The APS index theorem, domain-wall fermions, and global anomaly inflow

Shinichiroh Matsuo

2021-06-21 11:00–11:50 JST

SUSTech-Nagoya workshop on Quantum Science

Nagoya University, Japan

This talk is based on joint works

- arXiv:1910.01987 appeared in Comm. Math. Phys. 380 (2020)
- arXiv:2012.03543

of three mathematicians and three physicists:

- Mikio Furuta
- Mayuko Yamashita
- Shinichiroh Matsuo
- Hidenori Fukaya
- Tetsuya Onogi
- Satoshi Yamaguchi

Main theorem

Theorem (FFMOYY, CMP 2020)

For $m \gg 0$, we have a formula

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

- The Atiyah-Patodi-Singer index is expressed in terms of the η-invariant of domain-wall fermion Dirac operators.
- The original motivation comes from the bulk-edge correspondence of topological insulators in condensed matter physics and global anomaly inflow.
- The proof is based on a Witten localisation argument.
- We also have a formula for the mod-two APS index of real skew-adjoint operators. See arXiv:2012.03543.

Index and Eta

Let X be a closed manifold and $S \to X$ a hermitian bundle. Assume $\mathbf{dim}\, X$ is even. Assume S is $\mathbb{Z}/2$-graded: there exists $\gamma\colon \Gamma(S) \to \Gamma(S)$ such that $\gamma^2 = \mathbf{id}_S$.

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $D\colon \Gamma(S) \to \Gamma(S)$ be a 1st order elliptic differential operator. Assume D is odd and self-adjoint:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \text{ and } D_- = (D_+)^*.$$

Definition

$$\begin{aligned} \operatorname{Ind} D &:= \dim \operatorname{Ker} D_+ - \dim \operatorname{Ker} D_- \\ &= \dim \operatorname{Ker} D_+ - \dim \operatorname{Coker} D_+ \end{aligned}$$

Fix $m \in \mathbb{R} \setminus \{0\}$ and consider

$$D + m\gamma = \begin{pmatrix} m & D_- \\ D_+ & -m \end{pmatrix} : \Gamma(S) \to \Gamma(S).$$

This is self-adjoint but no longer odd; thus, its spectrum is real but not symmetric around 0. For $\mathrm{Re}(z) \gg 0$, let

$$\eta(D + m\gamma)(z) := \sum_{\lambda_j} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z},$$

where $\{\lambda_j\} = \mathrm{Spec}(D + m\gamma)$. Note that $\lambda_j \neq 0$ for any j.

Definition

$$\eta(D + m\gamma) := \eta(D + m\gamma)(0).$$

The eta invariant describes the overall asymmetry of the spectrum of a self-adjoint operator.

Proposition

For any $m > 0$, we have a formula

$$\operatorname{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

This formula might be unfamiliar; however, we can prove it easily, for example, by diagonalising D^2 and γ simultaneously. We will explain another proof later.

We will generalise this formula to handle compact manifolds with boundary and the Atiyah-Patodi-Singer index.

Proposition

For any $m > 0$, we have a formula

$$\mathrm{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

We will generalise this formula to handle compact manifolds with boundary and the Atiyah-Patodi-Singer index by using domain-wall fermion Dirac operators.

Theorem (FFMOYY, CMP 2020)

For $m \gg 0$, we have a formula

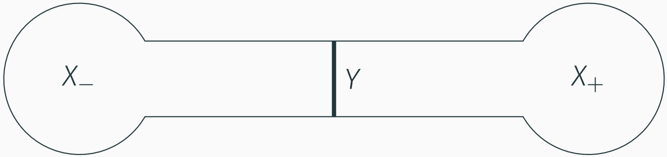
$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

Next, we review the Atiyah-Patodi-Singer index.

The Atiyah-Patodi-Singer index

Let $Y \subset X$ be a separating submanifold that decomposes X into two compact manifolds X_+ and X_- with common boundary Y. Assume Y has a collar neighbourhood isometric to $(-4, 4) \times Y$.

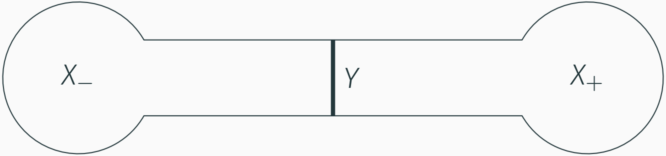
$$(-4, 4) \times Y \subset X = X_- \bigcup_Y X_+$$



Assume $S \to X$ and $D\colon \Gamma(S) \to \Gamma(S)$ are standard on $(-4, 4) \times Y$ in the sense that there exists a hermitian bundle $E \to Y$ and a self-adjoint elliptic operator $A\colon \Gamma(E) \to \Gamma(E)$ such that $S = \mathbb{C}^2 \otimes E$ and

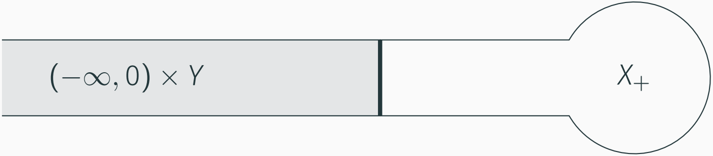
$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix}$$

on $(-4, 4) \times Y$.



Assume also A has no zero eigenvalues.

Let $\widehat{X_+} := (-\infty, 0] \times Y \cup X_+$.



We assumed D is translation invariant on $(-4, 4) \times Y$:

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix}.$$

Thus, $D|_{X_+}$ naturally extends to $\widehat{X_+}$, which is denoted by $\widehat{D}$.

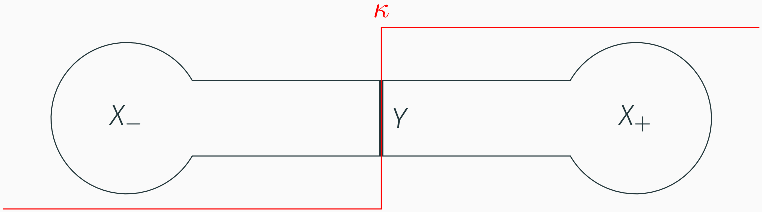
This is Fredholm if A has no zero eigenvalues.

Definition (Atiyah-Patodi-Singer index)

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) := \mathrm{Ind}(\widehat{D})$$

Domain-wall fermion Dirac operators

Let $\kappa \colon X \to \mathbb{R}$ be a step function such that $\kappa \equiv \pm 1$ on $X_{\pm}$.



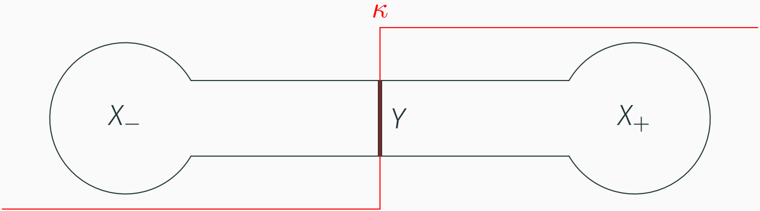
Definition

For $m > 0$,

$$D + m\kappa\gamma \colon \Gamma(S) \to \Gamma(S)$$

is called a domain-wall fermion Dirac operator.

$D + m\kappa\gamma$ is self-adjoint but not odd.



$$D = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix} \text{ on } (-4,4) \times Y$$

Proposition

If $\mathrm{Ker}\, A = \{0\}$, *then* $\mathrm{Ker}(D + m\kappa\gamma) = \{0\}$ *for* $m \gg 0$.

Next we will define $\eta(D + m\kappa\gamma)$.

The eta invariant of domain-wall fermion Dirac operators

Since $\mathrm{Ker}(D + m\kappa\gamma) = \{0\}$, there exists a constant $C_m > 0$ such that $\mathrm{Ker}(D + m\kappa\gamma + f) = \{0\}$ if $\|f\|_2 < C_m$.

Corollary of the variational formula of the eta invariant

Assume both $m\kappa\gamma + f_1$ and $m\kappa\gamma + f_2$ are smooth with $\|f_1\|_2 < C_m$ and $\|f_2\|_2 < C_m$. Then, we have

$$\eta(D + m\kappa\gamma + f_1) = \eta(D + m\kappa\gamma + f_2).$$

Definition

For any f with $\|f\|_2 < C_m$ and $m\kappa\gamma + f$ smooth, we set

$$\eta(D + m\kappa\gamma) := \eta(D + m\kappa\gamma + f).$$

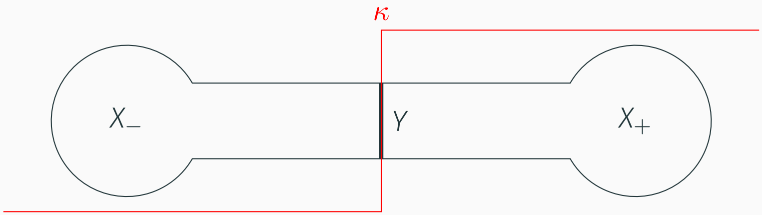
Main theorem

Main theorem

Theorem (FFMOYY, CMP 2020)

For $m \gg 0$, we have a formula

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$



- The Atiyah-Patodi-Singer index is expressed in terms of the η-invariant of domain-wall fermion Dirac operators.
- The original motivation comes from physics.
- The proof is based on a Witten localisation argument.

The proof of a toy model

Toy model

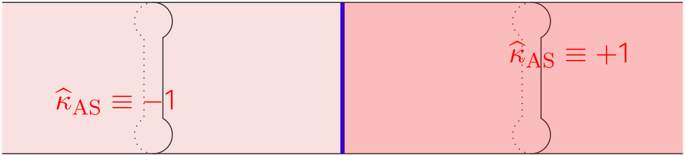
Proposition

For any $m > 0$, we have a formula

$$\mathrm{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

As a warm-up, we will prove this formula in the spirit of our proof of the main theorem.

Let $\widehat{\kappa}_{\mathrm{AS}} \colon \mathbb{R} \times X \to \mathbb{R}$ be a step function such that $\widehat{\kappa}_{\mathrm{AS}} \equiv 1$ on $(0, \infty) \times X$ and $\widehat{\kappa}_{\mathrm{AS}} \equiv -1$ on $(-\infty, 0) \times X$.



We consider $\widehat{D}_m \colon L^2(\mathbb{R} \times X; S \oplus S) \to L^2(\mathbb{R} \times X; S \oplus S)$ defined by

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

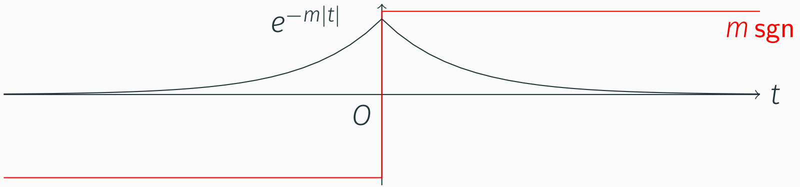
This is a Fredholm operator.

Model case: the Jackiw-Rebbi solution on $\mathbb{R}$

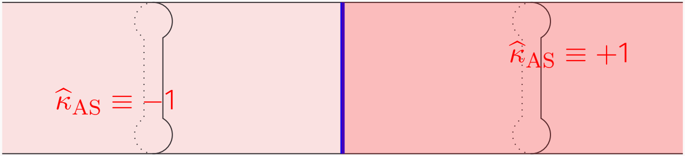
For any $m > 0$, we have

$$\frac{d}{dt} e^{-m|t|} = -m \,\mathrm{sgn}\, e^{-m|t|},$$

where $\mathrm{sgn}(\pm t) = \pm 1$. As $m \to \infty$, the solution concentrates at 0.



$$\begin{pmatrix} 0 & \partial_t + m\,\mathrm{sgn} \\ -\partial_t + m\,\mathrm{sgn} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ e^{-m|t|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$



$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) - \partial_t & 0 \end{pmatrix}$$

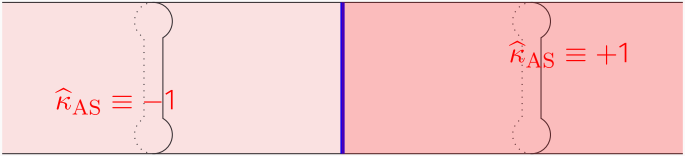
$$(e^{-m|t|})' = -m \operatorname{sgn} e^{-m|t|}$$

Proposition (Product formula)

$$\mathrm{Ind}(D) = \mathrm{Ind}(\widehat{D}_m)$$

Assume $D\phi = 0$. Set $\phi_\pm := (\phi \pm \gamma\phi)/2$. Then, we have

$$\begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) - \partial_t & 0 \end{pmatrix} \begin{pmatrix} e^{-m|t|}\phi_- \\ e^{-m|t|}\phi_+ \end{pmatrix} = 0.$$



$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) - \partial_t & 0 \end{pmatrix}$$

Proposition (APS formula)

$$\operatorname{Ind}(\widehat{D}_m) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}$$

- Note that $D + m\widehat{\kappa}_{\mathrm{AS}}(\pm 1, \cdot)\gamma = D \pm m\gamma$.
- Perturb $\widehat{\kappa}_{\mathrm{AS}}$ slightly near $\{0\} \times X$ to get a smooth operator.
- Use the Atiyah-Patodi-Singer index theorem on $\mathbb{R} \times X$.
- Since $\dim \mathbb{R} \times X$ is odd, the constant term in the asymptotic expansion of the heat kernel vanishes.

Proposition

$$\mathrm{Ind}(D) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\gamma) - \partial_t & 0 \end{pmatrix}$$

By the product formula, we have

$$\mathrm{Ind}(D) = \mathrm{Ind}(\widehat{D}_m).$$

By the APS formula, we have

$$\mathrm{Ind}(\widehat{D}_m) = \frac{\eta(D + m\gamma) - \eta(D - m\gamma)}{2}.$$

The proof of the main theorem

Outline of the proof

Theorem (FFMOYY, CMP 2020)

For $m \gg 0$, we have a formula

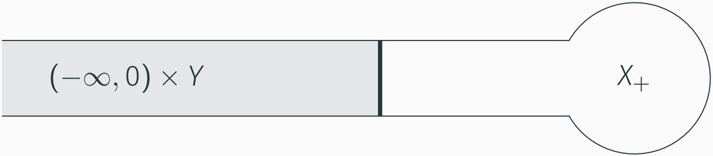
$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

The proof is modelled on the original embedding proof of the Atiyah-Singer index theorem.

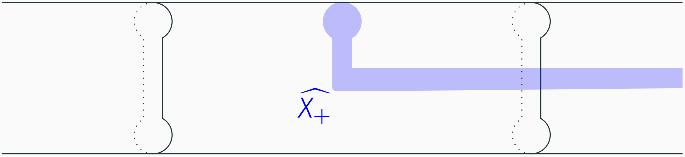
1. Embed $\widehat{X_+}$ into $\mathbb{R} \times X$.
2. Extend both $\widehat{D}$ on $\widehat{X_+}$ and $D + m\kappa\gamma$ on $\{10\} \times X$ to $\mathbb{R} \times X$.
3. Use the product formula, the APS formula, and a Witten localisation argument.

Embedding of $\widehat{X_+}$ into $\mathbb{R} \times X$

$\widehat{X_+} := (-\infty, 0] \times Y \cup X_+$.

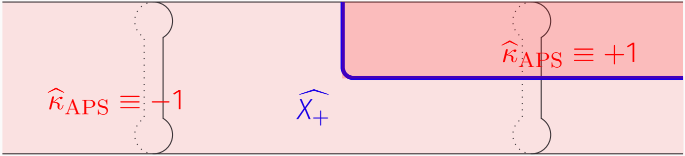


We can embed $\widehat{X_+}$ into $\mathbb{R} \times X$ as follows:



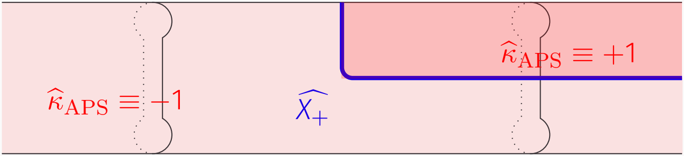
Extension of $\widehat{D}$ and $D + m\kappa\gamma$ to $\mathbb{R} \times X$

$(\mathbb{R} \times X) \setminus \widehat{X_+}$ has two connected components. We denote by $(\mathbb{R} \times X)_-$ the one containing $\{-10\} \times X_+$ and by $(\mathbb{R} \times X)_+$ the other half. Let $\widehat{\kappa}_{\mathrm{APS}} \colon \mathbb{R} \times X \to [-1, 1]$ be a step function such that $\widehat{\kappa}_{\mathrm{APS}} \equiv \pm 1$ on $(\mathbb{R} \times X)_\pm$.



We consider

$$\widehat{\mathcal{D}}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

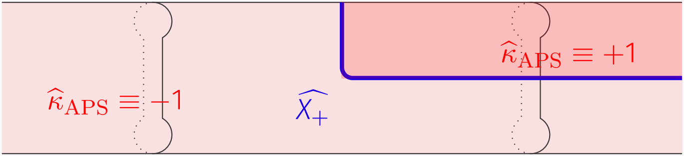


$$\widehat{\mathcal{D}}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

$$\widehat{\kappa}_{\mathrm{APS}} \equiv \kappa \text{ on } \{10\} \times X.$$

Proposition (APS formula)

$$\mathrm{Ind}(\widehat{\mathcal{D}}_m) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}$$



$$\widehat{\mathcal{D}}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) + \partial_t \\ (D + m\widehat{\kappa}_{\mathrm{APS}}\gamma) - \partial_t & 0 \end{pmatrix}.$$

The restriction of $\widehat{\mathcal{D}}_m$ to a tubular neighbourhood of $\widehat{X_+}$ is isomorphic to

$$\begin{pmatrix} 0 & (\widehat{D} + m\,\mathbf{sgn}\,\gamma) + \partial_t \\ (\widehat{D} + m\,\mathbf{sgn}\,\gamma) - \partial_t & 0 \end{pmatrix}$$

on $\mathbb{R} \times \widehat{X_+}$ near $\{0\} \times \widehat{X_+}$, where $\widehat{D}$ is the extension of $D|_{X_+}$ to $\widehat{X_+}$.

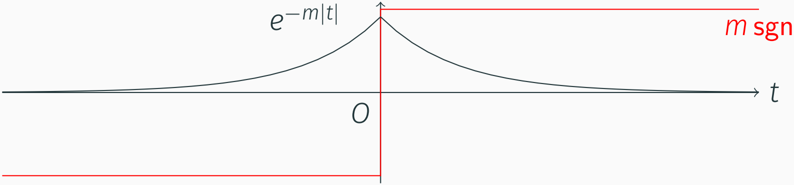
Witten localisation

Theorem (Witten localisation)

For $m \gg 0$, we have

$$\mathrm{Ind}(\widehat{\mathcal{D}}_m) = \mathrm{Ind}\begin{pmatrix} 0 & (\widehat{D} + m\,\mathrm{sgn}\,\gamma) + \partial_t \\ (\widehat{D} + m\,\mathrm{sgn}\,\gamma) - \partial_t & 0 \end{pmatrix}.$$

The proof is too technical to state here, but the idea is simple.



$$\begin{pmatrix} 0 & \partial_t + m\,\mathrm{sgn} \\ -\partial_t + m\,\mathrm{sgn} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ e^{-m|t|} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Proposition (Product formula)

$$\operatorname{Ind}\begin{pmatrix} 0 & (\widehat{D} + m \operatorname{sgn}\gamma) + \partial_t \\ (\widehat{D} + m \operatorname{sgn}\gamma) - \partial_t & 0 \end{pmatrix} = \operatorname{Ind}(\widehat{D})$$

Theorem (FFMOYY, CMP 2020)

For $m \gg 0$, we have a formula

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$

By definition, we have $\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \mathrm{Ind}(\widehat{D})$.
By the product formula, we have

$$\mathrm{Ind}(\widehat{D}) = \mathrm{Ind}\begin{pmatrix} 0 & (\widehat{D} + m\,\mathrm{sgn}\,\gamma) + \partial_t \\ (\widehat{D} + m\,\mathrm{sgn}\,\gamma) - \partial_t & 0 \end{pmatrix}.$$

By the Witten localisation argument, for $m \gg 0$, we have

$$\mathrm{Ind}\begin{pmatrix} 0 & (\widehat{D} + m\,\mathrm{sgn}\,\gamma) + \partial_t \\ (\widehat{D} + m\,\mathrm{sgn}\,\gamma) - \partial_t & 0 \end{pmatrix} = \mathrm{Ind}(\widehat{\mathcal{D}}_m).$$

By the APS formula, we have

$$\mathrm{Ind}(\widehat{\mathcal{D}}_m) = \frac{\eta(D + m\kappa\gamma) - \eta(D - m\gamma)}{2}.$$