Gapless Edges of 2+1D Topological Orders

Holiverse Yang

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SUSTech

Reference

Liang Kong and Hao Zheng:

A mathematical theory of gapless edges of 2+1D topological orders I

A mathematical theory of gapless edges of 2+1D topological orders II

Wei-Qiang Chen, Chao-Ming Jian, Liang Kong, Yi-Zhuang You and Hao Zheng:

A topological phase transition on the edge of the 2d $\mathbb{Z}_2$ topological order

Yalei Lu, Holiverse Yang, Liang Kong and Wei-Qiang Chen:

A self-dual gapless edge of the 2d $\mathbb{Z}_N$ topological order (in preparation)

Many years after Wen introduced the notion of topological order (TO)[Wen:89], it has been well-known that [Fredenhagen-Rehren-Schroer:89, Moore-Seiberg:89, Frohlich-Gabbiani:90, Kitaev:05]

A 2+1D (anomaly-free) TO can be described mathematically by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is a **unitary modular tensor category** (UMTC) and c is the chiral central charge such that $c^{top_{\mathcal{C}}} = c \pmod 8$.

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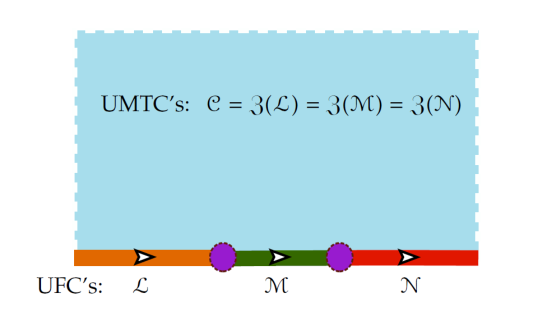
We would not introduce the complete mathematical definition of UMTC here. However, we will see the categorical structure of UMTC appears naturally in physics. For more details, one can read a note of topological orders written by Zhang and Kong.[Kong, Zhang: arXiv:2205.05565],



A 2+1D TO $(\mathcal{C}, 0)$ admitting a gapped edge is called a non-chiral 2+1D TO. A non-chiral bulk might have several different gapped boundaries. A 1+1D gapped boundary is described by a unitary fusion category (UFC) $\mathcal{L}$. A 2+1D TO has a gapped edge, is equivalent to say that the bulk UMTC is the Drinfeld center of a UFC, then the central charge c of bulk UMTC must be zero.

If the gapped edge of a non-chiral 2+1D TO exists, then the following relation between the boundary and bulk was known.

Theorem (Kitaev, Kong: arXiv:1104.5047)

The 2+1D bulk phase is uniquely determined by the anomalous 1+1D TO on its boundary, and the UMTC $\mathcal{C}$ is given by the Drinfeld center of the UFC $\mathcal{L}$, i.e. $\mathcal{C} \simeq \mathfrak{Z}(\mathcal{L})$

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Remark

There are counterexamples that a UMTC with 0 central charge cannot be a center of a UFC, which means a bulk TO with 0 central charge might not have a gapped boundary. In physics, we might consider a Chern-Simons TQFT to construct such a counterexample.

Above all, we have a complete mathematical description of gapped edges of a 2+1D TO. The natural question is whether there is a similar theory for gapless edges. In more details, we would ask:

Question

1. What is the mathematical description of a gapless edge of $(\mathcal{C}, c)$?
2. Does the boundary-bulk relation still hold for gapless edges?
3. What is the mathematical description of a 0d gapless domain wall between two gapless 1d edges?

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In [Kong, Wen, Zheng: arXiv:1702.00673], they proved that a bulk TO in any dimensions should be given by the center of its boundary, regardless whether the boundary is gapped or gapless, and whether the mathematical description of a gapless boundary is. It suggests us that the gapped and gapless edge of 2+1D TOs should be unified in a mathematical language.

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For question 2, the answer is yes. One can read more details in [Kong, Zheng: arXiv:1912.01760]and [Kong, Yuan, Zhang, Zheng: arXiv:2104.03121]. In the following talk of Zhang, he will introduce the second work.

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If we consider a gapless edge at a fixed point in the sense of the long-wave length limit, the edge modes will be scale invariant and therefore also conformal invariant. So a gapless edge should be described by the data of a conformal field theory (CFT) at that fixed point.

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We will only consider the (**unitary**) **rational** CFT here, which is described by a (**unitary**) **rational vertex operator algebra** (VOA). Roughly speaking, a VOA consists of field operators as vector space and their operator product expansion (OPE) as multiplication, which both respect the local conformal invariance.

In the following talk given by Kong, there will be more details of VOA and CFT.

A 2+1D (not necessarily chiral) TO can have a chiral gapless edge, the chiral edge modes are states in a chiral CFT with the central charge c. For example, the invertible TO has so called E_8 edge states, the corresponding chiral CFT is the E_8 level 1 WZW model which has only one trivial primary field.

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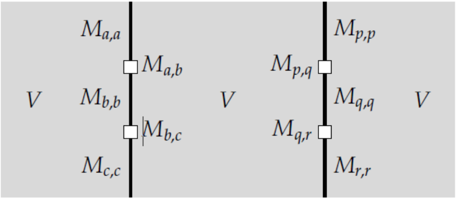
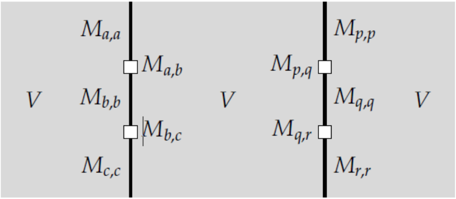
An interesting example of a non-chiral gapless edge is the gapless edge of the 2d toric code. We will give a detailed description of this edge after we introduce the completely mathematical description of chiral and non-chiral gapless edges.



A chiral gapless edge is given by the following data

- The local quantum symmetry V which is a unitary rational VOA living in the 1+1D world sheet.
- The category ${}^{\mathcal{B}}\mathcal{X}$ of 0+1D topological defect lines consisting of
  - objects are labels $a, b, c \ldots$ for each segment of line, which are also objects in $\mathcal{X}$. We will call $\mathcal{X}$ the **underlying category**.
  - a hom space is a 0D defect $M_{x,y} \in \mathcal{B} := \mathrm{Mod}_V$ which is the category of V-modules. We will call $\mathcal{B}$ the **background category**.



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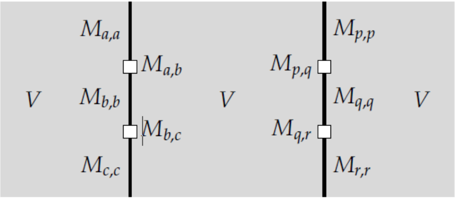
- The local quantum symmetry V which is a unitary rational VOA living in the 1+1D world sheet.
- The category ${}^{\mathcal{B}}\mathcal{X}$ of 0+1D topological defect lines consisting of
    - the identity $\iota_x : V \to M_{x,x}$ is induced by the V-invariant condition, which means that V can transparently move on the 1+1D world sheet except 0D defect.
    - the composition $M_{y,z} \otimes_V M_{x,y} \to M_{x,z}$ is a morphism in Mod_V induced by an intertwining operator, which represents the vertical fusion of 0D defects.



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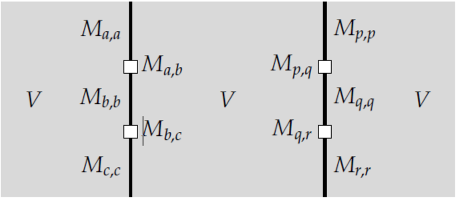
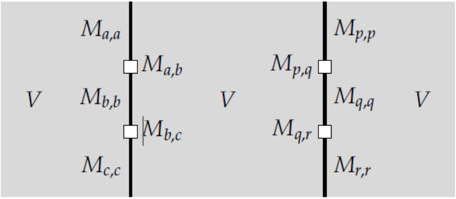
- The local quantum symmetry V which is a unitary rational VOA living in the 1+1D world sheet.
- The category ${}^{\mathcal{B}}\mathcal{X}$ of 0+1D topological defect lines consisting of
  - a monoidal structure which represents the horizontal fusion of topological defect lines.
  - a unitary structure.



As a consequence, we conclude that $\mathcal{X}^\sharp$ is a Mod_V-enriched unitary fusion category. We will denote a chiral gapless edge by a pair $(V, ^{\mathcal{B}}\mathcal{X})$. For more details of enriched unitary fusion categories and boostrap analysis, one can read [Kong, Zheng: arXiv:1905.04924].



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We set the background category of a gapped domain wall M by the category **H** of finite dimensional Hilbert spaces, thus we can rewrite this domain wall as $(\mathbb{C}, {}^{\mathbf{H}}M)$. Therefore gapped and chiral gapless domain walls can be unified in the enriched category language.

To construct a chiral gapless edge, note that for a UMTC $\mathcal{B}$, ${}^{\mathcal{B}}\mathcal{B}$ is a $\mathcal{B}$-enriched UFC. Hence if $\mathcal{B}$ can be written as $B = \mathrm{Mod}_V$ for some unitary rational VOA V, we will get a chiral gapless edge $(V, {}^{\mathcal{B}}\mathcal{B})$ of the 2d TO $(\mathcal{B}, c)$.

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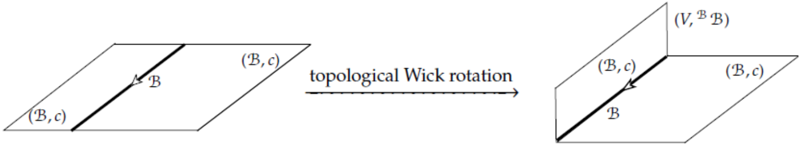
We have a so-called **topological Wick rotation** to geometrically describe this gapless edge.



One can imagine that we first choose a trivial domain wall $\mathcal{B}$ in the 2d TO $(\mathcal{B}, c)$ and then flip one side from the spatial dimension to the temporal dimension. The chiral gapless edges constructed like this is called the **canonical construction**.

Using the topological Wick rotation, we find a gapless edge is determined by the domain wall between the background category $\mathcal{B}$ and the 2d bulk $\mathcal{C}$, it is actually a $\mathcal{B}$-$\mathcal{C}$-bimodule.

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Theorem (Kong, Zheng: arXiv:1905.04924)

Gapped and chiral gapless edges of a 2d topological order $(\mathcal{C}, c)$ are precisely described and classified by pairs $(V, {}^{\mathcal{B}}\mathcal{X})$, where

- V is a unitary rational VOA of central charge c such that $B := \mathrm{Mod}_V$ is a UMTC;
- $\mathcal{X}$ is a closed fusion $\mathcal{B}$-$\mathcal{C}$-bimodules;
- ${}^{\mathcal{B}}\mathcal{X}$ is the $\mathcal{B}$-enriched UFC obtained via the canonical construction

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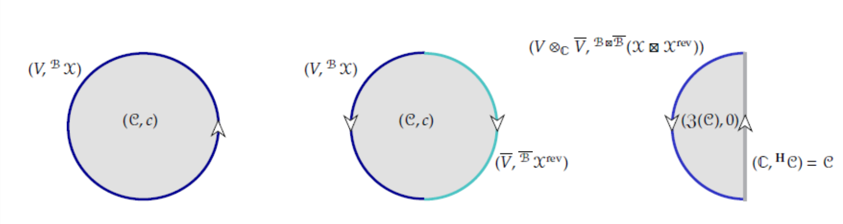
For non-chiral gapless edge, recall that a non-chiral CFT is a tensor product of a chiral CFT and an anti-chiral CFT. let V_L and V_R be unitary rational VOA's with central charge c_L and c_R, respectively, such that Mod_{V_L} and Mod_{V_R} are UMTC's. The following pair

$$(V_L \otimes_{\mathbb{C}} \overline{V}_R, {}^{\mathrm{Mod}_{V_L} \boxtimes \overline{\mathrm{Mod}_{V_R}}}(\mathrm{Mod}_{V_L} \boxtimes \mathrm{Mod}_{V_R}^{rev}))$$

defines a so-called **the canonical non-chiral gapless edge** of $(\mathrm{Mod}_{V_L} \boxtimes \overline{\mathrm{Mod}_{V_R}}, c_L - c_R)$. We will call V_L the **chiral symmetry**, $\overline{V}_R$ the **anti-chiral symmetry** and $V_L \otimes_{\mathbb{C}} \overline{V}_R$ the **non-chiral symmetry**.

Here is an example of non-chiral gapless edge constructed from a chiral gapless edge by folding. The following figures describe this process.



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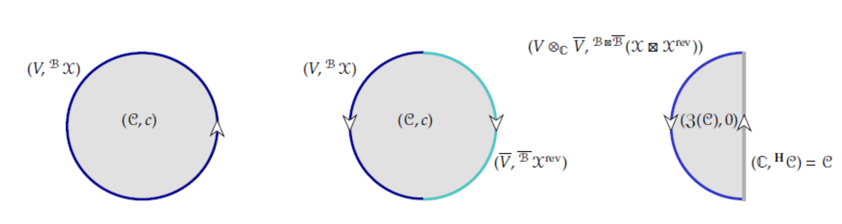

If we reverse the orientation of right side of circle, we will get a anti-chiral gapless edge $(\overline{V}, {}^{\overline{\mathcal{B}}}\mathcal{X}^{rev})$.

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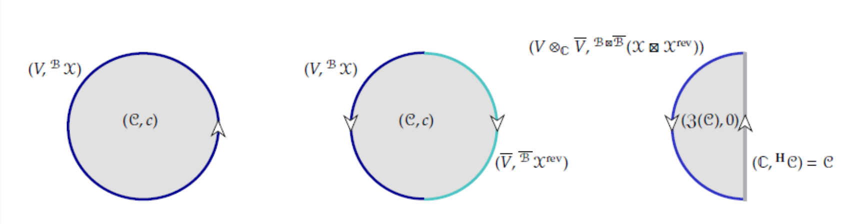

If we reverse the orientation of right side of circle, we will get a anti-chiral gapless edge $(\overline{V}, {}^{\overline{\mathcal{B}}}\mathcal{X}^{rev})$. Then fold the right side to the left side and squeeze together, we will get a non-chiral gapless edge $(V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{B}\boxtimes\overline{\mathcal{B}}}(\mathcal{X} \boxtimes \mathcal{X}^{rev}))$ of the 2d TO $(\mathfrak{Z}(\mathcal{C}), 0)$.

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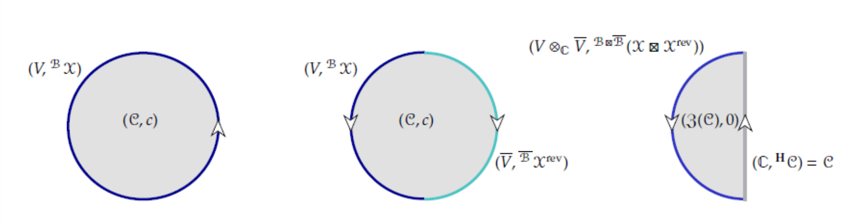

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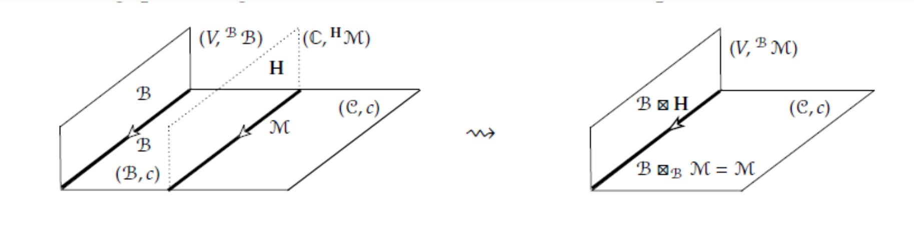
Just like fusing gapped domain walls between 2d TOs, we can also fuse a gapped domain wall to a (not necessarily chiral) gapless edge to get a new one.



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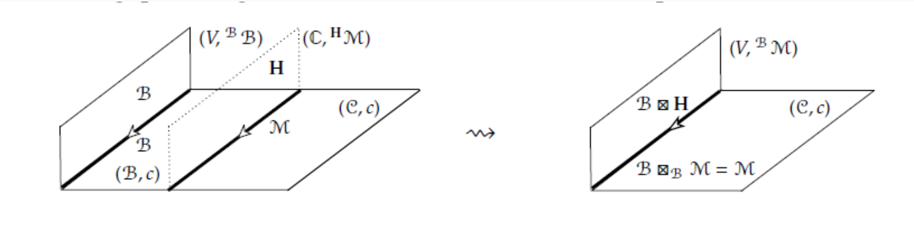

In the case depicted on the figure, after fusion, the background category of new edge is $\mathcal{B} \boxtimes \mathbf{H}$, and the underlying category of new edge is a fusion of the underlying categories of the domain wall and the underlying category of the edge, which is $\mathcal{B} \boxtimes_{\mathcal{B}} \mathcal{M} \simeq \mathcal{M}$.

For a more concrete example, let **Is** denote the Ising category, which is the module category of Ising VOA V_{Is}. We can construct a gappable non-chiral gapless edge for $(\mathfrak{Z}(\mathbf{Is}), 0)$ by the folding construction, which will give $(V_{Is} \otimes_{\mathbb{C}} \overline{V}_{Is}, {}^{\mathfrak{Z}(\mathbf{Is})}\mathfrak{Z}(\mathbf{Is}))$.

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We also know that in the double Ising TO $(\mathfrak{Z}(\mathbf{Is}), 0)$, there is a condensable algebra A such that the condensation of $\mathfrak{Z}(\mathbf{Is})$ by A is the toric code $(\mathfrak{Z}(\mathrm{Rep}(\mathbb{Z}_2)), 0)$ whose domain wall is $\mathfrak{Z}(\mathbf{Is})_A$, the category of right A-modules in $\mathfrak{Z}(\mathbf{Is})$.

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$$(V_{Is} \otimes_{\mathbb{C}} \overline{V}_{Is}, {}^{\mathfrak{Z}(\mathbf{Is})}\mathfrak{Z}(\mathbf{Is})) \boxtimes_{(\mathfrak{Z}(\mathbf{Is}),0)} \mathfrak{Z}(\mathbf{Is})_A = (V_{Is} \otimes_{\mathbb{C}} \overline{V}_{Is}, {}^{\mathfrak{Z}(\mathbf{Is})}\mathfrak{Z}(\mathbf{Is})_A)$$

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This gapless edge describe the critical point of phase tansition of two gapped edges $(\mathbb{C}, {}^{\mathbf{H}}\mathrm{Rep}(\mathbb{Z}_2))$, $(\mathbb{C}, {}^{\mathbf{H}}\mathrm{Vec}_{\mathbb{Z}_2)})$ of toric code.[Chen, Jian, Kong, You, Zheng: arXiv:1903.12334],

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In this case, we consider the so-called parafermion VOA V_{PF_n} and its module category PF_n. When n is odd, we prove that there is a condensable algebra B in double parafermion $\mathfrak{Z}(\mathrm{PF}_n)$ such that the condensation is $(\mathfrak{Z}(\mathrm{Rep}(\mathbb{Z}_n)), 0)$.

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Then after fusing the gapped domain wall between $\mathfrak{Z}(\mathrm{PF}_n)$ and $\mathfrak{Z}(\mathrm{Rep}(\mathbb{Z}_n))$ to the canonical non-chiral gapless edge of $\mathfrak{Z}(\mathrm{PF}_n)$, we will get:

$$(V_{PF_n} \otimes_{\mathbb{C}} \overline{V}_{PF_n}, {}^{\mathfrak{Z}(\mathrm{PF}_n)}\mathfrak{Z}(\mathrm{PF}_n)) \boxtimes_{(\mathfrak{Z}(\mathrm{PF}_n),0)} \mathfrak{Z}(\mathrm{PF}_n)_B = (V_{PF_n} \otimes_{\mathbb{C}} \overline{V}_{PF_n}, {}^{\mathfrak{Z}(\mathrm{PF}_n)}\mathfrak{Z}(\mathrm{PF}_n)_B)$$

which describes the critical point of the phase transition of completely symmetry breaking process of 1d gapped boundary of $\mathbb{Z}_n$ TO. In particular, when $n = 2$, $\mathrm{PF}_2 \simeq \mathbf{Is}$.

In the end of the story, there is also a classification theorem of non-chiral gapless edges that is similar to the chiral case.

Theorem (Kong, Zheng: arXiv:1912.01760)

Non-chiral gapless edges of a 2d topological order $(\mathcal{C}, c)$ are mathematically described and classified by pairs $(W, {}^{\mathcal{B}}\mathcal{X})$, where

- W is a unitary rational full field algebra with central charge (c_L, c_R).
- ${}^{\mathcal{B}}\mathcal{X}$ is the enriched monoidal category defined by the pair $(\mathcal{B}, \mathcal{X})$ via the canonical construction, where $\mathcal{B} := \mathrm{Mod}_W$ and $\mathcal{X}$ is a closed fusion $\mathcal{B}$-$\mathcal{C}$-bimodule.

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Non-chiral gapless edges of a 2d topological order $(\mathcal{C}, c)$ are mathematically described and classified by pairs $(W, {}^{\mathcal{B}}\mathcal{X})$, where

- W is a unitary rational full field algebra with central charge (c_L, c_R).
- ${}^{\mathcal{B}}\mathcal{X}$ is the enriched monoidal category defined by the pair $(\mathcal{B}, \mathcal{X})$ via the canonical construction, where $\mathcal{B} := \mathrm{Mod}_W$ and $\mathcal{X}$ is a closed fusion $\mathcal{B}$-$\mathcal{C}$-bimodule.

Notice that the chiral and non-chiral gapless edges can be unified together. If we set the anti-chiral part of a full field algebra to be $\mathbb{C}$, then we will get a chiral gapless edge.

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Notice that the chiral and non-chiral gapless edges can be unified together. If we set the anti-chiral part of a full field algebra to be $\mathbb{C}$, then we will get a chiral gapless edge. Moreover, recall that gapped and gapless edges can also be unified together in the language of enriched categories, in some sense, we can say that almost everthing is an enriched category.

Thanks for listening!