Sustech-Nagoya workshop on Quantum Science

A 2-Categorical Interpretation of Reconstruction Theorem for Weak Hopf Algebras

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based on joint work with Zhi-Hao Zhang (in preparation).

The notion of weak Hopf algebras (WHAs) orginates from the motivation to describe certain symmetries in quantum field theory and operator algebras [Böhm-Nill-Szlachányi:99][Mack&Schomerus:92][Schomerus:95]. As a "quantum symmetry", WHAs along with their representations have wide applications in the studies of both quantum phenomena[Kitaev-Kong:1104.5047] and novel algebraic structures[Nikshych&Vainerman:00].

Our goal today is to give a new interpretation/generalisation of a basic theorem on WHAs. This theorem is called the *reconstruction theorem* (or Tannaka-Krein duality) for WHAs, of which the original version is due to Prof. Takahiro Hayashi[Hayashi:math/9904073][Ostrik:math/0111139], who is also from Nagoya University. This theorem establishes a strong relation between WHAs and their representations.

Outline

- Basics on WHAs.
- The reconstruction theorem.
- An interpretation.

Basics on WHAs

(Finite dim'l) Weak Hopf algebras $H(M, u, \Delta, \varepsilon, S)$

$$M : H \otimes H \to H \quad u \colon \mathbb{C} \to H$$

$$\Delta \colon H \to H \otimes H \quad \varepsilon \colon H \to \mathbb{C} \quad S \colon H \to H$$

- (H, M, u) is a (finite dimensional) associative algebra.
- (H, Δ, ε) is a coassociative coalgebra:

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \qquad (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta.$$

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- Δ preserves multiplication (where $\tau \colon H \otimes H \to H \otimes H, a \otimes b \mapsto b \otimes a$):

$$\Delta M = (M \otimes M)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta).$$

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$$\Delta M = (M \otimes M)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta).$$
- M, u, Δ and ε satisfy:
$$(\Delta \otimes \mathrm{id})\Delta u = (\mathrm{id} \otimes M \otimes \mathrm{id})(\Delta \otimes \Delta)(u \otimes u) = (\mathrm{id} \otimes (M \circ \tau) \otimes \mathrm{id})(\Delta \otimes \Delta)(u \otimes u)$$
$$\varepsilon M(M \otimes \mathrm{id}) = (\varepsilon \otimes \varepsilon)(M \otimes M)(\mathrm{id} \otimes \Delta \otimes \mathrm{id}) = (\varepsilon \otimes \varepsilon)(M \otimes M)(\mathrm{id} \otimes (\tau \circ \Delta) \otimes \mathrm{id}).$$

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- The antipode S is an algebra-antihomomorphism and also a coalgebra-antihomomorphism, subject to certain compatibility conditions.

A simple example: groupoid algebra

$\mathcal{G}$: a finite groupoid. Define a weak Hopf algebra $\mathbb{C}[\mathcal{G}] := \mathrm{span}\{g \mid g \in \mathrm{Mor}(\mathcal{G})\}$ with

$$M(g_1 \otimes g_2) = \begin{cases} g_1 g_2 & \text{if } g_1 \text{ can be left composed to } g_2; \\ 0 & \text{otherwise.} \end{cases}$$

$$u(1) = \sum_{a \in \mathrm{Ob}(\mathcal{G})} \mathrm{id}_a$$

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}.$$

Remark. When $\mathcal{G}$ has only one object, i.e. is a group, then $\mathbb{C}[\mathcal{G}]$ is the group algebra and is a basic example of *Hopf algebras*.

The two canonoical subalgebras

An interesting fact is that there are two canonical subalgebras of a WHA H. To see this, first note that there are two projections

$$p_L\colon H \to H,\ x \mapsto \varepsilon(x1_1)1_2$$

$$\text{and} \qquad p_R\colon H \to H,\ x \mapsto 1_1\varepsilon(1_2x),$$

where we have used Sweedler's notation $\Delta(1) \equiv 1_1 \otimes 1_2$. Then we have the following facts:

1. The images $H_L := p_L(H)$ and $H_R := p_R(H)$ are closed under multiplication and contains $1 \in H$, thus are **subalgebras** of H. Moreover, they are **separable**.
2. The algebras H_L and H_R are anti-isomorphic, i.e., $H_R \cong H_L^{\mathrm{op}}$.
3. The algebras H_L and H_R **mutually commute** with each other, i.e., for $x \in H_L$ and $y \in H_R$, we have $xy = yx$.

The two canonoical subalgebras (cont'd)

4. There is $S(H_L) = H_R, S(H_R) = H_L$, i.e., the antipode “transfers” the two subalgebras to each other.

The case: groupoid algebra

Consider the groupoid algebra $\mathbb{C}[\mathcal{G}]$. For $g\colon a \to b \in \mathrm{Mor}(\mathcal{G})$, the two projections read

$$p_L\colon g \mapsto \mathrm{id}_a$$
$$p_R\colon g \mapsto \mathrm{id}_b,$$

and consequently

$$H_L = H_R = \mathrm{span}\{\mathrm{id}_a \mid a \in \mathrm{Ob}(\mathcal{G})\} = \mathbb{C}^{\oplus |\mathrm{Ob}(\mathcal{G})|}.$$

The projections p_L and p_R are hence called the **source** and **target** maps respectively, while subalgebras H_L and H_R the **source** and **target** algebras.

Remark. In this case the source and target algebras are commutative algebras, however this is *not* true in a generic WHA.

The representation category of a WHA and the reconstruction theorem

The representation category of H

Let H be a WHA and let ${}_H\mathrm{Mod}$ denote the category of left modules over H. Then the aforementioned facts on the two subalgebras lead to the following statement:

Fact

There is a forgetful functor $U\colon {}_H\mathrm{Mod} \to {}_{H_L \otimes H_R}\mathrm{Mod} = {}_{H_L}\mathrm{Mod}_{H_L}$.

Here the functor U acts in the following way: given a left H-module V, the left $H_L \otimes H_R$-module $U(V)$ has commuting left actions $(x, v) \mapsto x.v$ and $(y, v) \mapsto y.v$ for $x \in H_L, y \in H_R$.

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Note that ${}_{H_L}\mathrm{Mod}_{H_L}$ is a monoidal category whose tensor product is given by the relative tensor product over H_L. Using axioms of WHA, one can also show that

Fact (non-trivial)

(1)The category ${}_H\mathrm{Mod}$ carries a monoidal structure; (2)The functor U is monoidal.

The reconstruction theorem

The reconstruction theorem

Now I have introduced the "$\rightsquigarrow$" in the following figure.

$$\boxed{\text{WHA } H} \rightsquigarrow \boxed{\begin{array}{c} \text{pair } ({}_H\mathrm{Mod}, F\colon {}_H\mathrm{Mod} \to {}_{H_L}\mathrm{Mod}_{H_L}) \\ {}_H\mathrm{Mod}\colon \text{ monoidal category} \\ H_L\colon \text{ separable algebra} \\ F\colon \text{ monoidal functor} \end{array}}$$

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Theorem (Reconstruction theorem for WHAs)

The "$\rightsquigarrow$" has an inverse.

The reconstruction theorem

What should the inverse of "$\rightsquigarrow$" achieve? The inverse of "$\rightsquigarrow$" should read

$$\boxed{\text{WHA } H} \leftsquigarrow \boxed{\begin{array}{c} \text{pair } (\mathcal{C}, F\colon \mathcal{C} \to {}_R\mathrm{Mod}_R) \\ \mathcal{C}\colon \text{ monoidal category} \\ R\colon \text{ separable algebra} \\ F\colon \text{ monoidal functor} \end{array}}.$$

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Now I briefly show you how "$\leftrightsquigarrow$" can be realized, i.e., "proof" the theorem.

The reconstruction theorem (cont'd)

Define a long forgetful functor:

$$\widetilde{F}\colon\ \mathcal{C} \xrightarrow{F} {}_R\mathrm{Mod}_R \xrightarrow{f} \mathrm{Vec}\ ,$$

where Vec denotes the category of $\mathbb{C}$-vector spaces and f forgets the R-bimodule action. Then there is a standard construction $H := \mathrm{End}(\widetilde{F})$ such that ${}_H\mathrm{Mod} = \mathcal{C}$ as categories, where $\mathrm{End}(\widetilde{F})$ denotes the algebra of endo-natural transformations of $\widetilde{F}$.

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It still remains to give the weak Hopf algebra structure on H. However, I will skip most of it, and only guide you to see that there are two subalgebras of H isomorphic to R and R^{op} respectively. For $r \in R$, define natural transformations $r^\sharp, r^\flat : \widetilde{F} \Rightarrow \widetilde{F}$ as follows:

$$(r^\sharp)_X\colon\ \widetilde{F}(X) \to \widetilde{F}(X) \qquad (r^\flat)_X\colon\ \widetilde{F}(X) \to \widetilde{F}(X)$$
$$v \mapsto r.v \qquad\qquad v \mapsto v.r,$$

where $X \in \mathcal{C}$, and the (left and right) actions of r on v is defined in the bimodule $F(X)$.

The reconstruction theorem (cont'd)

Define sets

$$R^\sharp := \{r^\sharp \mid r \in R\}, \quad R^\flat := \{r^\flat \mid r \in R\},$$

then it is not hard to see that $R^\sharp, R^\flat \subset \mathrm{End}(\widetilde{F}) = H$ are subalgebras isomorphic to R, R^{op} respectively.

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then it is not hard to see that $R^\sharp, R^\flat \subset \mathrm{End}(\widetilde{F}) = H$ are subalgebras isomorphic to R, R^{op} respectively. Not surprisingly, if we work out all structures of H, which allows us to compute H_L and H_R, then $H_L = R^\sharp$ and $H_R = R^\flat$. Thus we have

$$\boxed{U\colon {}_H\mathrm{Mod} \to {}_{H_L}\mathrm{Mod}_{H_L}} = \boxed{F\colon \mathcal{C} \to {}_R\mathrm{Mod}_R}.$$

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We have shown $\rightsquigarrow \circ \leftsquigarrow = \mathrm{id}$. The other direction is not hard to verify, and we leave it to the intrigued audience. Hence we have "proved" the reconstruction theorem, up to some technical issues such as that "monoidal category" really means "finite tensor category".

An interpretation

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I want to emphasize that this reconstruction theorem is essentially a 2-categorical phenomenon. Define the following 2-category $2\mathrm{Vec}$:

1. 0-,1-,2-morphisms are separable algebras, bimodules and bimodule maps respectively. More precisely, we define the 1-hom-category from R to R' to be ${}_{R'}\mathrm{Mod}_R$ for separable algebras R, R'.
2. Composition of 1-morphisms are given by relative tensor product. More precisely, the composition functor ${}_{R''}\mathrm{Mod}_{R'} \times {}_{R'}\mathrm{Mod}_R \to {}_{R'}\mathrm{Mod}_R$ is defined to be $\otimes_{R'}$.

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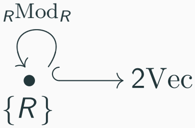
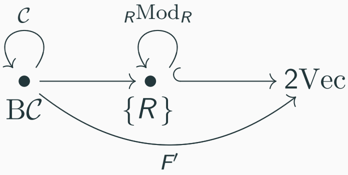
Then it is easy to see that the monoidal category ${}_R\mathrm{Mod}_R$ can be viewed as the looping of the one-object full sub-2-category $\{R\}$ of $2\mathrm{Vec}$, as is shown in the graph below.



An interpretation (cont'd)

In particular, the data of the pair $(\mathcal{C}, F\colon \mathcal{C} \to {}_R\mathrm{Mod}_R)$ are precisely the data of a pair $(\mathrm{B}\mathcal{C}, F'\colon \mathrm{B}\mathcal{C} \to 2\mathrm{Vec})$, where $\mathrm{B}\mathcal{C}$ and F' are defined in the following figure.



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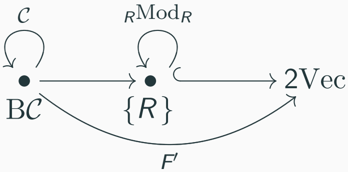

Thus it is equivalent to say that the reconstruction starts really from the pair $(\mathrm{B}\mathcal{C}, F')$. A natural question is: **can one reconstruct some algebraic data from more general 2-functors to $2\mathrm{Vec}$?**

Remark. Why do we care about $2\mathrm{Vec}$? The 2-category $2\mathrm{Vec}$ is 2-equivalent to the category of finite semisimple $\mathbb{C}$-linear categories, $\mathbb{C}$-linear functors and natural transformations, which deserves to be viewed as the categorification of Vec.

An interpretation (cont'd)

The answer is yes. Let $\mathcal{D}$ be a 2-category and let $U\colon \mathcal{D} \to 2\mathrm{Vec}$ be a 2-functor. We denote the hom-categories in $\mathcal{D}$ by $\{\mathcal{D}(a,b)\}_{a,b\in \mathrm{Ob}(\mathcal{D})}$, and the local 1-functors of U by $\{U_{a,b}\colon \mathcal{D}(a,b) \to {}_{U_b}\mathrm{Mod}_{U_a}\}_{a,b\in \mathrm{Ob}(\mathcal{D})}$. Then we have a 1-functor

$$\widetilde{U_{a,b}}: \ \mathcal{D}(a,b) \xrightarrow{U_{a,b}} {}_{U_b}\mathrm{Mod}_{U_a} \xrightarrow{f} \mathrm{Vec}$$

for objects $a, b \in \mathcal{D}$, where f denotes the functor forgeting the bimodule actions.

Then the standard construction tells us that the algebra $H_{a,b} := \mathrm{End}(\widetilde{U_{a,b}})$ satisfies that ${}_{H_{a,b}}\mathrm{Mod} = \mathcal{D}(a,b)$ as categories. Moreover, there are algebra homomorphisms $U_b \to H_{a,b}$ and $U_a^{\mathrm{op}} \to H_{a,b}$. As a matter of fact, the whole algebraic data reconstructed from U can be formulated.

Definition (first half)

A ***weak Hopf algebroid*** $\mathcal{H}$ consists of the following data:

1. A set of objects $\mathrm{Ob}(\mathcal{H})$;
2. For objects $a, b \in \mathrm{Ob}(\mathcal{H})$, there is an algebra $H_{a,b}$;
3. For objects $a, b, c \in \mathrm{Ob}(\mathcal{H})$, a *generalized comultiplication* $\Delta^{abc}\colon H_{a,c} \to H_{b,c} \otimes H_{a,b}$;
4. For object $a \in \mathrm{Ob}(\mathcal{H})$, a counit $\varepsilon^a\colon H_{a,a} \to \Bbbk$;
5. For objects $a, b \in \mathrm{Ob}(\mathcal{H})$, a *generalized antipode* $S_{a,b}\colon H_{a,b} \to H_{b,a}$,

which satisfy the following conditions for all $a, b, c, d \in \mathrm{Ob}(\mathcal{H})$

- Coassociativity and counitality hold (hence $\{H_{a,b}\}_{a,b\in\mathrm{Ob}(\mathcal{H})}$ forms a *cocategory*):

$$(\Delta^{bcd} \otimes \mathrm{id}) \circ \Delta^{abd} = (\mathrm{id} \otimes \Delta^{abc}) \circ \Delta^{acd}, \quad (\varepsilon^b \otimes \mathrm{id}) \circ \Delta^{abb} = \mathrm{id} = (\mathrm{id} \otimes \varepsilon^a) \circ \Delta^{aab};$$

Definition (second half)

- The generalized comultiplication $\Delta^{abc}\colon H_{a,c} \to H_{b,c} \otimes H_{a,b}$ preserves multiplication:
$$\Delta^{abc} M_{a,c} = (M_{b,c} \otimes M_{a,b})(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta^{abc} \otimes \Delta^{abc}),$$
where $M_{a,c}$ denotes the multiplication of $H_{a,c}$, etc.;
- The generalized comultiplications and counits satisfy other compatiblities:
$$(\mathrm{id} \otimes M_{b,c} \otimes \mathrm{id})(\Delta^{bcd} \otimes \Delta^{abc})(u_{b,d} \otimes u_{a,c}) = (\Delta^{bcd} \otimes \mathrm{id})\Delta^{abd} u_{a,d} =$$
$$(\mathrm{id} \otimes (M_{b,c}\tau) \otimes \mathrm{id})(\Delta^{bcd} \otimes \Delta^{abc})(u_{b,d} \otimes u_{a,c})$$
$$(M_{a,b} \otimes \varepsilon^a M_{a,a})(\mathrm{id} \otimes \Delta^{aab} u_{a,b} \otimes \mathrm{id}) = (\mathrm{id} \otimes \varepsilon^a M_{a,a})(\Delta^{aab} \otimes \mathrm{id})$$
$$(\varepsilon^a M_{a,a} \otimes M_{a,b})(\mathrm{id} \otimes (\tau \Delta^{aab} u_{a,b}) \otimes \mathrm{id}) = (\mathrm{id} \otimes \varepsilon^a M_{a,a})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{aab})$$
$$(\varepsilon^a M_{a,a} \otimes M_{c,a})(\mathrm{id} \otimes \Delta^{caa} u_{c,a} \otimes \mathrm{id}) = (\varepsilon^a M_{a,a} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{caa})$$
$$(\mathrm{id} \otimes \varepsilon^a M_{a,a})(\mathrm{id} \otimes (\tau \Delta^{caa} u_{c,a}) \otimes \mathrm{id}) = (\varepsilon^a M_{a,a} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\Delta^{caa} \otimes \mathrm{id});$$

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$$(\varepsilon^a M_{a,a} \otimes M_{a,b})(\mathrm{id} \otimes (\tau \Delta^{aab} u_{a,b}) \otimes \mathrm{id}) = (\mathrm{id} \otimes \varepsilon^a M_{a,a})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{aab})$$
$$(\varepsilon^a M_{a,a} \otimes M_{c,a})(\mathrm{id} \otimes \Delta^{caa} u_{c,a} \otimes \mathrm{id}) = (\varepsilon^a M_{a,a} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{caa})$$
$$(\mathrm{id} \otimes \varepsilon^a M_{a,a})(\mathrm{id} \otimes (\tau \Delta^{caa} u_{c,a}) \otimes \mathrm{id}) = (\varepsilon^a M_{a,a} \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\Delta^{caa} \otimes \mathrm{id});$$
- The antipode $S_{a,b}\colon H_{a,b} \to H_{b,a}$ is an algebra anti-homomorphism compatible with the comultiplications, and satisfies certain conditions.

Definition (second half)

- The generalized comultiplication $\Delta^{abc}\colon H_{a,c} \to H_{b,c} \otimes H_{a,b}$ preserves multiplication:
$$\Delta^{abc} M_{a,c} = (M_{b,c} \otimes M_{a,b})(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta^{abc} \otimes \Delta^{abc}),$$
where $M_{a,c}$ denotes the multiplication of $H_{a,c}$, etc.;
- The generalized comultiplications and counits satisfy other compatiblities:
$$(\mathrm{id} \otimes M_{b,c} \otimes \mathrm{id})(\Delta^{bcd} \otimes \Delta^{abc})(u_{b,d} \otimes u_{a,c}) = (\Delta^{bcd} \otimes \mathrm{id})\Delta^{abd} u_{a,d} =$$
$$(\mathrm{id} \otimes (M_{b,c}\tau) \otimes \mathrm{id})(\Delta^{bcd} \otimes \Delta^{abc})(u_{b,d} \otimes u_{a,c})$$
$$(M_{a,b} \otimes \varepsilon^{a} M_{a,a})(\mathrm{id} \otimes \Delta^{aab} u_{a,b} \otimes \mathrm{id}) = (\mathrm{id} \otimes \varepsilon^{a} M_{a,a})(\Delta^{aab} \otimes \mathrm{id})$$
$$(\varepsilon^{a} M_{a,a} \otimes M_{a,b})(\mathrm{id} \otimes (\tau \Delta^{aab} u_{a,b}) \otimes \mathrm{id}) = (\mathrm{id} \otimes \varepsilon^{a} M_{a,a})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{aab})$$
$$(\varepsilon^{a} M_{a,a} \otimes M_{c,a})(\mathrm{id} \otimes \Delta^{caa} u_{c,a} \otimes \mathrm{id}) = (\varepsilon^{a} M_{a,a} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^{caa})$$
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Remark. A one-object weak Hopf algebroid is a weak Hopf algebra.

Theorem

One can reconstruct a weak Hopf algebroid from a pair $(\mathcal{D}, U \colon \mathcal{D} \to 2\mathrm{Vec})$ where $\mathcal{D}$ is a 2-category and U is a 2-functor.

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Theorem (Part II)

Given a weak Hopf algebroid $\mathcal{H}$, there is a pair $(\mathrm{Mod}(\mathcal{H}), \mathcal{F}_{\mathcal{H}})$ where $\mathrm{Mod}(\mathcal{H})$ is a 2-category and $\mathcal{F}_{\mathcal{H}}\colon \mathrm{Mod}(\mathcal{H}) \to 2\mathrm{Vec}$ is a 2-functor.

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Given a weak Hopf algebroid $\mathcal{H}$, there is a pair $(\mathrm{Mod}(\mathcal{H}), \mathcal{F}_{\mathcal{H}})$ where $\mathrm{Mod}(\mathcal{H})$ is a 2-category and $\mathcal{F}_{\mathcal{H}}\colon \mathrm{Mod}(\mathcal{H}) \to 2\mathrm{Vec}$ is a 2-functor.

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This theorem generalizes the reconstruction theorem for WHAs. More precisely, if we take $\mathcal{D}$ to be a one-object 2-category, then we recover the latter. This provides a way for understanding the orginal theorem, especially the appearance of the category ${}_R\mathrm{Mod}_R$ there.

Some remarks on weak Hopf algebroids

- Let $\mathcal{H}$ be a weak Hopf algebroid. For any object $a \in \mathrm{Ob}(\mathcal{H})$, the algebra $H_{a,a}$ is a weak Hopf algebra. We denote the canonical subalgebra $(H_{a,a})_L \subset H_{a,a}$ by H_a.
- The data of the algebra homomorphisms $H_b \to H_{a,b}$ and $H_a^{\mathrm{op}} \to H_{a,b}$ can be obtained by structures of $\mathcal{H}$: they are the following maps restricted on H_b and H_a respectively

$$
\begin{aligned}
&p_L \colon H_{b,b} \to H_{a,b},\ x \mapsto \varepsilon^b(x 1_1^{ab}) 1_2^{ab} \\
\text{and} \quad &p_R \colon H_{a,a} \to H_{a,b},\ x \mapsto 1_1^{ab} \varepsilon^a(1_2^{ab} x).
\end{aligned}
$$

- Some examples of weak Hopf algebras can have its weak Hopf algebroid counterpart.
- In particular, the weak Hopf algebra studied in Kitaev and Kong (2012) is indeed a substructure of a weak Hopf algebroid, which can be used to describe more general boundary excitations in Levin-Wen models[Levin&Wen:cond-mat/0404617].

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