Pseudo standard entanglement structure cannot be distinguished from standard entanglement structure

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SUSTech-Nagoya workshop on Quantum Science
arXiv:2203.07968

Brief Summary

- Entanglement structures (ES)
  - is a possible structure of quantum composite system in General Probabilistic Theories (GPTs)
  - is not uniquely determined even if local structures are equivalent to the standard quantum theory.
- Pseudo standard entanglement structure (PSES) :
  - self-duality (via repeatability)
  - small error verification of all maximally entangled states
- Problem : Is there any possibility of PSES except for the standard entanglement structure?

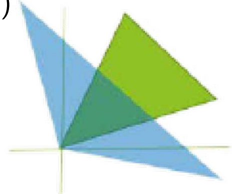
$\rightarrow$ infinitely many!

$\rightarrow$ perfect discrimination of non-orthogonal states

- Variety of ES with group symmetry
  - global unitary symmetry determines ES as the standard entanglement structure (SES)

Preliminary:Structure of Positive cones

- GPT is a generalization of probabilistic models including quantum theory
- positive (proper) cone $\mathcal{K} \subset \mathcal{V}$ ($\mathcal{V}$:real vector space)
  - closed convex set with non-empty interior
  - $\forall x \in \mathcal{K}$, $\forall r \geq 0$, $rx \in \mathcal{K}$
  - $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$



- positive cone corresponds to "positivity"
- GPT extends "positivity" in quantum theory by positive cone

Preliminary:Definition of Quantum Theory

Model of standard quantum theory on composite system $\mathcal{H}_A \otimes \mathcal{H}_B$
(here we call it "standard entanglement structure (SES)")

- State ρ (density matrix)
  - $\rho \in \mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$ (POVM)
  - $M_i \in \mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$

    $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0$ $(\forall \rho \in \mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B))$
  - $\sum_i M_i = I$
- Probability to get a measurement outcome
  - outcome i is measured with probability $\mathrm{Tr}\,\rho M_i$

$\rightarrow$ the SES is defined by $\mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$

$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$

$\mathcal{T}(\mathcal{H})$: set of Hermitian matrices on $\mathcal{H}$

$\mathcal{T}_+(\mathcal{H})$: set of positive semi-definite matrices on $\mathcal{H}$

Definition of Entanglement Structure

A model of GPTs is a generalization $\mathcal{T}_+(\mathcal{H}) \to \mathcal{K}$
$\mathcal{K}$: positive cone with $\mathrm{SEP}(A;B) \subset \mathcal{K} \subset \mathrm{SEP}^*(A;B)$

- State ρ
  - $\rho \in \mathcal{K}$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$
  - $M_i \in \mathcal{K}^*$

  $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0$ $(\forall \rho \in \mathcal{K})$
  - $\sum_i M_i = I$

$$\mathrm{SEP}(A;B) := \{\Sigma_i A_i \otimes B_i \mid A_i \in \mathcal{T}_+(\mathcal{H}_A), B_i \in \mathcal{T}_+(\mathcal{H}_B)\},$$

$$\mathrm{SEP}^*(A;B) := \{Y \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid \mathrm{Tr}\,XY \geq 0 \ {}^\forall X \in \mathrm{SEP}(A;B)\}$$

- Probability to get a measurement outcome
  - outcome i is measured with probability $\mathrm{Tr}\,\rho M_i$

$\rightarrow$ ES is defined by a cone $\mathcal{K}$

- Main Motivation : Characterization of a cone $\mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$

Motivation : Peculiarity of the SES

$\mathcal{K}^* := \{Y \in \mathcal{T}(\mathcal{H}) \mid \mathrm{Tr}\, XY \geq 0\ \forall X \in \mathcal{K}\}$

- Self-duality
  - self-dual : $\mathcal{K}^* = \mathcal{K}$
  - In the case $\mathcal{K} = \mathrm{SEP}(A; B)$, non-orthogonal separable states are distinguishable [AYH2019] (because of non-self-duality)
- Symmetry of Global Unitary operators
  - global unitary operators generate all pure states from a single pure state
- self-duality + homogeneity $\rightarrow$ Eucledian Jordan Algebra (in cones) [JNW1934] ($\rightarrow$ SES in our setting [this paper])
- self-duality + symmetric condition $\rightarrow$ SES (in ESs) [this paper]

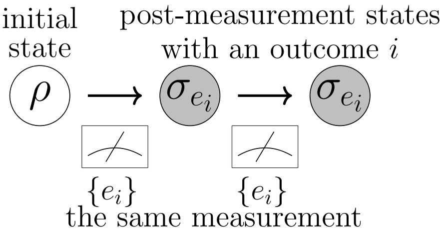
Q. Is symmetric condition essentially necessary?
  - no example of self-dual ES is known

Q. more operational characterization can?
  - repeatability
  - verification of maximally entangled states

[JNW1934] P. Jordan, J. v. Neumann, and E. Wigner, Annals of Mathematics, 35(1):29-64, (1934).
[AYH2019] H. Arai, Y. Yoshida, and M. Hayashi, J. Phys. A 52,

Repeatability and pre-duality

- Repeatability
  - ensures that post-measurement states cannot be changed by the same measurement



- there exists a correspondence $e_i \mapsto \sigma_{e_i}$

$\rightarrow$ Repeatability derives pre-duality, i.e., $\mathcal{K} \supset \mathcal{K}^*$

Self-dual modification

Main Theorem 1

Given any pre-dual cone $\mathcal{K}$, there exists a self-dual scone $\tilde{\mathcal{K}}$ such that $\mathcal{K} \supset \tilde{\mathcal{K}} \supset \mathcal{K}^*$.

- Due to this theorem, self-duality is a natural consequence of repeatability.
- For proofs, we applying Zorn's Lemma for a certain ordered set ($\rightarrow$The proof is not constructive)
- Different pre-dual cone cannot always be modified to different self-dual cone (This problem is solved by following theorem)

Main Theorem 2 (weak version)

Given a exact hierarchy of pre-dual cone $\mathcal{K}_1 \supsetneq \cdots \supsetneq \mathcal{K}_n$, there exists exactly different self-dual cones $\tilde{\mathcal{K}}_i$ such that $\mathcal{K}_i \supset \tilde{\mathcal{K}}_i \supset \mathcal{K}_i^*$.

Verification of maximally entangled states and ϵ-undistinguishability

Because the error probability of verification (hypothesis testing) is estimated by trace norm $\|\cdot\|_1$, we introduce the following value for $\mathcal{K}$

$$D(\mathcal{K}) := \max_{\sigma\in\mathrm{ME}} \min_{\rho\in\mathcal{K}} \|\sigma - \rho\|_1$$

- if $\mathcal{K}$ contains all maximally entangled states, then $D(\mathcal{K}) = 0$
- ϵ-undistinguishability
  - $D(\mathcal{K}) \leq \epsilon$
  - if $\mathcal{K}$ is ϵ-undistinguishable, we cannot deny the possibility of $\mathcal{K}$ by any verification of maximally entangled states with ϵ error

Definition (ϵ-Pseudo Standard Entanglement Structure)

If an ES $\mathcal{K}$ satisfies self-duality and ϵ-undistinguishability, we say that $\mathcal{K}$ is an ϵ-PSES.

- SES is an example of ϵ-PSES for $\epsilon \geq 0$

Q. Is there any other example of ϵ-PSES for small ϵ ? $\rightarrow$ Yes!

The existence of PSESs

- in order to apply main theorem 2, we construct an exact hierarchy of pre-dual cones
- in this paper, we construct $\mathcal{K}_r$ for a parameter $r > 0$
  - we show that $\mathcal{K}_r$ is pre-dual and $\mathcal{K}_r \supsetneq \mathcal{K}_{r'}$ for $r' < r \leq r_0$
  - $\mathcal{K}_0$ is equivalent to the SES
- $\tilde{\mathcal{K}}_r$ with each r is exactly different
- $D(\tilde{\mathcal{K}}_r) \leq D(\mathcal{K}_r^*) \leq 2\sqrt{\frac{2r}{2r+1}} \leq \epsilon$ for $r \leq \frac{\epsilon^2}{2(4-\epsilon^2)}$

$\rightarrow$ $\tilde{\mathcal{K}}_r$ is a ϵ-PSES.

Main Theorem 3

Given $\epsilon > 0$, there exists infinite models of ϵ-PSES.

Construction of pre-dual cones

$$\mathrm{MEOP}(A;B) := \left\{ \vec{E} = \{|\psi_k\rangle\langle\psi_k|\}_{k=1}^{d^2} \,\middle|\, \langle\psi_k|\psi_l\rangle = \delta_{kl}, \right.$$
$$\left. |\psi_k\rangle\langle\psi_k| : \text{ maximally entangled state on } \mathcal{H}_A \otimes \mathcal{H}_B \right\}.$$

$$\mathrm{NPM}_r(\mathcal{P}) := \left\{ \rho = -\lambda E_1 + (1+\lambda)E_2 + \frac{1}{2}\sum_{k=3}^{d^2} E_k \right|$$
$$\left. 0 \le \lambda \le r, \vec{E} = \{E_k\} \in \mathcal{P} \right\} \quad \text{for a subset } \mathcal{P} \subset \mathrm{MEOP}(A;B),$$
$$\mathcal{K}_r^{(0)}(\mathcal{P}) := \mathrm{SES}(A;B) + \mathrm{NPM}_r(\mathcal{P}),$$
$$\mathcal{K}_r(\mathcal{P}) := \left(\mathcal{K}_r^{(0)*}(\mathcal{P}) + \mathrm{NPM}_r(\mathcal{P}) \right)^* .$$

- Simply, $\mathcal{K}_r^*$ is convex hull of a subset of $\mathcal{T}_+$ and a subset of non-positive matrices with the above form
- for any $\mathcal{P} \subset \mathrm{MEOP}(A;B)$ and any $0 < r \le r_0$, $\mathcal{K}_r(\mathcal{P})$ is pre-dual

Non-orthogonal state discrimination in PSESs

Definition (state discrimination)

A family of states $\{\rho_k\}_{k=1}^n$ in $\mathcal{K}$ are perfectly distinguishable if there exists a measurement $\{M_k\}_{k=1}^n$ in $\mathcal{K}$ such that $\operatorname{Tr} \rho_k M_l = \delta_{kl}$.

- In SES, $\{\rho_k\}$ are perfectly distinguishable iff ρ_k are orthogonal, i.e., $\operatorname{Tr} \rho_k \rho_l = 0$ for $k \neq l$

Main theorem 4

In $\tilde{\mathcal{K}}_r(\mathcal{P}_0)$ with a certain subset $\mathcal{P}_0 \in \mathrm{MEOP}(A;B)$, there are a measurement $\{M_1, M_2\}$ and states ρ_1, ρ_2 such that $\operatorname{Tr} \rho_i M_j = \delta_{i,j}$ and $\operatorname{Tr} \rho_1 \rho_2 > 0$

$\rightarrow$ For any $\epsilon > 0$, there is an ϵ-PSES that contains a measurement discriminating two non-orthogonal states perfectly

- Conjecture: Any self-dual ES contains non-orthogonal distinguishable states except for the SES

Construction of the measurement and the state

$$\mathcal{P}_0 := \{\vec{P}, \vec{P}'\}, \quad P_1' = P_2, P_2' = P_1, P_k' = P_k \ (k \geq 3)$$

$$M_1(r;\vec{P}) = -rP_1 + (1+r)P_2 + \frac{1}{2}\sum_{k\geq 3} P_k,$$

$$M_2(r;\vec{P}) = (1+r)P_1 - rP_2 + \frac{1}{2}\sum_{k\geq 3} P_k,$$

$$\rho_1 := |\phi_1\rangle\langle\phi_1|, \quad \rho_2 := |\phi_2\rangle\langle\phi_2|,$$

$$|\phi_1\rangle := \sqrt{\frac{r}{2r+1}}|\psi_1\rangle + \sqrt{\frac{r+1}{2r+1}}|\psi_2\rangle,$$

$$|\phi_2\rangle := \sqrt{\frac{r+1}{2r+1}}|\psi_1\rangle + \sqrt{\frac{r}{2r+1}}|\psi_2\rangle.$$

$|\psi_k\rangle$:
eigen vector of P_k

Entanglement structures with group symmetry

$$\mathrm{GU}(A;B) := \{g \in \mathrm{GL}(\mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)) \mid g(\cdot) := U^\dagger(\cdot)U,$$
$$U \text{ is a unitary matrix on } \mathcal{H}_A \otimes \mathcal{H}_B\}.$$

$$\mathrm{LU}(A;B) := \{g \in \mathrm{GL}(\mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)) \mid g(\cdot) := (U_A^\dagger \otimes U_B^\dagger)(\cdot)(U_A \otimes U_B),$$
$$U_A, U_B \text{ are unitary matrices on } \mathcal{H}_A, \mathcal{H}_B\}$$

- G-symmetric cone $\mathcal{K}$: $g(x) \in \mathcal{K}$ for any $x \in \mathcal{K}$ and any $g \in G$.

Main theorem 5

For ES $\mathcal{K}$, $\mathcal{K}$ is $\mathrm{GU}(A;B)$-symmetric iff $\mathcal{K} = \mathrm{SES}$

- $\mathrm{GU}(A;B)$-symmetry is weaker condition than homogeneity for the derivation of Jordan Algebra
- Symmetric condition is essential for derivation of the SES
- However, operationally natural condition is $\mathrm{LU}(A;B)$-symmetry
- It is open whether there exists self-daul and $\mathrm{LU}(A;B)$-symmetric ES except for the SES

Summary and open problems

- There are many possibilities of ESs different from the SES.
- There exists infinite examples of ϵ-PSESs
  - self-duality
  - ϵ-undistinguishability

- Some ϵ-PSES can discriminate non-orthogonal states

OPEN. Any ϵ-PSES can discriminate non-orthogonal states except for the SES?

- $\mathrm{GU}(A;B)$-symmetry derives the SES from ESs

OPEN. there exists $\mathrm{LU}(A;B)$-symmetric ϵ-PSES except for the SES ?