THE JONES POLYNOMIAL OF A KNOT: THE BIRTH OF QUANTUM TOPOLOGY

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WORKSHOP ON QUANTUM
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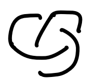
JONES POLYNOMIAL (V. JONES ~1983)

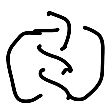
$J : \{$  $\} \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$ = polynomials in $q^{1/2}$ (and $q^{-1/2}$) with integer coefficients

Knots, links

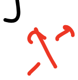
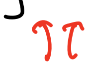
Example

$J_{\bigcirc}(q) = 1$

J  $(q) = -q^{-4} + q^{-3} + q^{-2}$

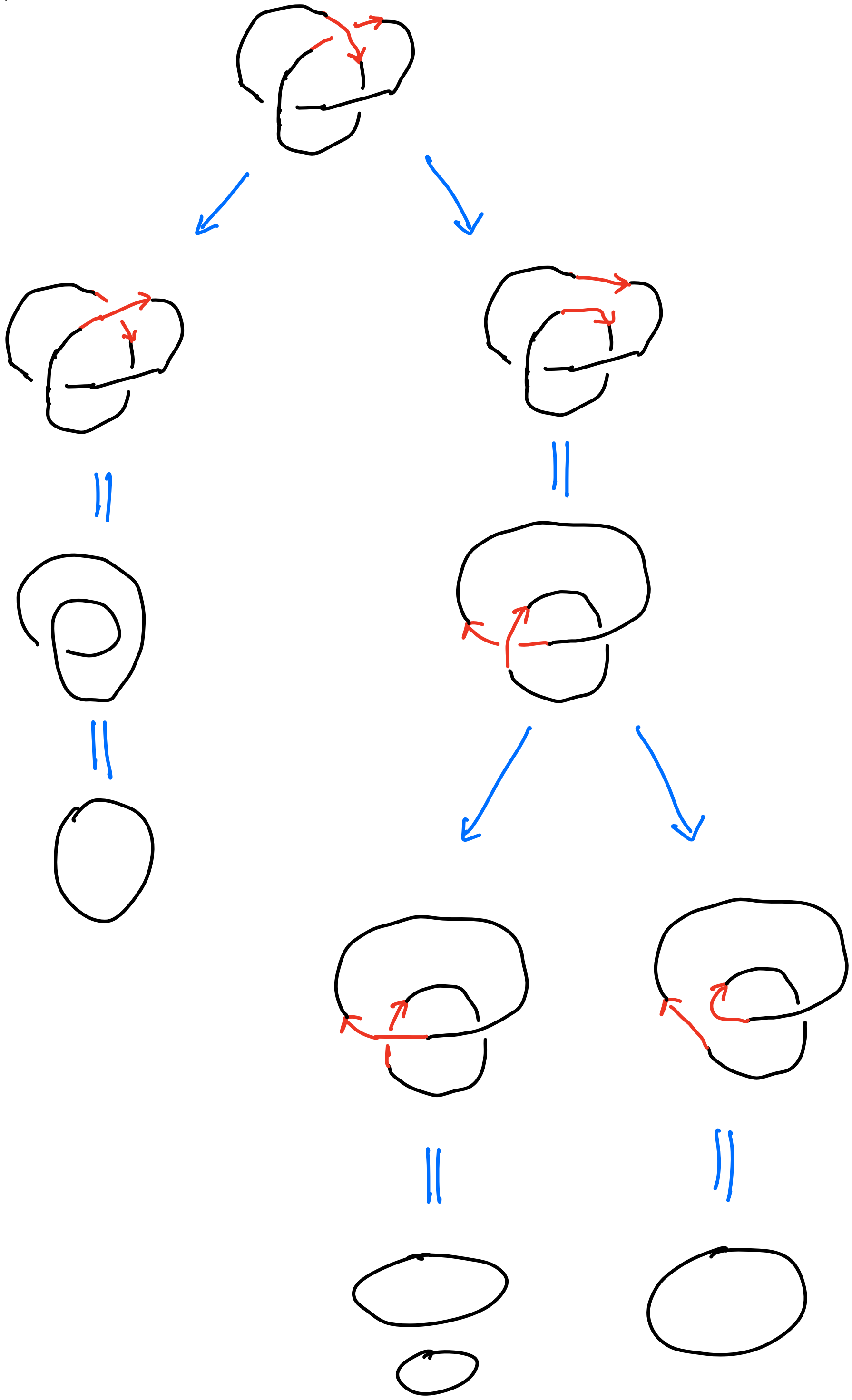
J  $(q) = q^{-2} - q^{-1} + 1 - q + q^{2}$

Rule (linear system of equations)

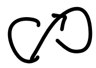
$q\, J$  $(q) - q^{-1} J$  $(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) J$  (q)

$J_{\bigcirc}(q) = 1$ (initial condition

Example



Every physicist knows: $\infty - \infty = 0 \; 0$!

qJ  $(q) - q^{-1}J$  $(q) = (q^{1/2} - q^{-1/2})J_{OO}(q)$

So $\quad J_{OO}(q) = \dfrac{q - q^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$

<u>Special value</u> $\quad q = 1$

$J_K(1) = 1$ for all knots

$J_L(1) = 0$ for all links with at most 2 components

<u>Russian school</u> Kirillov, Reshetikhin, Turaev Drienfeld,

<u>Japanese school</u> Jimbo, Miwa, ...   quantum groups

realized that the Jones polynomial is the simplest example of $sl_2\mathbb{C}$ (simplest simple Lie alg) and its fundamental $\mathbb{C}^2$-representation

$$sl_2\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{matrix} a+d=0 \\ a,b,c,d \in \mathbb{C} \end{matrix} \right\}, \text{ acts on } \mathbb{C}^2.$$

Lie algebra

$[x,y] = xy - yx$

However, there is one irreducible rep of $sl_2\mathbb{C}$ of dim N, namely $\mathbb{C}^N$. Think of

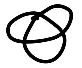
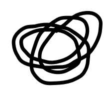
$\mathbb{C}^N = \{\, p(x,y) \mid \text{homogenous polys in } x,y \text{ of degree } N-1 \,\}$

$$\mathbb{C}^N = \mathrm{Sym}^{N-1}(\mathbb{C}^2) \subset \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N-1}$$

Thus there exists a colored Jones polynomial

$$J_{K,N}(q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \quad N = 1, 2, 3, \ldots$$

$$J_{K,1}(q) = 1 \qquad J_{K,2}(q) = J_K(q)$$

J  $_{,3}(q) \underset{\text{roughly}}{=} J$  (q) (2-parallel of knot)

$$+ J_{\emptyset}(q)$$

(eg $\mathbb{C}^3 + \mathbb{C} = \mathbb{C}^2 \otimes \mathbb{C}^2$ as $sl_2\mathbb{C}$ reps)

What is the colored Jones poly of a knot good for? A bit later.

<u>Example</u>

$$J_{4_1,N}(q) = \sum_{n=0}^{N-1} q^{-nN} (q^{N+1}; q)_n (q^{N-1}; q^{-1})_n$$

where $(x;q)_n = (1-x)(1-qx)\ldots(1-q^{n-1}x)$ is the quantum n-factorial

Witten (1989) CHERN-SIMONS PATH INTEGRAL

History: Atiyah asked Witten in Swansee, UK for a relation between the SU(2)-Jones polynomial and Yang-Mills theory (Donaldson theory) in 4-dimensions.

Witten's response: dimensionally reduce the 4D-Yang Mills, to 3-dimensions, thus to topological CS-theory. He filled a stack of napkins, with:

$$J_{K,N}\left(e^{\frac{2\pi i}{\ }}\right) = \int_{\mathcal{A}(S^3)} e^{2\pi i\ CS(A)}\, \Theta_{K,N}(A)\, dA$$

$\mathcal{A}(S^3) = \{$affine space of all su(2)-valued 1-forms on $S^3\}$ (SU(2)-connections)

$\Theta_K(A) =$ the trace of the holonomy going around K in the $\mathbb{C}^N$-representation of SU(2)

$$CS(A) = \int_{S^3} tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Further developments

- Asymptotic expansion of CS-theory at the trivial flat connection led to Vassiliev (ie finite-type) invariants of knots.

Eg $\left.\dfrac{d^{10}}{dq^{10}} J_{K,N}(q)\right|_{q=1}$ is a poly of N of degree 10 whose coefficients satisfy difference equations at 11 random points.

- Drinfeld, Kontsevich (1990, 92) constructed the universal Vassiliev invariant, and connected to nonabelian algebraic geometry (reps of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$)

- Relation to Alexander polynomial (classical knot invariant)

$$\Delta(t) - \Delta(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(t)$$

$$\Delta_{O}(t) = 1$$

Write $J_{K,N}(e^{h}) = \sum_{0 \le i \le j} a_{ij} N^{i} h^{j}$

Bar-Natan, G (1995) $\dfrac{1}{\Delta_K(e^{h})} = \sum_{i=0}^{\infty} a_{ii} h^{i}$

- Kashaev (1994) formulated the <u>Volume Conjecture</u> relating quantum topology to Thurston-hyperbolic geometry.

$$\lim_{N\to\infty} \frac{1}{N} \log |J_{K,N}(e^{\frac{2\pi i}{N}})| = \mathrm{vol}(S^3 - K)$$

↳ suitably normalized

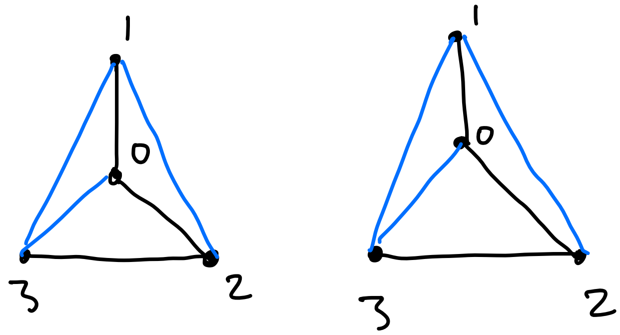
<u>Example</u>

$$J_{4_1,N}(e^{\frac{2\pi i}{N}}) = \sum_{n=0}^{N-1} (q;q)_n (q^{-1};q^{-1})_n \quad , \quad q = e^{2\pi i/N}$$

$$\sim e^{\frac{\mathrm{vol}(4_1)}{2\pi} N}$$

where
$\mathrm{vol}(4_1) = 2\,\mathrm{Im}\,\mathrm{Li}_2(e^{2\pi i/6}) = 2.0298832128 1\ldots$

Thurston

$S^3 -$  $=$ 

tet \ edge	01	02	03	12	13	23
0	0	0	1	1	1	0
1	0	0	1	1	1	0

tet \ face	012	013	023	123
0	0	1	2	3
1	2	1	0	3

edges 0 = black
1 = blue

The hyperbolic metric on 4_1 is given by the gluing of 2 ideal tetrahedra with dihedral angles $2\pi/6$ at each edge.
Their shapes $e^{2\pi i/6}$ enters in their volume.

q-difference equations for the colored Jones polynomial

The sequence $J_{K,N}(q)$ $N=1,2,3,\ldots$ satisfies a linear recursion relation with coefficients polys in q and q^N.

(G-Thang Le 2005)

For 4_1 the recursion is second order inhomogeneous of the form

$$a_2(q^n,q)J_{K,n+2}(q)+a_1(q^n,q)J_{K,n+1}(q)+a_0(q^n,q)J_{K,n}(q)=b(q,q^n)$$

where a_2, a_1, a_0, b are Laurent polynomials.

Let $LJ_n(q)=J_{n+1}(q)$ $\qquad LM=qML$

$MJ_n(q)=q^nJ_n(q)$

<u>Conj</u> (AJ Conjecture, G 2004)

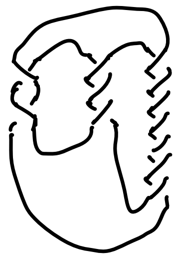
If $P(L,M,q)J=b(q,q^n)$

then the polynomial $P(L,M,1)$ defines the variety of $SL_2\mathbb{C}$-representations of π_1(knot complement)

Thm (G, 2011) Let $d_{K,n} = \deg_q J_{K,n}(q)$

Then $d_{K,n}$ is a quadratic quasipoly.

Ex For $(-2,3,7)$ pretzel knot



$$d_{(-2,3,7),n} = \left\lfloor \frac{37}{8}n^2 + \frac{17}{2}n \right\rfloor$$

$$= \frac{37}{8}n^2 + \frac{17}{2}n + \varepsilon(n)$$

where $\varepsilon(n) = \begin{cases} 0 & n \equiv 0 \mod 4 \\ \frac{1}{8} & n \equiv 1 \mod 4 \\ \frac{1}{2} & n \equiv 2 \mod 4 \\ \frac{1}{8} & n \equiv 3 \mod 4 \end{cases}$

Conjecture (Slope Conjecture G 2011)

The quadratic coeff of $d_{K,n}$ is 4 times a slope of an incompressible surface in knot complement

True for $(-2,3,7)$ $\frac{37}{2}$ is such.

q-series and analytic functions

In the last 5 years, several nonpolynomial knot invariants have been introduced by physicists and mathematicians

- 3D index (Dimofte-Gaiotto-Gukov, G, Kashaev-G)
- state integrals (Kashaev et al)
- factorially divergent formal power series (Dimofte-G, G-Zagier)

I will give only one example, a pair of q-series with integer coefficients associated to the 4_1 knot

$$g_m(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}+nm}}{(q;q)_n^2}$$

$$G_m(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}+nm}}{(q;q)_n^2}\left(2m + E_1(n) + 2\sum_{j=1}^{n} \frac{1+q^j}{1-q^j}\right)$$

where $E_1(q) = 1 - 4\sum_{n=1}^{\infty} q^n/(1-q^n)$ is the weight-1-Eisenstein series

Thank you for
your attention!