-n/2}} \longrightarrow P_\lambda(x;q,t) \quad \text{Macdonald polynomials}$$

$$g_1 = \frac{q^{1/2} - q^{-1/2}}{t^{1/2} - t^{-1/2}} \qquad g_2 = g_1 \cdot \frac{q^{1/2} + q^{-1/2}}{t^{1/2} + t^{-1/2}}$$

$$P_{\square}(x;q,t) = p_1, \qquad P_{\substack{\square\\\square}}(x;q,t) = \frac{1}{2}p_1^2 - \frac{1}{2}p_2,$$

$$P_{\square\square}(x;q,t) = \frac{1}{g_2 + g_1^2}\left(g_1^2 p_2 + g_2 p_1^2\right)$$

$$= \frac{(t^{1/2} + t^{-1/2})(t^{1/2} - t^{-1/2})}{2(q^{1/2}t^{1/2} - q^{-1/2}t^{-1/2})}\left(\frac{q^{1/2} - q^{-1/2}}{t^{1/2} - t^{-1/2}} p_2 + \frac{q^{1/2} + q^{-1/2}}{t^{1/2} + t^{-1/2}} p_1^2\right)$$

$$= \frac{1}{2(1 - qt)}\left((1-q)(1+t)\,p_2 + (1+q)(1-t)\,p_1^2\right) //$$

3. Macdonald polynomials and the duality

$$D_N^r = \sum_{\substack{I \subset \{1,\cdots N\} \\ |I| = r}} t^{r(r-1)/2} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} \Gamma_i$$

$$\Gamma_i \, x_j = q^{\delta_{ij}} x_j \Gamma_i \qquad (q\text{-difference operator})$$

Macdonald - Ruijsenaar difference operators (1987)

$$D_N^1 = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \Gamma_i$$

FACT $\{ D_N^r \}_{r=1}^{N}$ are mutually commuting and

$P_\lambda(x; q, t)$ are simultaneous eigenfunctions of D_N^r.

$$D_N(X;q,t) := \sum_{r=0}^{N} D_N^r X^r$$

$$D_N(X;q,t)\,P_\lambda(x;q,t) = \prod_{i=1}^{N}(1+y_i X)\,P_\lambda(x;q,t)$$

$$\underline{y_i = q^{\lambda_i} t^{N-i} \Rightarrow q^\lambda t^\rho}$$

OR

$$D_N^r(q,t)\,P_\lambda(x;q,t) = \underbrace{P_{(1^r)}(y)}_{\text{eigenvalues}}\,P_\lambda(x;q,t)$$

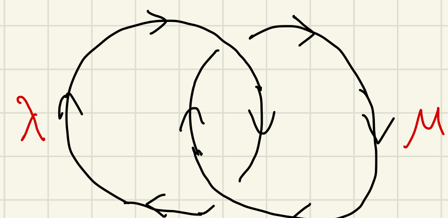
$$\| $$

$e_r(y)$ elementary symmetric poly.

<u>A remarkable identity due to Macdonald</u>

$$\frac{P_\lambda(q^\mu t^\rho)}{P_\lambda(t^\rho)} = \frac{P_\mu(q^\lambda t^\rho)}{P_\mu(t^\rho)} \longleftrightarrow$$

Normalized homological inv. of the Hopf link



Up to the normalization (or the gauge trf.) this is symmetric under $\lambda \leftrightarrow \mu$

All of these (and some other evidences, s. a. the Pieri formula) suggests an existence of "the mother function"

$$M(x, y; q, t) = M(y, x; q, t)$$

such that $\quad P_\lambda(x; q, t) = M(x, \underset{\sim\sim\sim}{q^\lambda t^\rho}; q, t)$

$$\Rightarrow P_\lambda(q^\mu t^\rho; q, t) = M(q^\mu t^\rho, q^\lambda t^\rho; q, t)$$

An answer was given by Noumi - Shiraishi (2012)

$$M_N(x, y; q, t) = \sum_{\{m_{ij}\}} C_N(m_{ij}, y; q, t) \prod_{1 \le i < j \le N} \left(\frac{x_j}{x_i}\right)^{m_{ij}}$$

summation over the positive cone of the roots of A_{N-1}

$$\begin{cases} m_{ij} = 0 & (i \ge j) \\ m_{ij} \in \mathbb{Z}_{\ge 0} \end{cases}$$

as a (formal) infinite Laurent series in $x_1, \cdots, x_N$

$$C_N(m_{ij}, y; q, t) = \prod_{k=2}^{N} \prod_{1 \le i < j \le k} \frac{(q^{m_{ij}^{(k)}} t\, y_j / y_i ; q)_{m_{ik}}}{(q^{m_{ij}^{(k)}} q\, y_j / y_i ; q)_{m_{ik}}}$$

$$\prod_{k=2}^{N} \prod_{1 \le i \le j < k} \frac{(q^{-m_{jk} + m_{ij}^{(k)}} q y_j / t y_i ; q)_{m_{ik}}}{(q^{-m_{jk} + m_{ij}^{(k)}} y_j / y ; q)_{m_{ik}}}$$

$$m_{ij}^{(k)} = \sum_{k<a} (m_{ia} - m_{ja}) \qquad (u;q)_n = \prod_{i=0}^{n-1} (1 - u q^k)$$

Thm (Noumi - Shiraishi)

$$P_\lambda(x; q, t) = x^\lambda M_N(x, q^\lambda t^\rho ; q, t)$$

$$\Phi_N(x, y; q, t) := \prod_{1 \le i < j \le N} \frac{(q y_j / y_i ; q)_\infty}{(q y_j / t y_i ; q)_\infty} \cdot M_N(x, y; q, t)$$

$$\Longrightarrow \quad \Phi_N(x, y; q, t) = \Phi_N(y, x; q, t)$$

$$P_\lambda(x;q,t) = x^\lambda M_N(x, q^\lambda t^\rho; q, t)$$ LII

make P_λ symmetric

The infinite series becomes a finite Laurent polynomial

$$\Phi_N(x,y;q,t) := \prod_{1\le i<j\le N} \frac{(q y_j/y_i; q)_\infty}{(q y_j/t y_i; q)_\infty} \cdot M_N(x,y;q,t)$$

agrees with a vertex function (A. Okounkov) or a vortex counting function mirror sym. of 3D SUSY theory

Example $N = 2$ $\quad m_{12} \Rightarrow m$

$$C_2(m, y; q, t) = \frac{\left(t\frac{y_2}{y_1}; q\right)_m \left(q^{-m}\frac{q}{t}; q\right)_m}{\left(q\frac{y_2}{y_1}; q\right)_m (q^{-m}; q)_m} = \frac{\left(t\frac{y_2}{y_1}; q\right)_m (t;q)_m}{\left(q\frac{y_2}{y_1}; q\right)_m (q;q)_m}\left(\frac{q}{t}\right)^m$$

$\rightsquigarrow$ q-hypergeometric series

$$C_2(m, y; q, t) = \frac{(t\frac{y_2}{y_1}; q)_m (t; q)_m}{(q\frac{y_2}{y_1}; q)_m (q; q)_m} \left(\frac{q}{t}\right)^m$$

For $\lambda = (\lambda_1 \geq \lambda_2)$ $\quad y_2/y_1 = t^{-1} q^{\lambda_2 - \lambda_1 \leq 0}$

$$(t\frac{y_2}{y_1}; q)_m = \prod_{i=0}^{m-1} (1 - q^{\lambda_2 - \lambda_1 + i + 1}) = 0 \quad (m \geq \lambda_1 - \lambda_2)$$

$$P_\lambda(x_1, x_2; q, t) = x_1^{\lambda_1} x_2^{\lambda_2} \sum_{m=0}^{\lambda_1 - \lambda_2} \frac{(q^{\lambda_2 - \lambda_1}; q)_m (t; q)_m}{(\frac{q}{t} q^{\lambda_2 - \lambda_1}; q)_m (q; q)_m} \left(\frac{q}{t}\frac{x_2}{x_1}\right)^m$$

$$P_{\square} = x_1 \left(1 + \frac{(1 - q^{-1})(1 - t)}{(1 - t^{-1})(1 - q)} \left(\frac{q}{t}\frac{x_2}{x_1}\right)\right) = x_1 + x_2 \quad //$$

$$P_{\square\square} = x_1^2 \left[1 + \frac{(1 - q^{-2})(1 - t)}{(1 - q^{-1}t^{-1})(1 - q)} \left(\frac{q}{t}\frac{x_2}{x_1}\right) + \frac{(1 - q^{-2})(1 - q^{-1})(1 - t)(1 - tq)}{(1 - q^{-1}t^{-1})(1 - t^{-1})(1 - q)(1 - q^2)} \left(\frac{q}{t}\frac{x_2}{x_1}\right)^2\right]$$

$$= x_1^2 + \frac{(1 + q)(1 - t)}{1 - qt} x_1 x_2 + x_2^2 \quad //$$

4 "Non-Kerov" deformation

The coefficients of the mother function $M_N(x, y; q, t)$ are products of

$$(u;q)_n = \frac{(u;q)_\infty}{(q^n u;q)_\infty}.$$

$$(u;q)_\infty = \prod_{k=0}^{\infty}(1-uq^k) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{u^n}{1-q^n}\right)$$

$\exists$ multi-parameter generalization

$$(u;q_1,q_2,\cdots q_r)_\infty = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{u^n}{(1-q_1^n)(1-q_2^n)\cdots(1-q_r^n)}\right)$$

$$\Gamma(u;q,p) = \frac{(pq/u;q,p)_\infty}{(u;q,p)_\infty}$$ elliptic gamma function

$$\lim_{p\to 0}\Gamma(u;q,p) = (u;q)_\infty^{-1}$$

//

Our proposal : replace

$$(u;q)_n \longrightarrow \textcircled{H}(u;q,p)_n = \frac{\Gamma(q^n u;q,p)}{\Gamma(u;q,p)}$$

$$\textcircled{H}(u;q,p)_n = \prod_{k=0}^{n-1} \vartheta_p(q^k u)$$

← Odd theta function with the elliptic modulus "p"

$$\boxed{\hat{P}_\lambda(x;q,t,p) = m_\lambda + \sum_{\lambda>\mu} C_{\lambda\mu}(q,t,p)\, m_\mu}$$

$C_{\lambda\mu}(q,t,p)$ are expressed in terms of

$$\zeta_k(z) := \frac{\vartheta_p(q^k z)\,\vartheta_p(tz)}{\vartheta_p(q^{k-1}tz)\,\vartheta_p(qz)} = \prod_{i=1}^{k-1} \eta(q^i z)$$

$$\eta(z) := \frac{\vartheta_p(qz)\,\vartheta_p(t/q\, z)}{\vartheta_p(tz)\,\vartheta_p(z)}$$

Example

$|\lambda| = 2 \qquad \ydiagram{2} > \ydiagram{1,1}$ $\qquad\qquad$ $|\lambda| = 3 \qquad \ydiagram{3} > \ydiagram{2,1} > \ydiagram{1,1,1}$

$$C = \begin{pmatrix} 1 & \xi_2(1) \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & \xi_2(1) + \xi_2(t) & \xi_2(1)\,\xi_3(1) \\ 0 & 1 & \xi_3(1) \\ 0 & 0 & 1 \end{pmatrix}.$$

An important issue is if $\hat{P}_\lambda$ is really symmetric.

For example, $\hat{P}_{\ydiagram{2,1}} \sim F_0 \left(x_1 x_2 x_3^2 + (x_1 x_2 + x_2 x_3 + x_3 x_1) x_4^2 \right)$

$$F_0 = \xi_2(1)^2 - \xi_2(1)\,\xi_2(qt) - \xi_2(1)\,\xi_2(q) + \xi_2(t)\,\xi_2(qt)$$

$F_0 = 0$ by some non-trivial identity of $\vartheta_p(z)$!

Why we call it "non-Kerov"?

The ring structure is qualitatively different.

$$\hat{P}_\lambda(x;q,t,\cdots)\,\hat{P}_\mu(x;q,t,\cdots) = \sum_{|\nu|=|\lambda|+|\mu|} C_{\lambda\mu}^{\nu}(q,t,\cdots)\,\hat{P}_\nu(x;q,t\cdots)$$

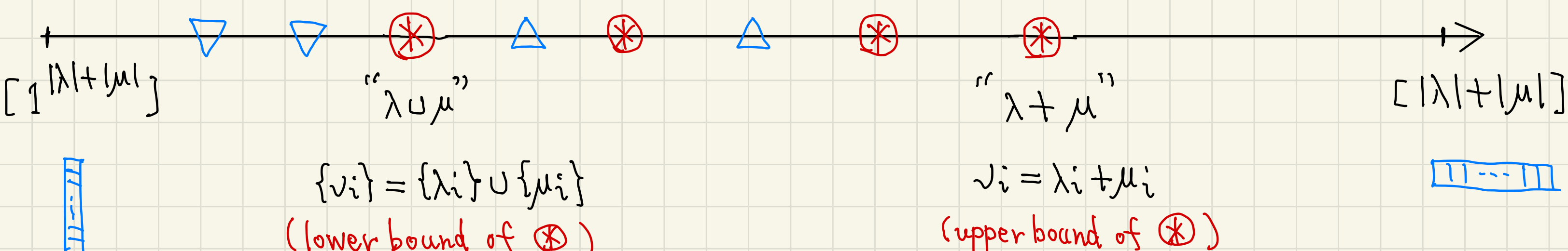
cf : $R(\lambda) \otimes R(\mu) = R_1 \oplus R_2 \oplus \cdots\cdots$ ⊛

(irreducible decomp. of gl_N rep.)

$$\deg \hat{P}_\lambda = |\lambda| = \sum_{i=1}^{N} \lambda_i$$

△ : Kerov

▽ : non-Kerov



Open problems

1. Prove that $\hat{P}_\lambda(x;q,t,\underline{p})$ is a symmetric polynomial. for any λ. (We have checked it for $|\lambda| \leq 5$)

   Conceptual understanding of theta fn. identities ?

2. Can the duality of $P_\lambda(x;q,t)$ survive ?

   It seems at least one more parameter "w" is required.

   $p \longleftrightarrow w$ under the duality.

   Affinization of x_i $\{x_1, \cdots x_N\} \longrightarrow \{x_i\}_{i\in\mathbb{Z}}$

$$x_{i+N} = w \cdot x_i$$