SUSTech-Nagoya workshop on Quantum Science

Higher Representation Theory of finite group

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Classical representation theory of finite group: group algebra

Fix an algberaically closed field $\Bbbk$ of characteristic 0. The **group algebra** of a finite group G, denoted as $\Bbbk[G]$, has the underlying vector space spanned by elements of G and its multiplication is induced from the group multiplication of G. In detail, $\Bbbk[G]$ has a Hopf algebra structure:

$$\begin{aligned} m : \Bbbk[G] \otimes \Bbbk[G] \to \Bbbk[G]; &\quad g \otimes h \mapsto gh \\ i : \Bbbk \to \Bbbk[G]; &\quad 1 \mapsto e \\ \Delta : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]; &\quad g \mapsto g \otimes g \\ \epsilon : \Bbbk[G] \to \Bbbk; &\quad g \mapsto 1 \\ S : \Bbbk[G] \to \Bbbk[G]; &\quad g \mapsto g^{-1} \end{aligned}$$

A G-representation is a left module over $\Bbbk[G]$, while a homomorphism between G-representations is a homomorphism between left $\Bbbk[G]$-modules. Let's denote the category of *finite dimensional* G-representations as $\mathrm{Rep}(G)$.

Classical representation theory of finite group: finite semisimplicity

Consider the pairing: $\langle \cdot, \cdot \rangle : \mathbb{k}[G] \otimes \mathbb{k}[G] \to \mathbb{k}; g \otimes h \mapsto \delta_e(g^{-1}h)$. It is non-degenerate and G-invariant, i.e.

- $\langle u, v \rangle = 0$ for all $u \in \mathbb{k}[G]$ implies $v = 0$;
- $\langle g.u, g.v \rangle = \langle u, v \rangle$ for any $g \in G$ and $u, v \in \mathbb{k}[G]$.

Thus, any G-submodule M of $\mathbb{k}[G]$ has a complementary G-submodule, $M^{\perp} := \{v \in \mathbb{k}[G] : \langle u, v \rangle = 0, \forall u \in M\}$, hence $\mathbb{k}[G]$ is finite semisimple.

Lemma

$\mathrm{Rep}(G)$ is a finite semisimple linear category.

Moreover, the Hopf algebra structure on $\mathbb{k}[G]$ induces a symmetric monoidal structure on $\mathrm{Rep}(G)$.

Classical representation theory of finite group: monoidal product

- monoidal product of two G-representations (V, ρ_V) and (W, ρ_W) is defined to be the tensor product of their underlying vector spaces, with an induced G-action $\rho_{V\otimes W}$ defined via $\rho_{V\otimes W}(g)(v \otimes w) := \rho_V(g)(v) \otimes \rho_W(g)(w)$ for any $g \in G$, $v \in V$ and $w \in W$;
- monoidal unit is the one-dimensional space $\Bbbk$ with the trivial G-action;
- associator, unitors and half-braiding for this monoidal product are the same as those for tensor product of underlying vector spaces.

The symmetric monoidal structure induces an $\mathbb{Z}_{\geq 0}$-algebra structure on isomorphism classes of simple objects in $\mathrm{Rep}(G)$, which we will refer to as the **representation ring** of $\mathrm{Rep}(G)$.

Classical representation theory of finite group: character

A function $\chi : G \to \mathbb{k}$ is a **class function** if $\chi(hgh^{-1}) = \chi(g)$ for any $g, h \in G$.

Given a finite dimensional G-representation (V, ρ), its **character** is defined to be the function $\mathrm{ch}_V : G \to \mathbb{k}; g \mapsto \mathrm{Tr}(\rho(g))$.

Group characters have the following properties: suppose V and W are two finite dimensional G-representations, then we have

1. $\mathrm{ch}_{V \oplus W} = \mathrm{ch}_V + \mathrm{ch}_W$,
2. $\mathrm{ch}_{V \otimes W} = \mathrm{ch}_V \cdot \mathrm{ch}_W$,
3. $\mathrm{ch}_{V^*} = \mathrm{ch}_V \circ S$, i.e. for any $g \in G$, we have $\mathrm{ch}_{V^*}(g) = \mathrm{ch}_V(g^{-1})$.

As a consequence, the algebra of characters is the representation ring with coefficients extended to field $\mathbb{k}$.

Classical representation theory of finite group: fixed point formula

Suppose (U, ρ) is a G-representation, then we denote the subrepresentation of G-fixed points as $U^G := \{u \in U : \rho(g)(u) = u, \forall g \in G\}$.

[Fixed point formula] Consider linear map $\pi : U \to U; u \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)(u)$. It turns out to be a projection, i.e. for $u \in U$, we have

$$\pi(\pi(u)) = \frac{1}{|G|^2} \sum_{g,h \in G} \rho(gh)(u) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(u) = \pi(u)$$

The image of this projection is U^G. Hence, we obtain

$$\dim U^G = \mathrm{Tr}(\pi) = \frac{1}{|G|} \sum_{g \in G} \mathrm{ch}_U(g)$$

Classical representation theory of finite group: class function

Representations are controlled by class functions

All class functions on G form a subalgebra of $\mathrm{Fun}(G)$, and characters of simple G-representations form a basis.

First, let's show $\{\mathrm{ch}_U : U \text{ simple representation (up to isomorphism)}\}$ is a linearly independent subset in the space of class functions. Define the pairing $\langle \cdot, \cdot \rangle$ on the space of class functions via $\langle \chi, \theta \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\theta(g)$.

For any simple G-representation U and V, we have

$$\langle \mathrm{ch}_U, \mathrm{ch}_V \rangle = \frac{1}{|G|} \sum_G (\mathrm{ch}_U \circ S) \cdot \mathrm{ch}_V = \frac{1}{|G|} \sum_G \mathrm{ch}_{U^* \otimes V} = \frac{1}{|G|} \sum_G \mathrm{ch}_{\mathrm{Hom}(U,V)}$$

$$= \dim \mathrm{Hom}(U, V)^G = \dim \mathrm{Hom}_G(U, V) = \begin{cases} 1, & U \cong V \\ 0, & U \ncong V \end{cases}$$

Classical representation theory of finite group: class function

where we apply the fact $\text{Hom}(U, V)^G = \text{Hom}_G(U, V)$, and the last equality follows from Schur's Lemma.

Finally, we need to show the linearly independent subset is maximal. By Artin-Wedderburn Theorem, the number of isomorphism classes of simple modules for a finite dimensional semisimple $\mathbb{k}$-algebra equals the dimension of its center.

Meanwhile, the center of group algebra, $Z(\mathbb{k}[G])$, turns out to be isomorphic to the algebra of class funcions on G.

Summary of classical representation theory of finite group

Simple objects in $\text{Rep}(G)$ are in one-to-one correspondence with basis elements in $Z(\mathbb{k}[G])$.

Higher representation theory of finite group

Next, let's review the categorification of the classical representation theory of finite group.

categorical level 0	**categorical level** 1
group algebra $\mathbb{k}[G]$	graded vector spaces Vec_G
$\mathrm{Rep}(G)$	$2\,\mathrm{Rep}(G)$
G-representation	linear category with G-action
tensor product of vector spaces	Deligne tensor product of linear categories
center of an algebra	Drinfel'd center of a tensor category
$Z(\mathbb{k}[G])$	$\mathcal{Z}_1(\mathrm{Vec}_G)$
class function on G	G-graded G-representation
irreducible G-character	Lagrangian algebra in $\mathcal{Z}_1(\mathrm{Vec}_G)$

Higher representation theory of finite group: algebras in braided category

Next, let's recall some important types of algebras within a braided monoidal category. Fix an ambient braided monoidal category $(\mathcal{B}, \otimes, I, \gamma)$, where we omit the associators and unitors of $\mathcal{B}$ due to a general coherence theorem, see [JS93].

An **algebra** in $\mathcal{B}$ consists of an object A in $\mathcal{B}$, multiplication $m : A \otimes A \to A$, unit $e : I \to A$ subject to the following conditions:

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes m} & A \otimes A \\
{\scriptstyle m \otimes \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow{\quad m \quad} & A
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\mathrm{id}_A \otimes e} & A \otimes A \\
{\scriptstyle e \otimes \mathrm{id}_A}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow{\quad m \quad} & A
\end{array}
$$

An algebra (A, m, e) in $\mathcal{B}$ is **commutative** if it satisfies an additional condition:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\gamma_{A,A}} & A \otimes A \\ {\scriptstyle m}\downarrow & \swarrow{\scriptstyle m} & \\ A & & \end{array}$$

An algebra (A, m, e) in $\mathcal{B}$ is **separable** if there exists $\Delta : A \to A \otimes A$ satistying $\mathrm{id}_A = m \circ \Delta$ and the following *Frobenius conditions*:

$$\begin{array}{ccccc} A \otimes A & & \xrightarrow{\mathrm{id}_A \otimes \Delta} & & A \otimes A \otimes A \\ & \searrow{\scriptstyle m} & & & \\ {\scriptstyle \Delta \otimes \mathrm{id}_A}\downarrow & & A & & \downarrow{\scriptstyle m \otimes \mathrm{id}_A} \\ & & & \searrow{\scriptstyle \Delta} & \\ A \otimes A \otimes A & & \xrightarrow{\mathrm{id}_A \otimes m} & & A \otimes A \end{array}$$

Higher representation theory of finite group: algebras in braided category

Given an algebra (A, m, e) in $\mathcal{B}$, a **left A-module in $\mathcal{B}$** consists of an object M in $\mathcal{B}$ with a left A-action $\rho : A \otimes M \to M$, subject to the following conditions:

$$
\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{\mathrm{id}_A \otimes \rho} & A \otimes M \\
{\scriptstyle m \otimes \mathrm{id}_M} \downarrow & & \downarrow {\scriptstyle \rho} \\
A \otimes M & \xrightarrow{\rho} & M
\end{array}
\qquad
\begin{array}{ccc}
M & & \\
{\scriptstyle e \otimes \mathrm{id}_M} \downarrow & \searrow\!\!\!\!= & \\
A \otimes M & \xrightarrow{\rho} & M
\end{array}
$$

Similarly, a **right A-module in $\mathcal{B}$** consists of an object N in $\mathcal{B}$ with a right A-action $\lambda : N \otimes A \to N$, subject to the following conditions:

$$
\begin{array}{ccc}
N \otimes A \otimes A & \xrightarrow{\mathrm{id}_N \otimes m} & N \otimes A \\
{\scriptstyle \lambda \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \lambda} \\
N \otimes A & \xrightarrow{\lambda} & N
\end{array}
\qquad
\begin{array}{ccc}
N & \xrightarrow{\mathrm{id}_N \otimes e} & N \otimes A \\
& \searrow\!\!\!\!= & \downarrow {\scriptstyle \lambda} \\
& & N
\end{array}
$$

A **left (resp. right) module homomorphism** is a morphism in $\mathcal{B}$ which commutes with the corresponding left (resp. right) A-actions.

Let's denote the category of *finite projective* left (resp. right) A-modules in $\mathcal{B}$ as ${}_A\mathcal{B}$ (resp. $\mathcal{B}_A$).

Theorem [Ost03, EGNO16, KZ17, etc.]

Given an algebra (A, m, e) in a tensor category $\mathcal{C}$, the category of right (resp. left) A-modules, $\mathcal{C}_A$ (resp. ${}_A\mathcal{C}$), is a left (resp. right) $\mathcal{C}$-module category.

Conversely, with some mild finiteness conditions, every left (resp. right) $\mathcal{C}$-module category is of this form.

In particular, when $\mathcal{C}$ is finite semisimple, $\mathcal{C}_A$ (or ${}_A\mathcal{C}$) is finite semisimple if and only if algebra A is separable.

When $\mathcal{B}$ is a braided tensor category, an algebra (A, m, e) in $\mathcal{B}$ is **indecomposable** if $\mathcal{B}_A$ is indecomposable as left $\mathcal{B}$-module category.

[Example] Artin-Wedderburn Theorem tells us that a separable algebra in Vec is a finite sum of matrix algebras. Moreover, an indecomposable separable algebra in Vec is just a matrix algebra.

[Example] An indecomposable separable algebra in $\mathrm{Rep}(G)$ is isomorphic to $V^* \otimes V$, for some simple G-representation. In particular, for one-dimensional simple G-representation V, we have $V^* \otimes V$ isomorphic to the trivial G-representation $\Bbbk$.

Theorem [Ost03]

Indecomposable separable algebras in $\mathrm{Rep}(G)$ are classified by pair (H, ϕ)'s, where H is a subgroup of G and $\phi \in \mathrm{H}^2(H; \Bbbk^\times)$.

An **étale algebra** is an indecomposable commutative separable algebra.

[Example] By Artin-Wedderburn Theorem, a commutative separable algebra in Vec is a finite sum of trivial algebra $\mathbb{k}$'s. Hence, up to isomorphism, the only étale algebra in Vec is the trivial algebra $\mathbb{k}$.

[Example] A commutative separable algebra in $\mathrm{Rep}(G)$ is a finite sum of trivial algebra together with a G-action. Hence, we can pick a basis closed under that G-action, i.e. a finite G-set. Moreover, if the algebra is indecomposable then the basis is a transitive G-set. As a result, an étale algebra in $\mathrm{Rep}(G)$ is isomorphic to the function algebra on a coset space, i.e. $\mathrm{Fun}(G/H)$ for some subgroup H.

Higher representation theory of finite group: étale algebras in $\mathcal{Z}_1(\mathrm{Vec}_G)$

Theorem [Dav10]

Étale algebras in $\mathcal{Z}_1(\mathrm{Vec}_G)$ are classified up to isomorphism by triple (H, N, ϕ)'s, where H is a subgroup of G, N is a normal subgroup of H and $\phi \in \mathrm{coker}(\mathrm{H}^2(H/N;\Bbbk^\times) \xrightarrow{\pi^*} \mathrm{H}^2(H;\Bbbk^\times))$, where $\pi : H \to H/N$ is the quotient homomorphism.

In particular, when $H = N$, we have $\phi \in \mathrm{H}^2(H;\Bbbk^\times)$; the corresponding étale algebras in $\mathcal{Z}_1(\mathrm{Vec}_G)$ are called Lagrangian algebras.

Corollary

Simple objects in $2\,\mathrm{Rep}(G)$, which by definition are Morita classes of indecomposable separable algebras in $\mathrm{Rep}(G)$, are in one-to-one correspondence with Lagrangian algebras in $\mathcal{Z}_1(\mathrm{Vec}_G)$.

Higher representation theory of finite group: Summary

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class function on G	G-graded G-representation
irreducible G-character	Lagrangian algebra in $\mathcal{Z}_1(\mathrm{Vec}_G)$

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