Strongly-fusion 2-category is grouplike

Chun-Yu Bai

June 2, 2022

Shenzhen Institute for Quantum Science and Engineering

This talk is based on Theo Johnson-Freyd and Matthew Yu's proof of the following result [TY20, arXiv:2010.07950]:

Theorem

If $\mathbf{C}$ *is a strong fusion 2-category, then the equivalence classes of indecomposable objects of* $\mathbf{C}$ *form a finite group under the fusion product. Similar to Schur's lemma in 1-category, the equivalence relation is "related by a nonzero morphism".*

This result was first introduced and proved on a physical level of rigor by Tian Lan, Liang Kong, Xiao-Gang Wen, [LKW17, arXiv:1704.04221].

The definition and basic theory of semisimple and multifusion 2-categories were first introduced in [DR18, arXiv:1812.11933]. Let's first review the main features. $\mathbb{C}$ is complex number field.

Definition

A 2-category **C** is $\mathbb{C}$-**linear** if all hom-sets of 2-morphisms are vector spaces over $\mathbb{C}$, and both vertical composition and horizontal composition of 2-morphisms are bilinear.

Definition

An object in a linear 2-category is **decomposable** if it is equivalent to a direct sum of nonzero objects, and **indecomposable** if it is nonzero and not decomposable.

Remark

A **simple object** X in a 2-category is one such that any injective 1-morphism $A \hookrightarrow X$ is either 0 or an equivalence. In finite semisimple 2-categories all indecomposable objects are simple [DR18, arXiv:1812.11933]. I will use the terms "simple" and "indecomposable" interchangeably.

In particular the objects which we consider in the 2-category will only be sums of finitely many simple objects. In our goal to define a semisimple 2-category, we present some definitions for the higher categorical generalization of the notion of idempotent complete, see [GJF19, arXiv:1905.09566].

Definition

A 2-category $\mathbf{C}$ is **locally idempotent complete** if all objects $A, B \in \mathbf{C}$, the 1-category $\hom_{\mathbf{C}}(A, B)$ is idempotent complete. It is **locally finite semisimple** if $\hom_{\mathbf{C}}(A, B)$ is furthermore a finite semisimple $\mathbb{C}$-linear category (i.e. an abelian $\mathbb{C}$-linear category with finitely many isomorphism classes of simple object and in which every object decomposes as a finite direct sum of simple objects).

In what follows, we will assume $\mathbf{C}$ is a locally idempotent complete 2-category.

Definition

A **condensation** $A \twoheadrightarrow B$ in 2-category $\mathbf{C}$ consists of the following data:

- 1-morphisms $f : A \to B$ and $g : B \to A$;
- 2-morphisms $\varepsilon : fg \Rightarrow \mathrm{id}_B$ and $\gamma : \mathrm{id}_B \to fg$ such that $\varepsilon\gamma = \mathrm{id}_{\mathrm{id}_B}$.

Definition

A **unital condensation** in 2-category $\mathbf{C}$ is a condensation $A \twoheadrightarrow B$ equipped with a 2-morphism $\eta : \mathrm{id}_A \Rightarrow gf$ such that $f \dashv g \equiv (f, g, \eta, \varepsilon)$ is an adjunction.

Definition

Let A be a simple object in $\mathbf{C}$. A monad $(p : A \to A, m : p \circ p \to p, u : \mathrm{id}_A \to p)$ in a 2-category is **separable** if there exists an (p, p)-bimodule map $c : p \to p \circ p$ such that $m \circ c = \mathrm{id}_p$.

Example

A separable monad in a 2-category with one object $\star$ is a separable algebra in the E_1-monoidal endomorphism category $\mathrm{End}(\star)$.

Definition

A separable monad (p, m, u) is **split** if there exist a unital condensation $(f, g, \varepsilon, \gamma, \eta)$ and $gf \simeq p$.

You can check this definition is well-defined. (hint:$m : gfgf \xrightarrow{\mathrm{id}_g\, \varepsilon\, \mathrm{id}_f} gf$; $u : \mathrm{id}_A \xrightarrow{\eta} gf$; $c : gf \xrightarrow{\mathrm{id}_g\, \gamma\, \mathrm{id}_f} gfgf$; $\varepsilon\gamma = \mathrm{id}_{\mathrm{id}_B}$ leads that monad (p, m, u) is separable).

Definition

A 2-category $\mathbf{C}$ is **2-idempotent complete** if it is

1. locally idempotent complete;
2. every separable monad splits.

Remark

Requiring the unitality of p differs slightly from the situation in 1-categories. In 1-categories there is an equality of $p^2 = p$ but there is no equality of 1 and p. [GJF19, arXiv:1905.09566] developed a nonunital version of separable monad for 2-categories and showed that if $\mathbf{C}$ has adjoints for 1-morphisms, then the notion of 2-idempotent completion above and in [GJF19, arXiv:1905.09566] agree.

Definition

A $\mathbb{C}$-linear 2-category **C** is **finite semisimple** if

1. it has finitely many isomorphism classes of simple objects;
2. it is locally finite semisimple;
3. has adjoints for all 1-morphisms;
4. has direct sums of objects;
5. is 2-idempotent complete.

Definition

A **multifusion 2-category** is a finite semisimple monoidal 2-category in which all objects have duals.

Remark

As noted in [DR18, arXiv: 1812.11933, Definition 2.1.6], in a multifusion 2-category, left and right duals are the same.

Definition

A multifusion 1-categories $\mathcal{C}$ is **fusion** if the endomorphism algebra $\Omega(\mathcal{C}, \mathbb{1}_{\mathcal{C}}) = \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{C}$, where $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ denotes the tensor unit.

There are two reasonable categorifications of 'fusion' when $\mathbf{C}$ is a multifusion 2-category:

Definition

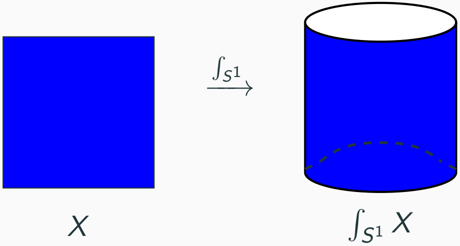
The stronger generalization, which we will call **strongly fusion**, is to ask that the endomorphism 1-category $\Omega(\mathbf{C}, \mathbb{1}_{\mathbf{C}}) = \mathrm{End}_{\mathbf{C}}(\mathbb{1}_{\mathbf{C}}) \simeq \mathrm{Vect}_{\mathbb{C}}$.

The weak notion, which we will call merely **fusion**, is to ask only that $\Omega^2(\mathbf{C}, \mathbb{1}_{\mathbf{C}}) = \mathrm{End}_{\Omega(\mathbf{C}, \mathbb{1}_{\mathbf{C}})}(\mathbb{1}_{\mathbb{1}_{\mathbf{C}}}) \simeq \mathbb{C}$, where $\mathbb{1}_{\mathbb{1}_{\mathbf{C}}} \in \Omega(\mathbf{C}, \mathbb{1}_{\mathbf{C}})$ is the tensor unit.

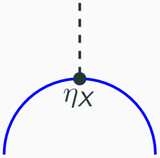
Remark

The particle-like topological excitations (which form the UMTC) are 0d domain walls between trivial 1d domain walls. The condition of strong fusion tells us $\mathbf{C}$ has no particle-like excitation.

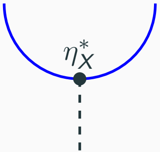
We now begin to develope the necessary graphical calculus in order to prove the main results. For an object $X \in \mathbf{C}$, we denote $\int_{S^1} X$ as the wrapping of X around a S^1. This integral is a map $\int_{S^1} : \mathbf{C} \to \Omega(\mathbf{C}, \mathbb{1}_{\mathbf{C}})$.



We now describe this operation $\int_{S^1}$ algebraically. Because we are working with a strongly fusion 2-category each object has a dual and we have a unit $\eta_X : \mathbb{1}_{\mathbf{C}} \to X \otimes X^*$. It corresponds to the half-circle:



Also, since all 1-morphisms have adjoints, there is a right adjoint (unital condensation) $\eta_X^* : X \otimes X^* \to \mathbb{1}_{\mathbf{C}}$:



All together, we find the algebraic definition:

$$\int_{S^1} X := \eta_X^* \circ \eta_X.$$

State-operator correspondence

In a multifusion 2-category **C** there is an isomorphism

$$\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X) \simeq \hom_{\Omega(\mathbf{C},\mathbb{1}_{\mathbf{C}})}(\mathrm{id}_{\mathbb{1}_{\mathbf{C}}}, \int_{S^1} X)$$

Proof.

The duality of X with X^* provides an equivalence of

$$\mathrm{End}_{\mathbf{C}}(X) \simeq \hom_{\mathbf{C}}(\mathbb{1}_{\mathbf{C}}, X \otimes X^*)$$

This equivalence identities id_X with η_X, and so in particular

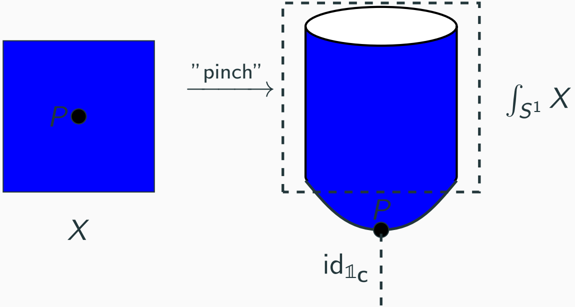
$$\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X) \simeq \mathrm{End}_{\hom(\mathbb{1}_{\mathbf{C}}, X\otimes X^*)}(\eta_X)$$

Since **C** is multifusion , all 1-morphisms $f : A \to B$ have adjoints $f^* : B \to A$, i.e. $\mathrm{End}_{\hom(A,B)}(f) \simeq \hom_{\mathrm{End}(A)}(\mathrm{id}_A, f^* \circ f)$. Taking $f = \eta_X$, with $A = \mathbb{1}_{\mathbf{C}}$ and $B = X \otimes X^*$ completes the proof. □

How to understand "state-operator correspondence

$\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X) \simeq \hom_{\Omega(\mathbf{C},\mathbb{1}_{\mathbf{C}})}(\mathrm{id}_{\mathbb{1}_{\mathbf{C}}}, \int_{S^1} X)$ " in geometry?

A element P in $\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X)$ is a "point" on X-sheet.



Every P gives an "way" from $\mathrm{id}_{\mathbb{1}_{\mathbf{C}}}$ to $\int_{S^1} X$.

Remark

Suppose $\mathbf{C}$ is fusion. According to the above isomorphism

$$\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X) \simeq \hom_{\Omega(\mathbf{C},\mathbb{1}_{\mathbf{C}})}(\mathrm{id}_{\mathbb{1}_{\mathbf{C}}}, \int_{S^1} X)$$

X is simple iff $\hom_{\Omega(\mathbf{C},\mathbb{1}_{\mathbf{C}})}(\mathrm{id}_{\mathbb{1}_{\mathbf{C}}}, \int_{S^1} X)$ is one-dimensional. This is self-consistent with the definition of when multifusion is fusion.

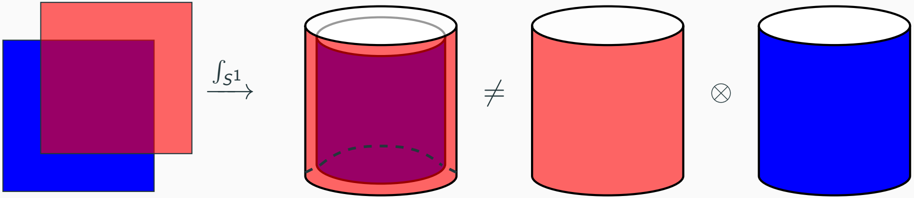
Theorem

Suppose $\mathbf{C}$ *is strong fusion. If* $X \in \mathbf{C}$ *is indecomposable, then* $\int_{S^1} X = \mathbb{C}$*.*

Proof.

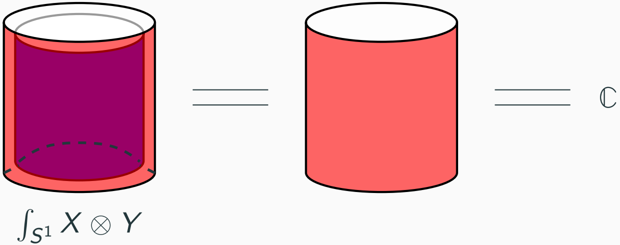
Since $\mathbf{C}$ is strong fusion, $\Omega(\mathbf{C}, \mathbb{1}_{\mathbf{C}}) \simeq \mathrm{Vect}_{\mathbb{C}}$. In order to self-consistent with the definition of strong fusion, X is indecomposable iff $\mathrm{End}_{\mathbf{C}}(X) \simeq \mathrm{Vect}_{\mathbb{C}}$. Then $\mathrm{End}_{\mathrm{End}_{\mathbf{C}}(X)}(\mathrm{id}_X) \simeq \mathrm{End}_{\mathrm{Vect}_{\mathbb{C}}}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$. So we must have $\int_{S^1} X = \mathbb{C}$. $\square$

We now consider the tensor product of two indecomposable objects $X \otimes Y$ mapped by the integral $\int_{S^1}$. This represents a cylinder within a cylinder as follows:



$X \otimes Y$ $\quad$ $\int_{S^1} X \otimes Y$ $\quad$ $\int_{S^1} X \otimes \int_{S^1} Y$

In general, we see that $\int_{S^1}$ is not monoidal: a cylinder within a cylinder is not the same as two adjacent cylinders. However, in the strongly fusion case, if X and Y are simple then we may collapse down the inner cylinder via the state operator map into the vaacum $\mathbb{C}$. We may then collapse the out cylinder.



So we have the following corollary:

Theorem

In a strong fusion 2-category, the tensor product of indecomposable objects is indecomposable.

Theorem

If $\mathbf{C}$ *is a strong fusion 2-category, then the equivalence classes of indecomposable objects of* $\mathbf{C}$ *form a finite group under the fusion product. Similar to Schur's lemma in 1-category, the equivalence relation is "related by a nonzero morphism".*

Proof.

If $X \in \mathbf{C}$ is a indecomposable object, then X^* is as well (since $\mathrm{End}(X) \simeq \mathrm{End}(X^*)$), and hence so is $X \otimes X^*$. Since $\eta_X : \mathbb{1}_{\mathbf{C}} \to X \otimes X^*$ is nonzero, the idecomposable objects $\mathbb{1}_{\mathbf{C}} \simeq X \otimes X^*$. The identity in group is given by $\mathbb{1}_{\mathbf{C}}$. The associativity is given by the associator of $\mathbf{C}$. In a strong fusion 2-category, the tensor product of indecomposable objects is indecomposable. Thus the equivalence of classes of indecomposable objects of $\mathbf{C}$ form a finite group under the fusion product. □

Reference

1, Fusion 2-categories with no line operators are grouplike, Theo and Matthew.

2, [DR18] Fusion 2-categories and a state-sum invariant for 4-manifolds, Douglas and Reutter;

3, [GJF19] Condensation in higher categories, Gaiotto and Theo;

4, [LKW17] Tian Lan, Liang Kong, and Xiao-Gang Wen. Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries.