No-go theorems for quantum resource purification: Universal theories and practical applications

Zi-Wen Liu
Perimeter Institute

SUSTech-Nagoya Workshop on Quantum Science

Joint work with Kun Fang (Institute for Quantum Computing, Baidu Research)

[1] arXiv:1909.02540; Phys. Rev. Lett. 125, 060405 (2020) Featured in Physics Editors' Suggestion

[2] arXiv:2010.11822

Outline

General results (state & channel)

- Universal limits on the accuracy of approximate quantum resource purification
- Logarithmic lower bounds on distillation cost

Approaches

- Quantum hypothesis testing monotone [1]
- Free component/resource weight [2]

Applications

- *Costs of magic state distillation and fault-tolerant quantum computing [1][2]
- *Accuracy of constrained quantum error correction [1][2]
- Noisy quantum circuit synthesis/compiling [2]
- Quantum communication/Shannon theory [2]

Metrology 2005.11918
Random codes 2102.11835

Background

- Quantum technologies (computing, communication, sensing…) potentially provide revolutionary advantages over conventional methods, if reliably scaled up
- Scientific experiments demand more and more precise controls over quantum systems

Noise is a notorious obstacle…

- Quantum systems and information is highly susceptible to noise effects: undesirable interactions with environments, imperfect controls, unstable memories…
- Generically make the system mixed, so that the system becomes unreliable for usage or lose power
- "Purifying" noisy q. systems — q. error correction, error mitigation, distillation… How well can they be done in practice?

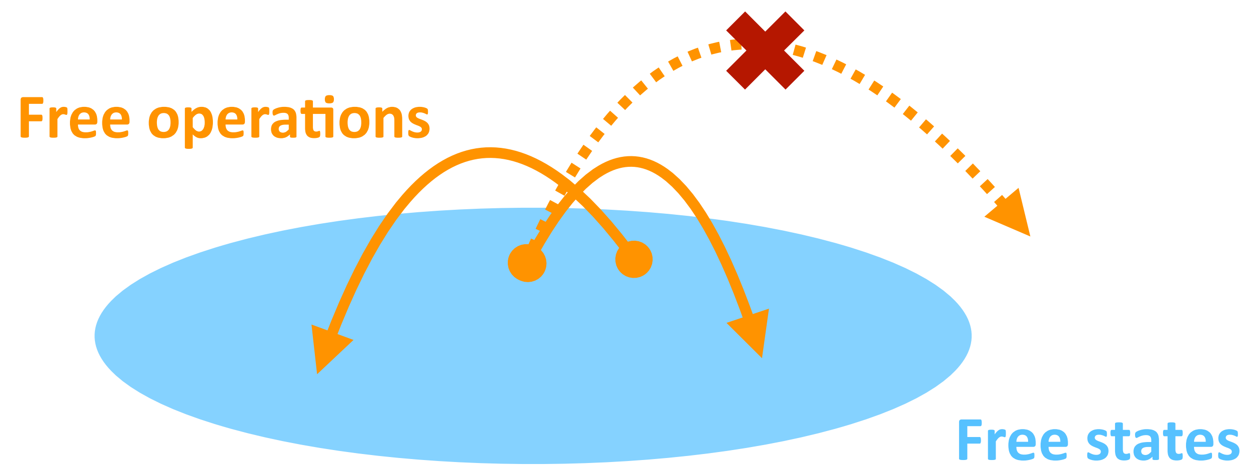
Constraints induce resources

- Practical scenarios always entail physical limitations, inducing values of "resources"
- Important QI examples:
  - For "distant labs" paradigm, LOCC is cheap; Entanglement emerge as "resource" since it can enable q. communication & cannot be created by LOCC
  - For fault-tolerance and classical simulation, stabilizer circuits are cheap; Magic states enable universal quantum computation
- Many other forms: coherence/superposition, asymmetry, athermality, Bell nonlocality, discord, steering, non-Gaussianity, "uncomplexity"...
- Manipulation of quantum resources underlies all kinds of QIP tasks

Resource theory

- "Resource theory" provides a universal, rigorous language: Manipulation of resources under "free" operations (see e.g. Chitambar/Gour RMP'19 for intro)
- Building blocks: Free states (objects) & Free operations (morphisms)
- **Golden rule** of free operation: Maps free state to free state
  - "Resource non-generating" operations
  - Includes all possible free operations (minimal condition of a nontrivial theory)

  ⇒ What is impossible under this condition is always impossible



Resource purification/distillation

Free operations

$$\rho \xrightarrow{\quad\quad} \approx \quad \psi$$ (One-shot)

(Probabilistic)

Noisy resource **Pure resource**

Later: Extension to channel (dynamical) theory

A very general type of task in practical QIP

Resource purification/distillation

Free operations

$$\rho \quad \xrightarrow{\approx} \quad \psi$$

(One-shot)

(Probabilistic)

Noisy resource **Pure resource**

No go!

Universal limits for

- Any well-defined resource theory
- Any pure non-free target
- Any free manipulation

Practical issues in QIP

- Noise
- Resource theory constraints
- Finite resources (One-shot setting)

Approach I: Divergence monotone

- Basic axiom of resource measures
  - Monotonicity: $$f(\mathcal{E}(\rho)) \leq f(\rho)$$

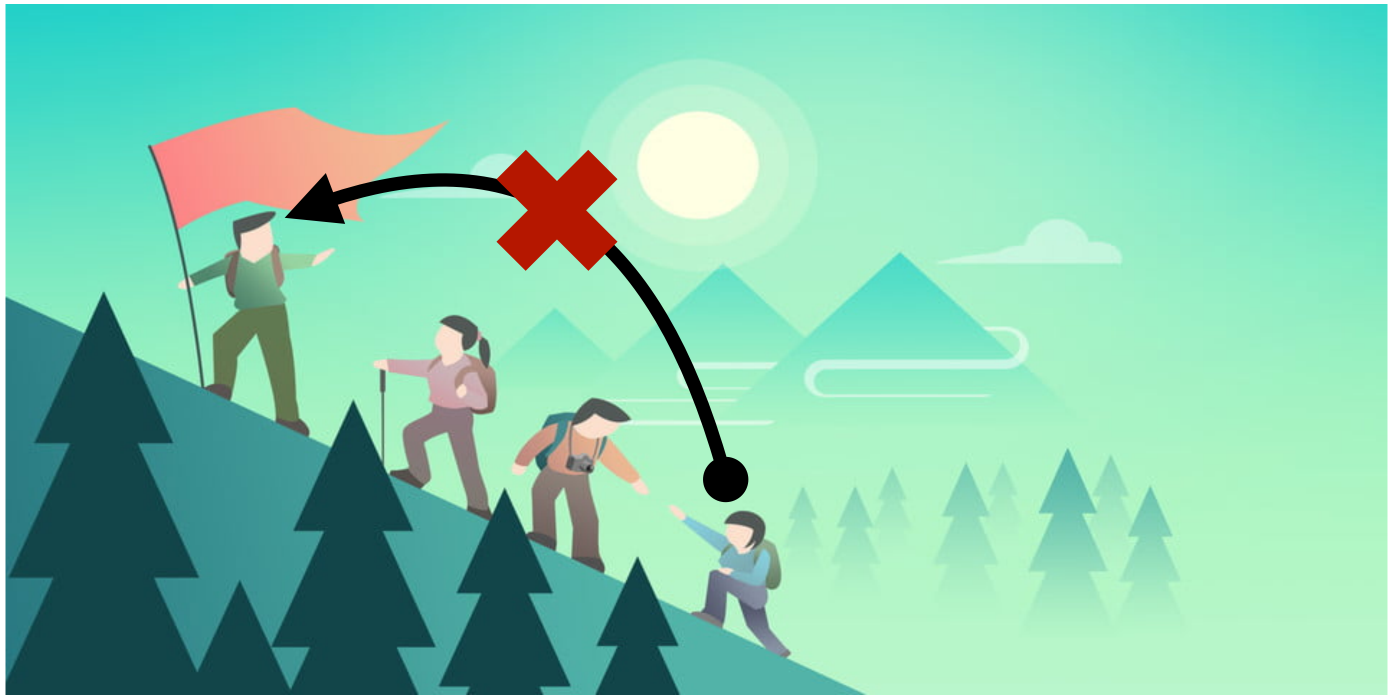
Any free operation

f: Resource measure

State $\longrightarrow \mathbb{R}$

e.g. entanglement entropy

- (Different) Monotones induce (different) partial orders



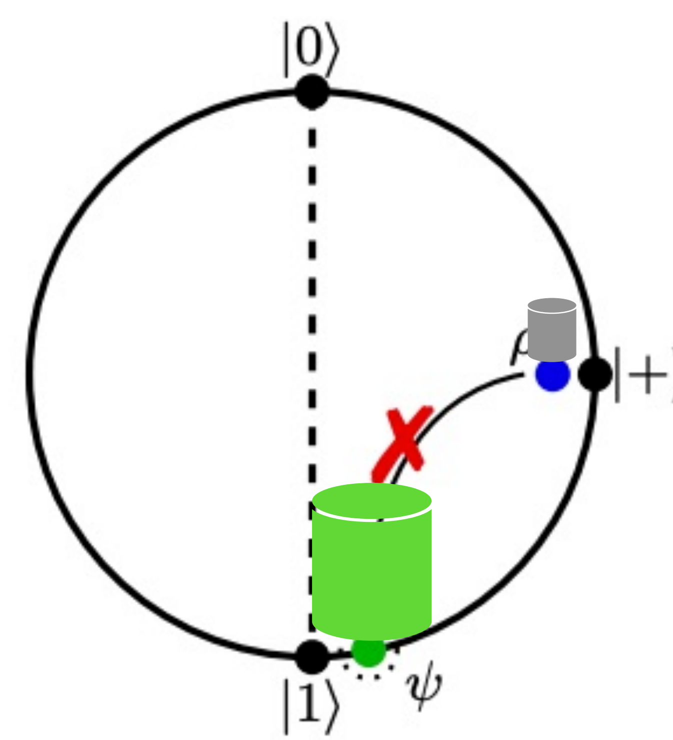
Approach I: Divergence monotone

- A peculiar type of general monotone ZWL/Bu/Takagi PRL'19 induced by "quantum hypothesis testing relative entropy" (a "smoothed" min-relative entropy) Buscemi/Datta TIT'10, Wang/Renner PRL'12

$$D_H^\epsilon(\rho\|\sigma) := \max_{0\le P\le I, \mathrm{Tr}\{P\rho\}\ge 1-\epsilon} (-\log \mathrm{Tr}\{P\sigma\})$$

  - Monotonicity due to data-processing inequality
  - Arises from one-shot distillation rates in general theories

- Consider min-entropy monotone: $\epsilon = 0$
  - Zero for noisy states
  - Non-zero for pure resource states
  - Smoothing → "Good" continuity



Approach I: Divergence monotone

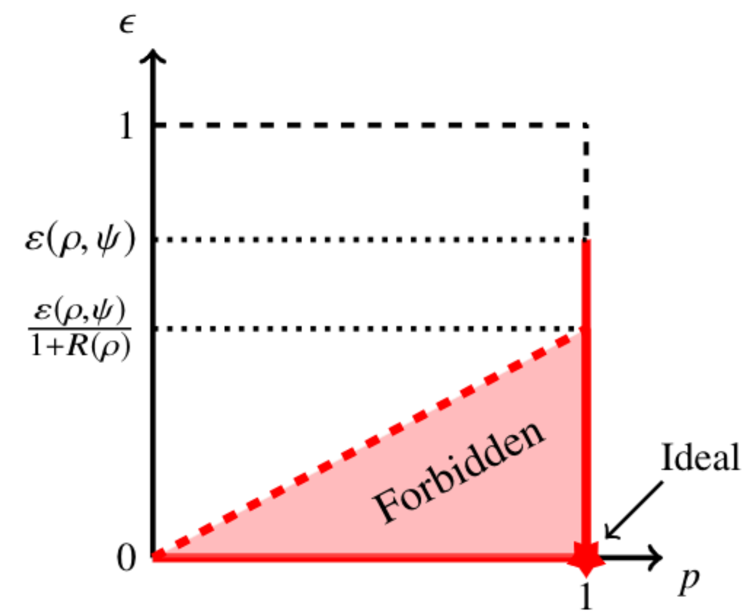
- With a little tweak we can also deal with probabilistic case (important in magic state distillation etc.)

Quantitative result (full-rank input):

An “uncertainty relation” (trade-off bound) between accuracy and success probability

Overlap with free states

$$\varepsilon(\rho, \psi) = \lambda_{\min}(\rho)(1 - f_\psi)$$



A take-home message

There are generic fundamental limits on (even probabilistic) quantum resource purification.

Distillation overhead

Widely studied setting: Given a supply of raw resources, **"distill"** high-quality target states

- A widely studied subroutine in QIP
  Entanglement/coherence distillation: QKD, communication, metrology…
  Magic state distillation: fault-tolerant quantum computing (later)

- Aim is to minimize the cost/overhead of the task

Free operations

$$\hat{\rho}^{\otimes n} \longrightarrow \approx \quad \psi$$

Noisy resource   Pure resource

An important figure of merit is the asymptotic scaling as $\epsilon \to 0$

Our bound implies $\quad n \geq \Omega(\log^{\gamma}(1/\epsilon)), \quad \gamma = 1$

⇒ A fundamental "price" for scalable FTQC and q.techs based on distillation

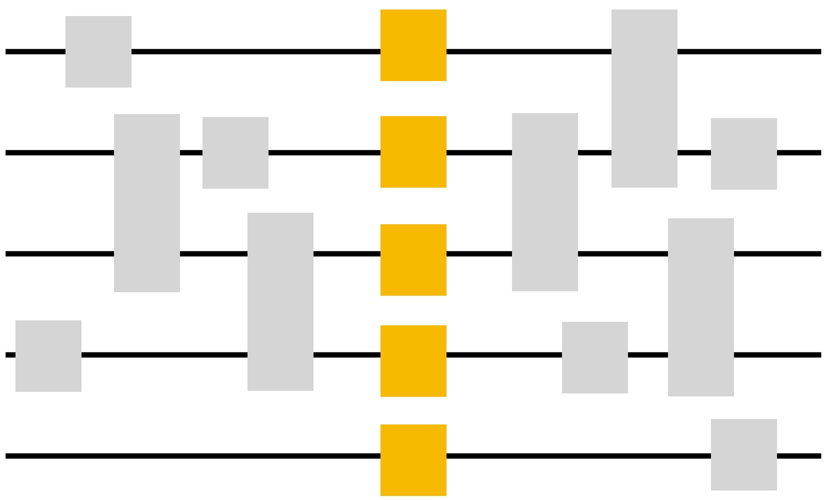
Channel resource theory

- Often quantum resources are channels/intrinsically dynamical
- Big recent efforts on extending resource theory framework to channels
- Basic analogies:

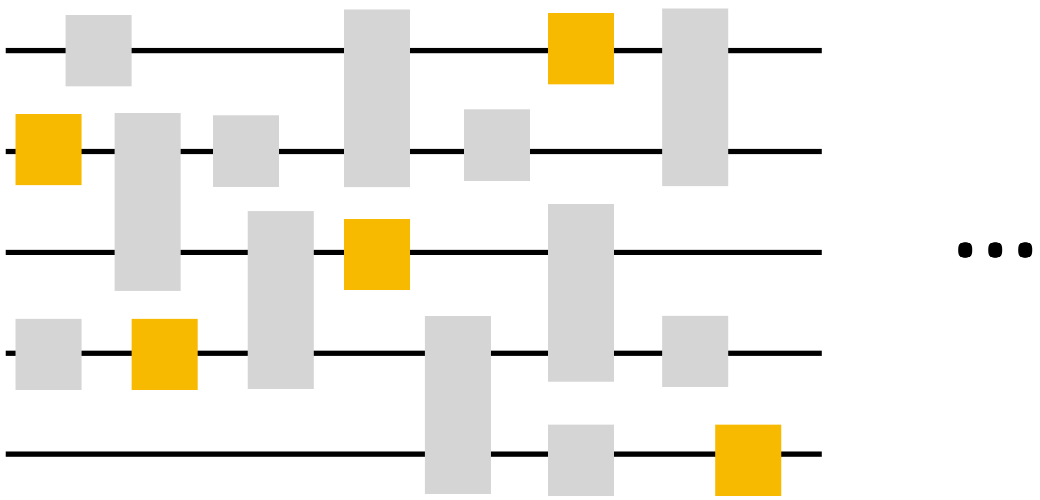
State theory	**Channel (dynamical) theory**
State (density operator)	Channel (cptp map)
Pure state [Purification]	Unitary channel [Unitary simulation]
Operation/channel	Supermap/superchannel/ quantum comb

Channel resource theory

- A crucial complication compared to state theory: Composition/utilization of multiple instances



Parallel



Adaptive "Comb"

General channel simulation, circuit synthesis...

General purification task

	Noisy	Free	Pure resource
State	$\rho_1, \rho_2, \ldots$	$\longrightarrow \approx$	Ψ
Channel	$\mathcal{N}_1, \mathcal{N}_2, \ldots$	$\longrightarrow \approx$	U

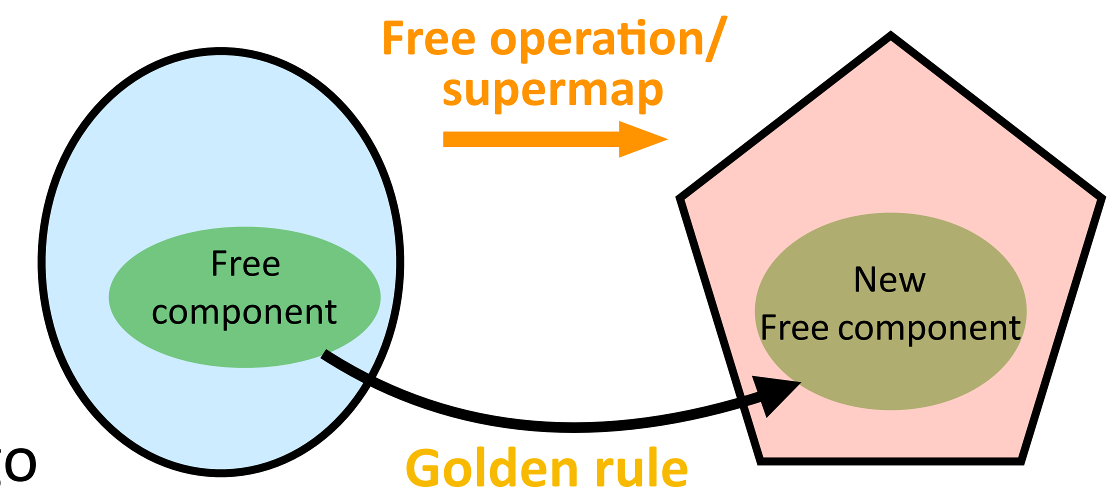
- $\mathcal{N}_1, \mathcal{N}_2, \ldots$: Noisy channels (open system)
- $\longrightarrow$: Free supermaps (free-channel-preserving maps)
- U: Unitary (noiseless) channel

▸ Different usage strategies!

Approach II: Free component

- When the input admits "free decomposition"

$$\rho = p\sigma + (1-p)\tau, \quad 0 < p < 1$$



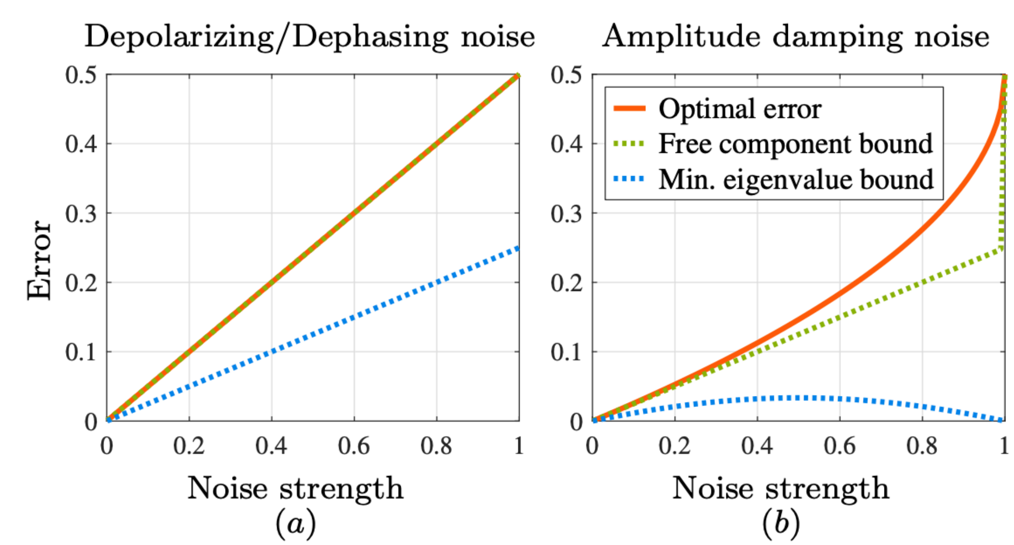
- Output must still have free component -> No-go

- Easily generalizable to channel theory (multiple inputs), logarithmic lower bound on overhead holds

- Better bound for states/known to be tight in certain cases

$$\epsilon \geq \tilde{\lambda}(\rho)(1-f_\psi)$$

$$\tilde{\lambda}(\rho) := \max \lambda \text{ s.t. } \rho - \lambda\sigma \geq 0, \sigma \in \mathcal{F}$$



See also Regula/Takagi 2010.11942

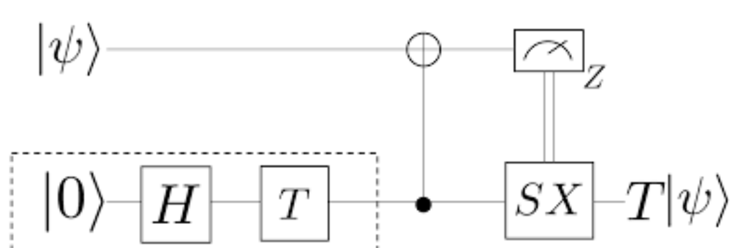
Magic state distillation

Fault-tolerant

- Clifford **+ T = Universality** $T = \mathrm{diag}\left(1, e^{i\pi/4}\right)$

Classically simulable

- A leading scheme for full fault tolerance Bravyi/Kitaev PRA'05… :
  Distill high-quality T-states offline
  (use "injection" gadgets that consume T-states to emulate T-gates)



T-state

$$|T\rangle = T|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$

MSD is the dominant source of resource cost. Understanding the achievable efficiency of MSD is of great theoretical and practical importance.

Magic state distillation

General formulation of the task: Given n noisy magic states ρ, output with probability p an m-qubit state τ s.t.

$$\mathrm{Tr}\ \tau_i T = \langle T|\tau_i|T\rangle \geq 1-\epsilon, \forall i = 1, ..., m$$

Theorem (Lower bound on T-state distillation total overhead)

$$n \geq \log_{\frac{1+R_G(\sigma)}{\lambda_{\min}(\sigma)}} \frac{\left((4-2\sqrt{2})^m - 1\right)p}{(4-2\sqrt{2})^m m\epsilon} \qquad n \geq \log_{\frac{2-\sqrt{2}}{2\zeta}} \frac{(4-2\sqrt{2})^m - 1}{(4-2\sqrt{2})^m m\epsilon}$$

Encodes noise strength

Magic state distillation

Comparison with known MSD protocols?

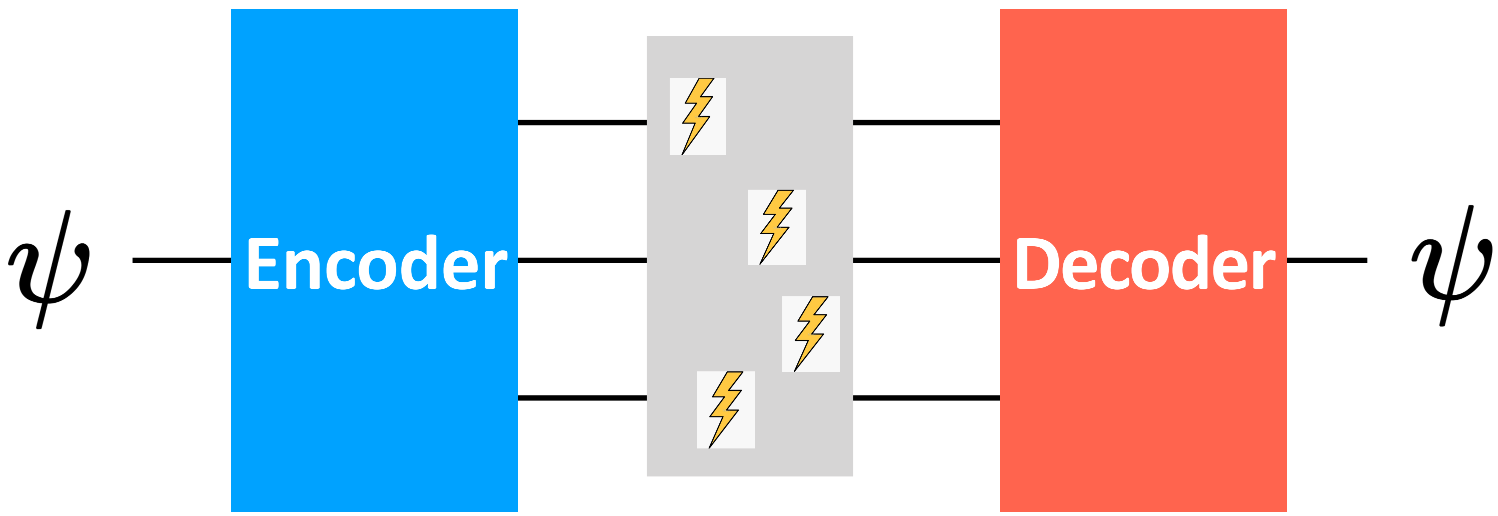
Commonly considered:

- Protocols: based on QEC with stabilizer codes
- Average overhead $n/m \in O(\log^\gamma(1/\epsilon)), \gamma =?$

- Any "sublogarithmic" (γ<1) protocol Hastings/Haah PRL'18, Krishna/Tillich PRL'19 must have exponentially diverging output size — although avg. overhead is considered low, the overall cost could blow up

- For any k≤d code, or any constant-size-output protocol, e.g. k=1 codes, avg. overhead has γ≥1 lower bound (best known γ→2, still a gap)

Constrained QEC



Important cases:

- Covariant code
- Stabilizer code

Purification formulations:

- State: Noisy physical state —decoder—> Pure logical state
- Channel: Noisy channel —encoder+decoder—> Identity channel

Covariant QEC

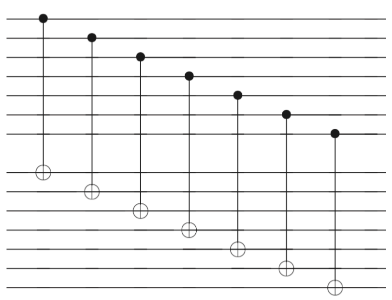
QEC + Symmetry

$$\mathcal{E}_{S\leftarrow L} \circ \mathcal{U}_{L,\theta} = \mathcal{U}_{S,\theta} \circ \mathcal{E}_{S\leftarrow L}, \forall\theta \qquad \mathcal{U}_{L/S,\theta}(\rho_L) = e^{-iH_{L/S}\theta}\rho_{L/S}e^{iH_{L/S}\theta}$$

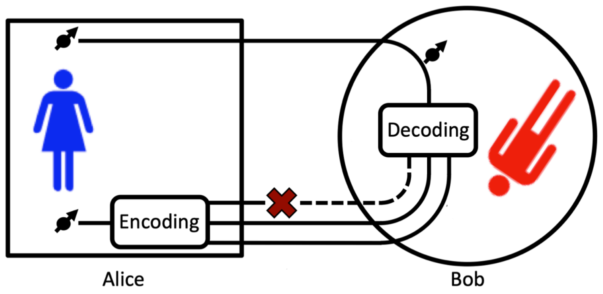
Encode -> Transform = Transform -> Encode

(Covariance)

Why care about covariant codes?



- Fault tolerance: Eastin-Knill theorem—No codes admit transversal implementation of a universal gate set



- Error-correcting “physical” (e.g. reference frame) information Hayden et al. ’17
- Quantum gravity: Bulk global symmetry conjectures Harlow/Ooguri PRL’19, Faist et al. PRX’20, black hole information problem with conservation laws Yoshida ’20…
- Condensed matter: Error correction in spin systems Brandao et al. PRL’19…

Covariant QEC

See also: Zhou/ZWL/Jiang 2005.11918
Kong/ZWL 2102.11835

- Continuous symmetries: U(1), SO(3), SU(2)…

- Lemma: When noise is covariant, covariant decoder is best

- No-purification theorems -> Cannot do perfect QEC with covariant codes
No-purification bounds -> Bounds on covariant QEC ("robust" Eastin-Knill theorem) c.f. Faist et al. PRX'20, Woods/Alhambra Quantum'20…

- A particularly good (asymmetry) monotone: RLD quantum Fisher information
Marvian Nat.Comm.'20

$$F_R(\rho) := \mathrm{Tr}\left(H\rho^2 H\rho^{-1}\right) - \mathrm{Tr}\left(\rho H^2\right)$$

Pure coherent states: infinite

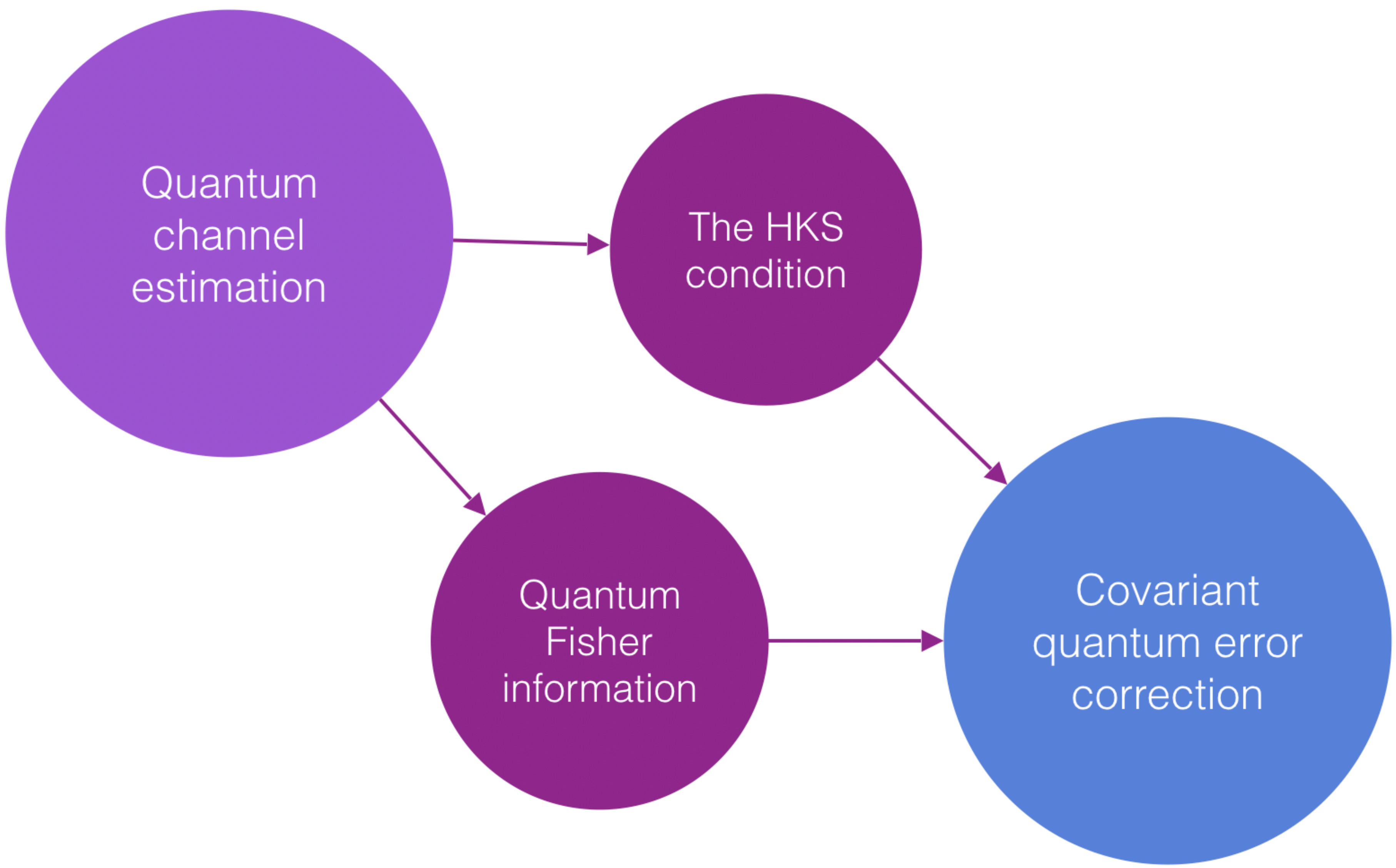
Lower bound: $\varepsilon \gtrsim \dfrac{(\Delta H_L)^2}{4F_{\mathfrak{R}}\left(\mathcal{N}_{S,\theta}\right)}.$

Covariant QEC & quantum metrology

Zhou/ZWL/Jiang 2005.11918

- Another lens: quantum metrology/channel estimation

  c.f. Kubica/Demkowicz-Dobrzanski 2004.11893



Covariant QEC & quantum metrology

Zhou/ZWL/Jiang 2005.11918

- Another lens: quantum metrology/channel estimation

  c.f. Kubica/Demkowicz-Dobrzanski 2004.11893

Consider the channel estimation of

$$\mathcal{N}_{S,\theta} = \mathcal{N}_S \circ \mathcal{U}_{S,\theta}.$$

If perfectly error-correcting covariant codes exist,

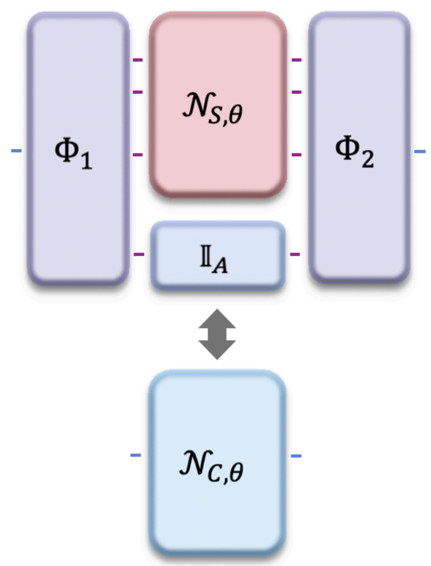
$$\mathcal{R}_{L\leftarrow S} \circ \mathcal{N}_{S,\theta} \circ \mathcal{E}_{S\leftarrow L} = \mathcal{U}_{L,\theta}$$

QFI has Heisenberg scaling

No-go: **Hamiltonian-in-Kraus-span (HKS)**

$$H_S \in \mathrm{span}\{K_{S,i}^\dagger K_{S,j}, \forall i,j\}$$

Lower bound: $$\varepsilon \gtrsim \frac{(\Delta H_L)^2}{4F_{\mathfrak{S}}^{\mathrm{reg}}(\mathcal{N}_{S,\theta})}.$$

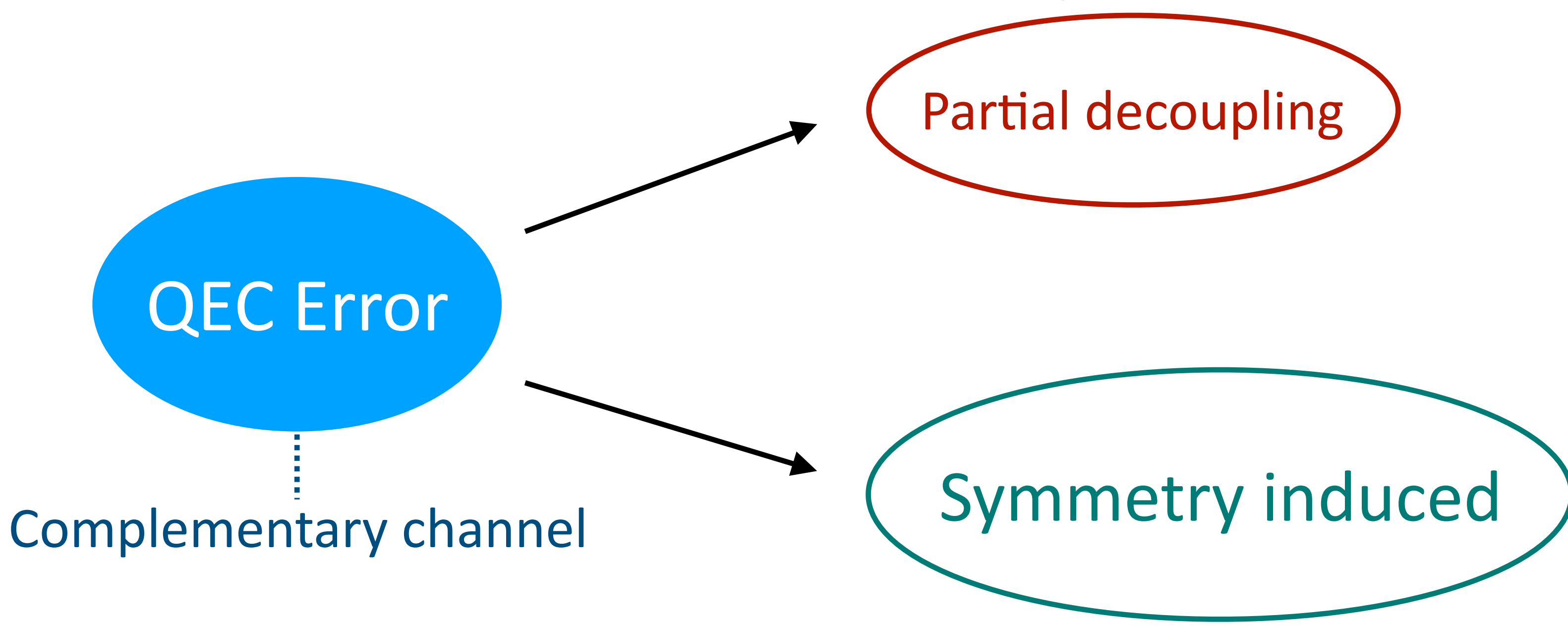


Optimal covariant QEC by random coding

Kong/ZWL 2102.11835

- A "universal" **achievability** result: U(1) covariant codes generated by Haar-random charge-conserving unitaries almost always saturate the lower bounds to leading order (in certain models)

$$\mathcal{H}^A = \bigoplus_{j=1}^{J} \mathcal{H}_j^{A_l} \otimes \mathcal{H}_j^{A_r}$$



Result: With overwhelming probability

$$\epsilon_{\text{Choi}} \leq \frac{\sqrt{kt}}{4n\sqrt{a(1-a)}}\left(1+O\left(\frac{k^2t^2}{n}\right)\right),$$

$$\epsilon_{\text{worst}} \leq \frac{k\sqrt{t}}{4n\sqrt{a(1-a)}}\left(1+O\left(\frac{kt^2}{n}\right)\right).$$

t: # of erasures
an: initial charge

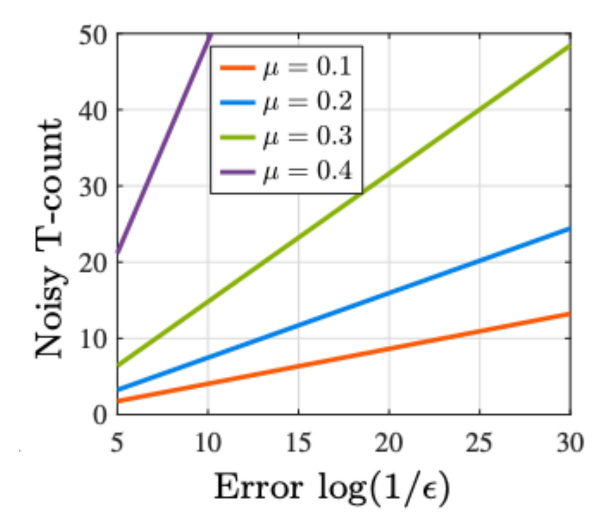
- Holds for U(1)-symmetric 2-designs. U(1)-symmetric random circuits?

More QIP applications

See 2010.11822

- Quantum Shannon theory (communication, channel capacities)
  - Core problem: the capability of channels to reliably transmit information
  - Quantum capacities: rates of simulating identity/unitary channels
  - Many different types: normal, entanglement-assisted, non-signalling-assisted…; Unified formulation: different types of superchannels/combs!

- Noisy circuit synthesis/compiling
  - Setting: decompose large unitary into elementary gates (e.g. Clifford + T)
  - Key problem: T-count? E.g. CCZ needs 4 T-gates
  - Practical version: Noisy T-count



Summary and outlook

- A comprehensive theory of universal practical limitations to tasks involving the purification of noisy quantum systems in all kinds of scenarios

- A new fundamental principle of quantum mechanics

- Profound practical implications: strong theoretical bounds on the practical costs of a broad range of quantum technologies—quantum computation, quantum error correction, quantum communication…

- Achievability (good answers in some specific cases)

- Many practical/physical applications await further developments

Thanks for your attention!

1909.02540	with Kun Fang
2010.11822	with Kun Fang
2005.11918	with Sisi Zhou, Liang Jiang
2102.11835	with Linghang Kong
210x.xxxxx	with Sisi Zhou