Sustech-Nagoya workshop on Quantum Science

Generalized Eilenberg-Watts calculus and 1d condensation theory

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Introduction

In this talk I will develop a kind of Eilenberg-Watts calculus for module categories over finite tensor categories. To do this I will introduce the notion of module ends(coends). Then I will apply these mathematical notions to the condensation theory of 1-dimensional topological order. In particular, I will explain the geometric construction of the 1d condensable algebra.

Preliminaries

- We fix $\mathbf{k}$ to be an algebraically closed field of characteristic 0. We use Vec to denote the category of finite dimensional vector spaces over $\mathbf{k}$ and $\mathbf{k}$-linear maps.
- All categories we use are finite categories over $\mathbf{k}$, i.e. categories equivalent to the category of finite dimensional modules over some finite dimensional $\mathbf{k}$-algebra.
- By a finite tensor category $\mathcal{C}$, we mean a finite monoidal rigid category whose tensor unit is simple. For an algebra $A \in \mathcal{C}$, we use $\mathrm{RMod}_A(\mathcal{C})$ (resp. $\mathrm{LMod}_A(\mathcal{C})$) to denote the category of right (reps. left) A-modules in $\mathcal{C}$. For two algebras $A, B \in \mathcal{C}$, we use $\mathrm{BMod}_{A|B}(\mathcal{C})$ to denote the category of A-B-bimodules. For simplicity, when $\mathcal{C} = \mathrm{Vec}$, we simply use the notation RMod_A.
- If $\mathcal{C}$ is a finite tensor category and $\mathcal{M}$, $\mathcal{N}$ are finite left $\mathcal{C}$-modules. We use $\mathrm{Fun}^R_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ (resp. $\mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$) to denote the category of right exact (resp. left exact) $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$.

Classical Morita Theory

Classical Morita theory studies the following problem: given two finite dimensional $\mathbf{k}$-algebras A and B, when do we have an equivalence (of $\mathbf{k}$-linear categories): $\mathrm{RMod}_A \simeq \mathrm{RMod}_B$?

This problem is now well-understood by the famous Eilenberg-Watts theorem:

Theorem (Eilenberg-Watts)

For two finite-dimensional $\mathbf{k}$-algebras A, B, There is a canonical equivalence of categories

$$\mathrm{BMod}_{A|B} \simeq \mathrm{Fun}^R_{\mathbf{k}}(\mathrm{RMod}_A, \mathrm{RMod}_B)$$

In particular, equivalences between RMod_A and RMod_B are classified by invertible A-B-bimodules.

Moreover, the equivalence in the Eilenberg-Watts theorem can be constructed explicitly as follows

$$\Phi : \mathrm{BMod}_{A|B}(\mathrm{Vec}) \rightleftarrows \mathrm{Fun}^R_{\mathbf{k}}(\mathrm{RMod}_A, \mathrm{RMod}_B) : \Psi$$
$$_AM_B \mapsto - \otimes_A M$$
$$F(A) \leftarrow\!\shortmid F$$

Here $F(A)$ is canonically equipped with a A-B-bimodule structure, so it lies in the category $\mathrm{BMod}_{A|B}$. It is direct to check that Φ and Ψ are inverse to each other.

Remark

We also have a dual version of Eilenberg-Watts theorem, which establishes an equivalence as follows

$$\mathrm{BMod}_{A|B} \simeq \mathrm{Fun}^L(\mathrm{RMod}_A, \mathrm{RMod}_B).$$

Coordinate-free version

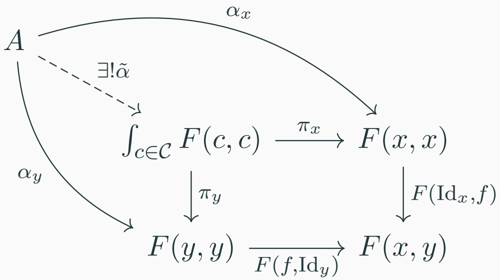
Now we want to develop a "coordinate-free" version of Eilenberg-Watts theorem. To be explicit, for two finite categories $\mathcal{M}$, $\mathcal{N}$ we want to find the following equivalence:

$$\mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N} \simeq \mathrm{Fun}_{\mathbf{k}}^{R}(\mathcal{M}, \mathcal{N})$$

We can find algebras A and B such that $\mathrm{RMod}_A \simeq \mathcal{M}$, $\mathrm{RMod}_B \simeq \mathcal{N}$. Under the "coordinates" A and B we can identify $\mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N}$ with $\mathrm{BMod}_{A|B}$, and by Eilenberg-Watts theorem we get the desired equivalence. However, the choices of A and B are not canonical, and neither is the identification $\mathcal{M}^{\mathrm{op}} \boxtimes \mathcal{N} \simeq \mathrm{BMod}_{A|B}$. What we really want is a "coordinate-free" version of Eilenberg-Watts theorem.

Review of (co)ends

Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. Let $\Delta : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D})$ be the diagonal functor which sends each object in d to the corresponding constant functor. The end of F is a pair $(\int_{c\in\mathcal{C}} F(c,c), \pi)$ where $\int_{c\in\mathcal{C}} F(c,c) \in \mathcal{D}$, $\pi : \Delta_{\int_{c\in\mathcal{C}} F(c,c)} \overset{\bullet\bullet}{\to} F$ is a dinatural transformation such that they are universal in the following sense: for each pair $(A, \alpha) \in (\Delta, \downarrow, F)$, there is a unique morphism $\tilde{\alpha} : A \to \int_{c\in\mathcal{C}} F(c,c)$ rendering the whole diagram commutative:



The notion of a coend is defined dually.

Example

Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Then we have

$$\mathrm{Nat}(F, G) = \int_{c \in \mathcal{C}} \hom_{\mathcal{D}}(F(c), G(c)).$$

Example (Yoneda and co-Yoneda Lemma)

For every functor $K : \mathcal{C} \to \mathcal{D}$, we have the following isomorphisms:

$$K(-) \cong \int^{c \in \mathcal{C}} \hom_{\mathcal{C}}(c, -) \bullet Kc; \quad K(-) \cong \int_{c \in \mathcal{C}} \hom_{\mathcal{C}}(-, c)^* \bullet Kc.$$

Here $\bullet$ means the canonical action of Vec over a $\mathbf{k}$-linear category.

Using the language of ends/coends, we can write down the coordinate-free version of Eilenberg-Watts theorem as follows:

Theorem

Let $\mathcal{C}$, $\mathcal{D}$ be finite categories. There are two pairs of adjoint equivalences of categories:

$$\Phi^l : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{D} \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}), \quad a \boxtimes b \mapsto \hom_{\mathcal{C}}(a, -) \bullet b$$
$$\Psi^l : \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \to \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{D}, \quad F \mapsto \int^{c \in \mathcal{C}} c \boxtimes F(c).$$

and

$$\Phi^r : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{D} \to \mathrm{Fun}^R(\mathcal{C}, \mathcal{D}), \quad a \boxtimes b \mapsto \hom_{\mathcal{C}}(-, a)^* \bullet b$$
$$\Psi^r : \mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \to \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{D}, \quad F \mapsto \int_{c \in \mathcal{C}} c \boxtimes F(c).$$

Generalization towards the $\mathcal{C}$-module case

For now we have been discussing finite categories, i.e. module categories over Vec. From now on we turn to study the Morita theory of finite module categories over some finite tensor category $\mathcal{C}$.

If $\mathcal{M}$ is a $\mathcal{C}$-module, then we may equip $\mathcal{M}^{\mathrm{op}}$ with different right $\mathcal{C}$-module structures using the rigidity of $\mathcal{C}$:

- We use $(\mathcal{M}^{\mathrm{op}|L}, \odot^L)$ to denote the right $\mathcal{C}$-module equipped the following action:

$$m \odot^L c := c^L \odot m$$

- We use $(\mathcal{M}^{\mathrm{op}|R}, \odot^R)$ to denote the right $\mathcal{C}$-module equipped the following action:

$$m \odot^R c := c^R \odot m$$

Theorem (Kong-Zheng:1507.00503)

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}, \mathcal{N}$ be finite left $\mathcal{C}$-modules, then there are equivalences of categories:

$$\Phi_r : \mathcal{M}^{\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \mathrm{Fun}^R_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \quad x \boxtimes_{\mathcal{C}} y \mapsto [-, x]^R \odot y$$

$$\Phi_l : \mathcal{M}^{\mathrm{op}|R} \boxtimes_{\mathcal{C}} \mathcal{N} \simeq \mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \quad x \boxtimes_{\mathcal{C}} y \mapsto [x, -] \odot y$$

We may understand this theorem by taking "coordinates". As a matter of fact, we can always find algebras $A, B \in \mathcal{C}$ such that $\mathcal{M} \simeq \mathrm{RMod}_A(\mathcal{C})$, $\mathcal{N} \simeq \mathrm{RMod}_B(\mathcal{C})$ as left $\mathcal{C}$-modules. Under these coordinates the equivalence Φ_r reads

$$\mathrm{BMod}_{A|B}(\mathcal{C}) \simeq \mathrm{Fun}^R_{\mathcal{C}}(\mathrm{RMod}_A(\mathcal{C}), \mathrm{RMod}_B(\mathcal{C})), \quad M \mapsto - \otimes_A M$$

which is completely parallel to the $\mathbf{k}$-linear theory.

The problem is that, we cannot write down the quasi-inverse of Φ_l and Φ_r in the coordinate-free form. To solve this problem, we need to introduce new constructions.

Definition

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a left $\mathcal{C}$-module. We equip $\mathcal{M}^{\mathrm{op}}$ with the right $\mathcal{C}$-module structure $\mathcal{M}^{\mathrm{op}|L}$. Let $(F, e) : \mathcal{M}^{\mathrm{op}|L} \times \mathcal{M} \to \mathcal{D}$ be a balanced $\mathcal{C}$-module functor with e being the balancing natural isomorphism. The **$\mathcal{C}$-module end of** F is a pair $(\int_{x\in\mathcal{M}}^{\mathcal{C}} F(x,x), \pi)$ with $\int_{x\in\mathcal{M}}^{\mathcal{C}} F(x,x) \in \mathcal{D}$ and $\pi : \Delta_{\int_{x\in\mathcal{M}}^{\mathcal{C}} F(x,x)} \overset{\bullet\bullet}{\to} F$ such that the following diagram is commutative:

$$\begin{array}{ccc} \int_{x\in\mathcal{M}}^{\mathcal{C}} F(x,x) & \xrightarrow{\pi_m} & F(m,m) \\ {\scriptstyle \pi_{c^L\odot m}}\downarrow & & \downarrow \\ F(c^L\odot m, c^L\odot m) & \xrightarrow{e_{m,c,c^L\odot m}} & F(m, c\odot c^L\odot m) \end{array}$$

and is terminal among all such pairs.

The notion of a $\mathcal{C}$-module coend is defined dually, which we denote by $\int_{\mathcal{C}}^{x\in\mathcal{M}} F(x,x)$. The two constructions should be regarded as some kind of modified end/coend which is compatible with the rigidity structure of $\mathcal{C}$. Their power shall be illustrated by the following examples:

Example

Let $\mathcal{M}, \mathcal{N}$ be left $\mathcal{C}$-modules. Let $(F, \eta^F), (G, \eta^G) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ be $\mathcal{C}$-module functors. Then

$$\int_{m\in\mathcal{M}}^{\mathcal{C}} \hom_{\mathcal{N}}(F(m), G(m)) \cong \mathrm{Nat}_{\mathcal{C}}(F, G).$$

The RHS means the space of $\mathcal{C}$-module natural transformations. Note that the functor $\hom_{\mathcal{N}}(F(-), G(-)) : \mathcal{M}^{\mathrm{op}|L} \times \mathcal{M} \to \mathrm{Vec}$ is equipped with a canonical balancing structure.

Example

- Let $(G, \eta^G) \in \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, then there are canonical isomorphisms of $\mathcal{C}$-module functors:

$$G(-) \cong \int_{\mathcal{C}}^{x\in\mathcal{M}} [x, -] \odot G(x), \quad G(-) \cong \int_{x\in\mathcal{M}}^{\mathcal{C}} [-, x]^R \odot G(x)$$

- Two finite tensor categories $\mathcal{C}$ and $\mathcal{D}$ are **Morita equivalent** if they can be connected by an invertible bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$, or equivalently, if there is an bimodule ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$ such that the monoidal functor $u : \mathcal{D}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{C}}^R(\mathcal{M}, \mathcal{M})$ is an equivalence. Now we may use the language of $\mathcal{C}$-module ends to write down the quasi-inverse of u. As a matter of fact we compute its right adjoint u^R, which reads:

$$u^R : \mathrm{Fun}_{\mathcal{C}}^R(\mathcal{M}, \mathcal{M}) \to \mathcal{D}^{rev}, \quad F \mapsto \int_{m\in\mathcal{M}}^{\mathcal{C}} [m, F(m)]_{\mathcal{D}^{rev}}$$

Using the notion of $\mathcal{C}$-module ends and coends, we are able to write down the generalized form of Eilenberg-Watts calculus.

Theorem (Generalized Eilenberg-Watts calculus)

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}, \mathcal{N}$ be finite left $\mathcal{C}$-modules. Then there are adjoint pairs of equivalences:

$$\Phi^l : \mathcal{M}^{\mathrm{op}|R} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \quad x \boxtimes_{\mathcal{C}} y \mapsto [x, -] \odot y$$
$$\Psi^l : \mathrm{Fun}^L_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathcal{M}^{\mathrm{op}|R} \boxtimes_{\mathcal{C}} \mathcal{N}, \quad F \mapsto \int_{\mathcal{C}}^{m \in \mathcal{M}} m \boxtimes_{\mathcal{C}} F(m).$$

and

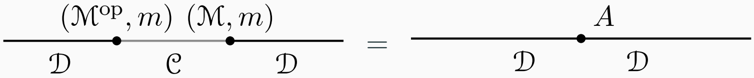
$$\Phi^r : \mathcal{M}^{\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{N} \to \mathrm{Fun}^R_{\mathcal{C}}(\mathcal{M}, \mathcal{N}), \quad x \boxtimes_{\mathcal{C}} y \mapsto [-, x]^R \odot y$$
$$\Psi^r : \mathrm{Fun}^R_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \to \mathcal{M}^{\mathrm{op}|L} \boxtimes_{\mathcal{C}} \mathcal{N}, \quad F \mapsto \int^{\mathcal{C}}_{m \in \mathcal{M}} m \boxtimes_{\mathcal{C}} F(m).$$

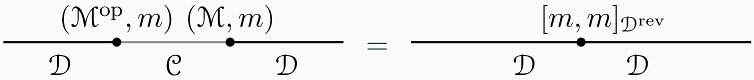
Physical application: 1d condensation theory

In physics, a 1d gapped topological order is described by a unitary fusion category, which is a special case of a finite tensor category.

We will explain the geometric construction of 1d condensable algebras. Let $\mathcal{C}$, $\mathcal{D}$ be Morita equivalent 1d topological orders separated by a 0d domain wall $(\mathcal{M}, m)$. Then $\mathcal{C}$ and $\mathcal{D}$ can condense to each other. The condensation process from $\mathcal{D}$ to $\mathcal{C}$ is controlled by a distinguished object in $\mathcal{D}$, called the **condensable algebra**. [Kong:1307.8244]

Intuitively, the condensable algebra which condenses $\mathcal{D}$ to $\mathcal{C}$ can be obtained by conducting a dimensional reduction process to a "$\mathcal{C}$-bubble" in $\mathcal{D}$.





We can now show that the condensable algebra is given by the internal $[m, m]_{\mathcal{D}^{\mathrm{rev}}}$

$$\mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{D}$$

$$\begin{aligned} m \boxtimes_{\mathcal{C}} m \mapsto [-, m]^R \odot m \mapsto & \int_{x \in \mathcal{M}}^{\mathcal{C}} [x, [x, m]^R \odot m]_{\mathcal{D}^{\mathrm{rev}}} \\ \cong & \int_{x \in \mathcal{M}}^{\mathcal{C}} [[x, m] \odot x, m]_{\mathcal{D}^{\mathrm{rev}}} \\ \cong & \, [\int_{\mathcal{C}}^{x \in \mathcal{M}} [x, m] \odot x, m]_{\mathcal{D}^{\mathrm{rev}}} \\ \cong & \, [m, m]_{\mathcal{D}^{\mathrm{rev}}} \end{aligned}$$

Thank You!