Probability distribution expressed by Racah hypergeometric orthogonal polynomial

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2021/06/24 13:00–13:50 (CST) 14:00–14:50 (JST)

SUSTech-Nagoya workshop on Quantum Science

Contents

Based on the collaboration [HHY]:

Masahito Hayashi (SUSTech/Nagoya), Akihito Hora (Hokkaido), S.Y., "Asymmetry of tensor product of asymmetric and invariant vectors arising from Schur-Weyl duality based on hypergeometric orthogonal polynomial", arXiv:2104.12635, 71pp.

Today, only mathematical (technical) part will be explained.

1. Conclusion and setting (9 pages)
   Based on §2 of our paper [HHY].
   1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$.
   1.2. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$.

–. Intermission

2. How to prove Main Theorem
3. Asymptotic behavior of $P_{n,m,k,l}$
4. Concluding remarks

1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$ (1/5)

The (generalized) hypergeometric series

$$ {}_{r+1}F_r\left[\begin{matrix} a_1,\ a_2,\ \ldots,\ a_{r+1} \\ b_1,\ b_2,\ \ldots,\ b_r \end{matrix}; z\right] := \sum_{i=0}^{\infty} \frac{(a_1)_i (a_2)_i \cdots (a_r)_i (a_{r+1})_i}{(b_1)_i (b_2)_i \cdots (b_r)_i (1)_i} z^i $$

with $(a)_i := a(a+1)\cdots(a+i-1)$ the rising factorial.

Theorem 1

Let $n, m, k, l \in \mathbb{Z}$ satisfy
$0 \leq 2m, k, l \leq n$, $M := m - l \geq 0$ and $N := n - m - k + l \geq 0$. Then

$$ p(x) := \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix}; 1\right] $$

gives a discrete probability distribution $P_{n,m,k,l}$ for $x \in \{0, 1, \ldots, n\}$.

$\binom{a}{k} := \frac{1}{k!} a(a-1)\cdots(a-k+1) \in \mathbb{Q}[a]$ for $k \in \mathbb{Z}_{\geq 0}$.

Our function again:

$$p(x) := \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix};1\right].$$

$\big(n, m, k, l \in \mathbb{Z},\ 0 \le 2m, k, l \le n,\ M := m-l \ge 0$ and $N := n-m-k+l \ge 0.\big)$

Immediate but non-trivial remarks:

- The ${}_4F_3$-term is expanded as $\displaystyle\sum_{i=0}^{M\wedge N} (-1)^i \frac{\binom{x}{i}\binom{n+1-x}{i}\binom{M}{i}\binom{N}{i}}{\binom{m}{i}\binom{n-m}{i}\binom{M+N}{i}}$.
  Theorem 1 says that this sum is non-negative for $0 \le x \le n$.
- Theorem 1 also says that the total sum is 1: $\sum_{x=0}^{n} p(x) = 1$, which is extended to a nontrivial identity in the next page.

1.1. Conclusion: The discrete probability distribution $P_{n,m,k,l}$ (3/5)

The probability distribution function (pdf) again:

$$P_{n,m,k,l}[X=x] = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix}-x, & x-n-1, & -M, & -N\\ -m, & m-n, & -M-N & \end{matrix};1\right].$$

$(n,m,k,l \in \mathbb{Z},\ 0 \le 2m, k, l \le n,\ M := m-l \ge 0$ and $N := n-m-k+l \ge 0.)$

Theorem 2

The cumulative distribution function (cdf) satisfies

$$P_{n,m,k,l}[X \le x] = \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} {}_4F_3\left[\begin{matrix}-x, & x-n, & -M, & -N\\ -m, & m-n, & -M-N & \end{matrix};1\right].$$

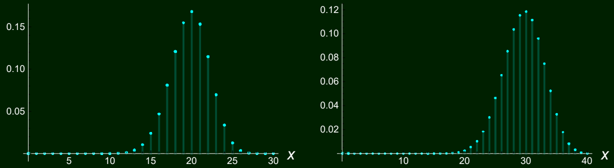
Moreover, we have $P[X \le m] = P[X \le m+1] = \cdots = P[X \le n] = 1.$

The ${}_4F_3$-term is expanded as $\displaystyle\sum_{i=0}^{M\wedge N} (-1)^i \frac{\binom{x}{i}\binom{n-x}{i}\binom{M}{i}\binom{N}{i}}{\binom{m}{i}\binom{n-m}{i}\binom{M+N}{i}}.$

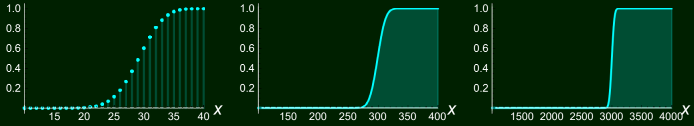
- For our distribution $P_{n,m,k,l}$, both pdf and cdf are ${}_4F_3$-series.
- There seems no distribution in literature whose pdf and cdf are both ${}_{r+1}F_r$-series.

distribution	pdf $\Pr[X=x]$	cdf $\Pr[X \le x]$
binomial	$\binom{n}{x} p^x (1-p)^x$	$\sim {}_1F_0\left[\begin{matrix} -n \\ - \end{matrix}; \frac{p}{1-p}\right]_{\le x}$
hypergeometric	$\binom{m}{x}\binom{n-m}{l-x}/\binom{n}{l}$	$\sim {}_3F_2\left[\begin{matrix} 1,\ x+1-m,\ x+1-l \\ x+2,\ n+x+2-m-l \end{matrix}; 1\right]$
our distribution	$\sim {}_4F_3\left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix}\right]$	$\sim {}_4F_3\left[\begin{matrix} -x,\ x-n,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix}; 1\right]$

($\sim$ denotes that some factor is suppressed.)



pdf $P_{n,m,k,l}[X = x]$ with
$(n, m, k, l) = (100, 30, 40, 20)$ in left and $(100, 40, 60, 30)$ in right.



cdf $P_{n,m,k,l}[X \leq x]$ with $(\frac{m}{n}, \frac{k}{n}, \frac{l}{n}) = (0.4, 0.6, 0.3)$,
$n = 100$ (left), 1000 (middle) and 10000 (right).

1.2. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $(\mathbb{C}^2)^{\otimes n}$ (1/4)

Consider the classical Schur-Weyl duality of SU(2) and $\mathfrak{S}_n$.

- $\mathrm{SU}(2) \curvearrowright \mathbb{C}^2$: the vector repr. of the special unitary group SU(2).
  $\mathrm{SU}(2) \curvearrowright (\mathbb{C}^2)^{\otimes n}$: the n-th fold tensor representation.
- $(\mathbb{C}^2)^{\otimes n} \curvearrowleft \mathfrak{S}_n$: permuting tensor factors by the symmetric group $\mathfrak{S}_n$.
- These two actions of SU(2) and $\mathfrak{S}_n$ commute:

$$\mathrm{SU}(2) \curvearrowright \mathcal{H} := (\mathbb{C}^2)^{\otimes n} \curvearrowleft \mathfrak{S}_n,$$

- The irreducible decomposition of the bimodule is

$$(\mathbb{C}^2)^{\otimes n} = \bigoplus_{x=0}^{\lfloor n/2 \rfloor} \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)}.$$

$\mathcal{U}_r$: the highest weight SU(2)-irrep of dimension r.
$\mathcal{V}_{(n-x,x)}$: the $\mathfrak{S}_n$-irrep corresponding to the partition $(n-x, x)$.

Examples of the irreducible decomposition of the SU(2)-$\mathfrak{S}_n$-bimodule

$$(\mathbb{C}^2)^{\otimes n} = \bigoplus_{x=0}^{\lfloor n/2 \rfloor} \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)},$$

for $n = 1, 2, 3$, using the basis $\mathbb{C}^2 = \mathbb{C}\,|0\rangle + \mathbb{C}\,|1\rangle$ and
$\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}E + \mathbb{C}F + \mathbb{C}H$, $E\,|0\rangle = |1\rangle$, $F\,|1\rangle = |0\rangle$, $H\,|b\rangle = (-1)^{b+1}\,|b\rangle$.

- $\mathbb{C}^2 = \mathcal{U}_2 \boxtimes \mathcal{V}_{(1,0)} = \mathbb{C}^2 \boxtimes \mathbb{C}_{\mathrm{triv}} = \mathbb{C}\,|0\rangle + \mathbb{C}\,|1\rangle$.
- $(\mathbb{C}^2)^{\otimes 2} = \mathcal{U}_3 \boxtimes \mathcal{V}_{(2,0)} \bigoplus \mathcal{U}_1 \boxtimes \mathcal{V}_{(1,1)} = \mathbb{C}^3 \boxtimes \mathbb{C}_{\mathrm{triv}} \bigoplus \mathbb{C} \boxtimes \mathbb{C}_{\mathrm{sgn}}$
  $= \big(\mathbb{C}\,|00\rangle + \mathbb{C}(|01\rangle + |10\rangle) + \mathbb{C}\,|11\rangle\big) \bigoplus \mathbb{C}(|01\rangle - |10\rangle)\big)$.
- $(\mathbb{C}^2)^{\otimes 3} = \mathcal{U}_4 \boxtimes \mathcal{V}_{(3,0)} \bigoplus \mathcal{U}_2 \boxtimes \mathcal{V}_{(2,1)} = \mathbb{C}^4 \boxtimes \mathbb{C}_{\mathrm{triv}} \bigoplus \mathbb{C}^2 \boxtimes \mathbb{C}^2_{\mathrm{std}}$
  $= \big(\mathbb{C}\,|000\rangle + \mathbb{C}(|001\rangle + \text{perm.}) + \mathbb{C}(|011\rangle + \text{perm.}) + \mathbb{C}\,|111\rangle\big)$
  $\bigoplus \Big[\ \ (\mathbb{C}(|001\rangle - |100\rangle) + \mathbb{C}(|011\rangle - |110\rangle))$
  $\oplus (\mathbb{C}(|010\rangle - |100\rangle) + \mathbb{C}(|011\rangle - |101\rangle))\Big]$.

1.2. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $(\mathbb{C}^2)^{\otimes n}$ (3/4)

- The decomp. $(\mathbb{C}^2)^{\otimes n} = \bigoplus_{x=0}^{\lfloor n/2 \rfloor} \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)}$ gives projectors

$$\mathrm{P}_x \colon \mathcal{H} = (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

Then any element $|v\rangle \in (\mathbb{C}^2)^{\otimes n}$, normalized for the standard hermitian pairing, gives a discrete probability

$$\Pr[X = x] := \langle v| \, \mathrm{P}_x \, |v\rangle \quad (x = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

- Our choice of the normalized element:

$$\left|\Xi_{n,m|k,l}\right\rangle := \left|1^l \, 0^{k-l}\right\rangle \otimes \left|\Xi_{n-k,m-l}\right\rangle \quad \in (\mathbb{C}^2)^{\otimes n},$$

$$\left|\Xi_{n-k,m-l}\right\rangle := \frac{1}{\binom{n-k}{m-l}^{1/2}} \sum_{w \in |1^{m-l} 0^{n-m-k+l}\rangle . \mathfrak{S}_{n-k}} w \quad \in (\mathbb{C}^2)^{\otimes (n-k)}.$$

1.2. Setting: The state $\Xi_{n,m|k,l}$ in the Schur-Weyl bimodule $(\mathbb{C}^2)^{\otimes n}$ (4/4)

Definitions again:

$$\mathsf{P}_x \colon (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

$$\left|\Xi_{n,m|k,l}\right\rangle := \left|1^l\, 0^{k-l}\right\rangle \otimes \frac{1}{\binom{n-k}{m-l}^{1/2}} \sum_{w \in \left|1^{m-l} 0^{n-m-k+l}\right\rangle . \mathfrak{S}_{n-k}} w \quad \in (\mathbb{C}^2)^{\otimes n}.$$

Main Theorem (coincise form of Theorem 1)

The discrete probability associated to $\left|\Xi_{n,m|k,l}\right\rangle$ coincides with $P_{n,m,k,l}$ in Theorem 1, i.e.,

$$\begin{aligned} &\left\langle \Xi_{n,m|k,l} \right| \mathsf{P}_x \left| \Xi_{n,m|k,l} \right\rangle \\ &= \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3\left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix} ; 1 \right] \end{aligned}$$

for $x = 0, 1, \ldots, \lfloor n/2 \rfloor$. $(M := m-l,\ N := n-m-k+l,\ M+N = n-k.)$

End of first half.

2. How to prove Main Theorem

1. Conclusion and setting
2. How to prove Main Theorem (7 pages)
   Based on §4 of our paper [HHY].
   2.1. Projector formula
   2.2. Gelfand pairs and zonal spherical functions.
   2.3. Hahn summation formula
   2.4. Main Theorem – Racah formula
3. Asymptotic behavior of $P_{n,m,k,l}$
4. Concluding remarks

2.1. Projector formula (1/2)

Recollection of Main Theorem

Using $M := m - l$ and $N := n - m - k + l$, define

$$\mathrm{P}_x \colon (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{U}_{n-2x+1} \boxtimes \mathcal{V}_{(n-x,x)} \quad (x = 0, 1, \ldots, \lfloor n/2 \rfloor).$$

$$\left|\Xi_{n,m|k,l}\right\rangle := \left|1^l\, 0^{k-l}\right\rangle \otimes \frac{1}{\binom{M+N}{M}^{1/2}} \sum_{w \in \left|1^M 0^N\right\rangle . \mathfrak{S}_{M+N}} w \quad \in (\mathbb{C}^2)^{\otimes n}.$$

Then we have

$$\left\langle \Xi_{n,m|k,l} \right| \mathrm{P}_x \left| \Xi_{n,m|k,l} \right\rangle$$
$$= \binom{n-k}{m-l} \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} {}_4F_3 \left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix} ; 1 \right].$$

We will calculate $\left\langle \Xi_{n,m|k,l} \right| \mathrm{P}_x \left| \Xi_{n,m|k,l} \right\rangle$ by $\mathfrak{S}_n$-representation theory.

2.1. Projector formula (2/2)

Regarding the decomposition as $\mathfrak{S}_n$-representation, we have

$$\mathsf{P}_x \colon (\mathbb{C}^2)^{\otimes n} \twoheadrightarrow \mathcal{V}_{(n-x,x)}{}^{\otimes \dim_{\mathbb{C}} \mathcal{U}_{n-\mathbf{2}x+\mathbf{1}}} = \mathcal{V}_{(n-x,x)}{}^{\otimes (n-2x+1)}.$$

To calculate $\left\langle \Xi_{n,m|k,l} \middle| \mathsf{P}_x \middle| \Xi_{n,m|k,l} \right\rangle$, we want some formula for P_x.

Representation theory of finite groups tells us:

Fact (projector formula)

Denoting by φ the $\mathfrak{S}_n$-action, we have

$$\mathsf{P}_x = \sum_{\sigma \in \mathfrak{S}_n} \frac{\dim_{\mathbb{C}} \mathcal{V}_{(n-x,x)}{}^{\otimes (n-2x+1)}}{|\mathfrak{S}_n|} \chi^{(n-x,x)}(\sigma)\, \varphi(\sigma)$$

with $\chi^{(n-x,x)}$ the character of the irreducible representation $\mathcal{V}_{(n-x,x)}$.

$\dim_{\mathbb{C}} \mathcal{V}_{(n-x,x)}$ is given by the well-known hook length formula.

Thus, we next want some formula for the part $\sum_\sigma \cdots \chi^{(n-x,x)}(\sigma)\varphi(\sigma)$.

2.2. Gelfand pairs and zonal spherical functions

Consider the subgroup $\mathfrak{S}_m \times \mathfrak{S}_{n-m} \subset \mathfrak{S}_n$.

The pair $(G, K) := (\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ is a Gelfand pair, i.e., the induced representation $\mathrm{Ind}_K^G \mathbb{C}_{\mathrm{triv}}$ has multiplicity free irreducible decomposition:
For this Gelfand pair, zonal spherical function $\omega_{(n-x,x)} \colon G \to \mathbb{C}$ is

$$\omega_{(n-x,x)}(g) := \frac{1}{|K|} \sum_{k \in K} \chi^{(n-x,x)}(kg^{-1}).$$

The value $\omega_{(n-x,x)}(g)$ depends only on the double coset KgK,
and we have the induced $\omega_{(n-x,x)} \colon K \backslash G / K \to \mathbb{C}$.

Fact [Delsarte 1973, 1978]

The set G/K, equipped with a certain distance function, has the structure of Johnson graph $J(n, m)$, which induces bijections

$$K \backslash G / K = \{K\text{-orbits of } J(n, m)\} = \{0, 1, \ldots, m\}.$$

2.3. Hahn summation formula

Zonal spherical function $\omega_{(n-x,x)}\colon K\backslash G/K \to \mathbb{C}$ is now totally determined by the values $\{\omega_{(n-x,x)}(i) \mid i = 0, 1, \ldots, m\}$.

Fact [Delsarte]

The value $\omega_{(n-x,x)}(i)$ is given by

$$\omega_{(n-x,x)}(i) = {}_3F_2\left[\begin{matrix}-i, & -x, & x-n-1 \\ & -m, & m-n\end{matrix}; 1\right] := \sum_{a\geq 0} \frac{(-i)_a(-x)_a(x-n-1)_a}{(1)_a(-m)_a(m-n)_a}.$$

The RHS is known as Hahn polynomial with variable i, degree x.

Hahn summation formula [HHY, Theorem 4.1.1]

Using $M := m - l$ and $N := n - m - k + l$, we have

$$\left\langle \Xi_{n,m|k,l} \right| \mathrm{P}_x \left| \Xi_{n,m|k,l} \right\rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \sum_{i=0}^{M\wedge N} \binom{M}{i}\binom{N}{i} \omega_{(n-x,x)}(i).$$

2.4. Main Theorem – Racach formula (1/2)

The Hahn summation formula is a double sum, and difficult to use for analysis.

$$\langle \Xi | \mathsf{P}_x | \Xi \rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \sum_{i=0}^{M \wedge N} \binom{M}{i} \binom{N}{i} {}_3F_2\left[\begin{matrix} -i, & -x, & x-n-1 \\ & -m, & m-n \end{matrix}; 1\right].$$

Racah formula (Main Theorem) [HHY, Theorem 4.2.1]

We have the following hypergeometric summation formula

$$\sum_{i=0}^{M \wedge N} \binom{M}{i} \binom{N}{i} {}_3F_2\left[\begin{matrix} -i, & -x, & x-n-1 \\ & -m, & m-n \end{matrix}; 1\right] = \binom{n-k}{m-l} R_x(M),$$

$$R_x(M) := {}_4F_3\left[\begin{matrix} -x, & x-n-1, & -M, & -N \\ -m, & m-n, & -M-N & \end{matrix}; 1\right].$$

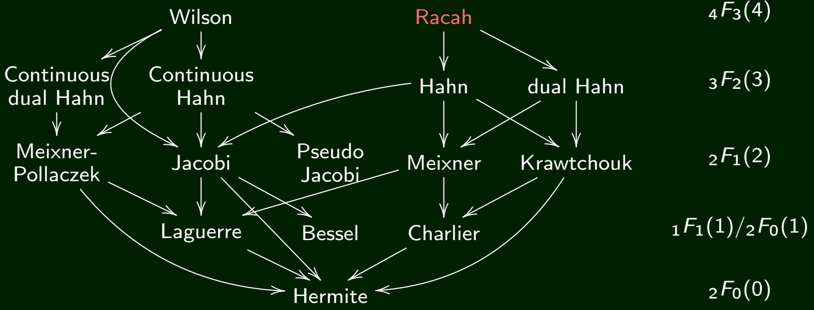
$R_x(M)$ is known as Racah polynomial. It yields Main Theorem:

$$\left\langle \Xi_{n,m|k,l} \right| \mathsf{P}_x \left| \Xi_{n,m|k,l} \right\rangle = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \binom{n-k}{m-l} R_x(M).$$

2.4. Main Theorem – Racah formula (2/2)

Profile of Racah polynomial

- Racah polynomial $R_x(z)$ of variable z and degree $x = 0, 1, \ldots, n$:
$$R_x(z; a, b, c, d) := {}_4F_3\left[\begin{matrix} -x,\ x+a+b+1,\ -z,\ z+c+d+1 \\ a+1,\ b+c+1,\ d+1 \end{matrix}; 1\right]$$
with $a + 1 = -n$ or $b + c + 1 = -n$ or $d + 1 = -n$.
- The family $\{R_x(z; a, b, c, d) \mid x = 0, 1, \ldots, n\}$ is orthogonal with respect to some discrete weight function $w(z)$: $\sum_{i=0}^{n} R_x(i) R_y(i) w(i) = \delta_{x,y}$.
- It sits in the top line of Askey scheme of hypergeometric orthogonal polynomials.



3. Asymptotic behavior of $P_{n,m,k,l}$

1. Conclusion and setting
2. How to prove Main Theorem (Racah formula)

$$P_{n,m,k,l}[X = x] = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \binom{n-k}{m-l} {}_4F_3\left[\begin{matrix} -x,\ x-n-1,\ -M,\ -N \\ -m,\ m-n,\ -M-N \end{matrix}; 1\right].$$

$(n, m, k, l \in \mathbb{Z},\quad 0 \leq 2m, k, l \leq n,\quad M := m - l \geq 0,\quad N := n - m - k + l \geq 0,$
$x \in \{0, 1, \ldots, n\}.)$

3. Asymptotic behavior of $P_{n,m,k,l}$ (3 pages)
   Based on §5 of our paper [HHY].
   3.1. What is Racah formula useful for?
   3.2. Central limit theorem
4. Concluding remarks

3.1. What is Racah formula useful for?

Racah polynomial R_x of degree x (and variable M) in Main Theorem

$$P_{n,m,k,l}[X=x] = \frac{\binom{n}{x}}{\binom{n}{m}} \frac{n-2x+1}{n-x+1} \binom{n-k}{m-l} R_x, \quad R_x := {}_4F_3\left[\begin{matrix} -x, & \dots \\ -m, & \dots \end{matrix}; 1\right]$$

is an orthogonal polynomial, and satisfies three-term recursive formula of the form $a_x R_{x+1} + b_x R_x + c_x R_{x-1} = 0$. It is rewritten as:

Three-term recursive formula [HHY, Lemma 4.3.3]

$p(x) = P_{n,m,k,l}[X=x]$ satisfies the recursive formula

$$A_x p(x+1) + B_x p(x) + C_x p(x-1) = 0,$$

$$A_x := \frac{(m-x)(n-m-x)(n-k-x)(n-x+1)}{(n-2x)(n-2x+1)} \frac{n-2x-1}{n-x} \frac{x+1}{n-x},$$

$$C_x := \frac{x(x-k-1)(m-x+1)(n-m-x+1)}{(n-2x+1)(n-2x+2)} \frac{n-2x+3}{n-x+2} \frac{x-1}{n-x+1}.$$

It enables us to do asymptotic analysis for $P_{n,m,k,l}$, $n \to \infty$.

3.2. Central limit theorem (1/2)

Consider the limit $n \to \infty$ with the ratios $\frac{m}{n}, \frac{k}{n}, \frac{l}{n}$ fixed. We use

$$\alpha = \frac{l}{n}, \quad \beta = \frac{m-l}{n}, \quad \gamma = \frac{k-l}{n}, \quad \delta = \frac{n-m-k+l}{n}.$$

Central limit theorem for generic type II limit [HHY, Thm 5.2.9]

In the above limit $n \to \infty$ with $\alpha + \gamma, \beta, \delta > 0$, we have

$$\lim_{n\to\infty} P_{n,m,k,l}\left[r \le \frac{X - n\mu}{\sqrt{n}\sigma} \le s\right] = \frac{1}{\sqrt{2\pi}} \int_r^s e^{-u^2/2}\, du$$

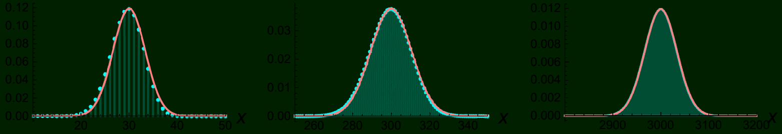
with μ and σ given by

$$\mu := \frac{1-\sqrt{D}}{2}, \quad \sigma := \sqrt{\frac{(\alpha+\gamma)\beta\delta}{D}}, \quad D := 1 - 4(\alpha\gamma + \alpha\delta + \beta\gamma).$$

We guessed the expectation value μ and the variance σ by taking a formal limit of the recursive formula $A_x p(x+1) + B_x p(x) + C_x p(x-1) = 0$ to get a differential equation

$$\frac{d}{dt} \log p(nt) \approx -\frac{t-\mu}{\sigma/\sqrt{n}} \quad (n \to \infty).$$

3.2. Central limit theorem (2/2)



Pdf $P_{n,m,k,l}[X = x]$ by cyan dots and the limit normal distribution by pink lines with $(\frac{m}{n}, \frac{k}{n}, \frac{l}{n}) = (0.4, 0.6, 0.3)$ fixed and $n = 100$ (left), 1000 (middle), 10000 (right). The limit distribution has $\mu = 0.3$ and $\sigma = 0.3354\ldots$.

4. Concluding remarks (1/2)

Conclusions again:

- We found a discrete probability distribution $P_{n,m,k,l}$ whose pdf is a Racah ${}_4F_3$-polynomial, and cdf is a ${}_4F_3$-polynomial. $\leftarrow$ the first (?) appearance of higher hypergeometric orthogonal polynomial in probability theory.
- Central limit theorem holds for generic type II limit: $n \to \infty$ with ratios $\frac{m}{n}, \frac{k}{n}, \frac{l}{n}$ fixed, satisfying a generic condition.

Topics in [HHY] not explained in this talk:

- Asymptotic analysis beyond central limit theorem [§5.5]
- Another limit of $P_{n,m,k,l}$: $n \to \infty$ with $\frac{m}{n}, k, l$ fixed. [§5.1]
- Meanings and applications in quantum information theory. [§1, §3]
- Computation using $\mathfrak{sl}_2$-Casimir operator. [§4.4, §5.5]
- q-analogue of the distribution $P_{n,m,k,l}$. [Appendix C]

4. Concluding remarks (2/2)

Logically we started with the distinguished element

$$\left|\Xi_{n,m|k,l}\right\rangle := \left|0^l 1^{k-l}\right\rangle \otimes \left|\Xi_{n-k,m-l}\right\rangle \in \mathcal{H} = (\mathbb{C}^2)^{\otimes n}$$

and succeeded in the computation of $\left\langle \Xi_{n,m|k,l}\right| \mathsf{P}_x \left|\Xi_{n,m|k,l}\right\rangle$, obtaining explicit and useful hypergeometric formulas.

However, at this moment, we do not have a conceptual reason why we were able to get nice formulas of the distribution.

Naive open problem

What property of the state $\left|\Xi_{n,m|k,l}\right\rangle$ enabled us to get nice formulas?

Is there some characterization of $\left|\Xi_{n,m|k,l}\right\rangle$ among all the normalized states of $\mathcal{H}$ so that the associated distribution can be expressed by a hypergeometric orthogonal polynomial?

(I expect some hidden "integrability" of the state $\left|\Xi_{n,m|k,l}\right\rangle$.)

Thank you for your attention.

Appendix: q-analogue of the distribution $P_{n,m,k,l}$

q-hypergeometric series and q-binomial coefficient:

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad [n]_q := 1+q+\cdots+q^{n-1},$$

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix}; q,\ z\right] := \sum_{i\geq 0} \frac{(a_1,\dots,a_r;q)_i}{(b_1,\dots,b_s;q)_i} z^i, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}.$$

[HHY, Theorems C.3.1, C.3.2]

Let $n,m,k,l \in \mathbb{Z}$ s.t. $0 \leq 2m, k, l \leq n$, $M := m-l, N := n-m-k+l \geq 0$. Then, for $q \in \mathbb{R}_{>0}$, the function

$$p(x|q) := \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q}{\begin{bmatrix} n \\ m \end{bmatrix}_q} q^x \frac{[n-2x+1]_q}{[n-x+1]_q} {}_4\phi_3\left[\begin{matrix} q^{-x}, & q^{x-n-1}, & q^{-M}, & q^{-N} \\ & q^{-m}, & q^{m-n}, & q^{-M-N} \end{matrix}; q,\ q\right]$$

defines a discrete probability distribution for $x \in \{0,1,\dots,n\}$, and

$$\sum_{u=0}^{x} p(x|q) = \begin{bmatrix} n-k \\ m-l \end{bmatrix}_q \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q}{\begin{bmatrix} n \\ m \end{bmatrix}_q} {}_4\phi_3\left[\begin{matrix} q^{-x}, & q^{x-n}, & q^{-M}, & q^{-N} \\ & q^{-m}, & q^{m-n}, & q^{-M-N} \end{matrix}; q,\ q\right].$$