Estimation of group action with energy constraint and its application to uncertainty relations on S^1 and S^3

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Com. Math. Phys. **347** 1495 (2016)

J. Phys. B **50** 015503 (2019)

Contents

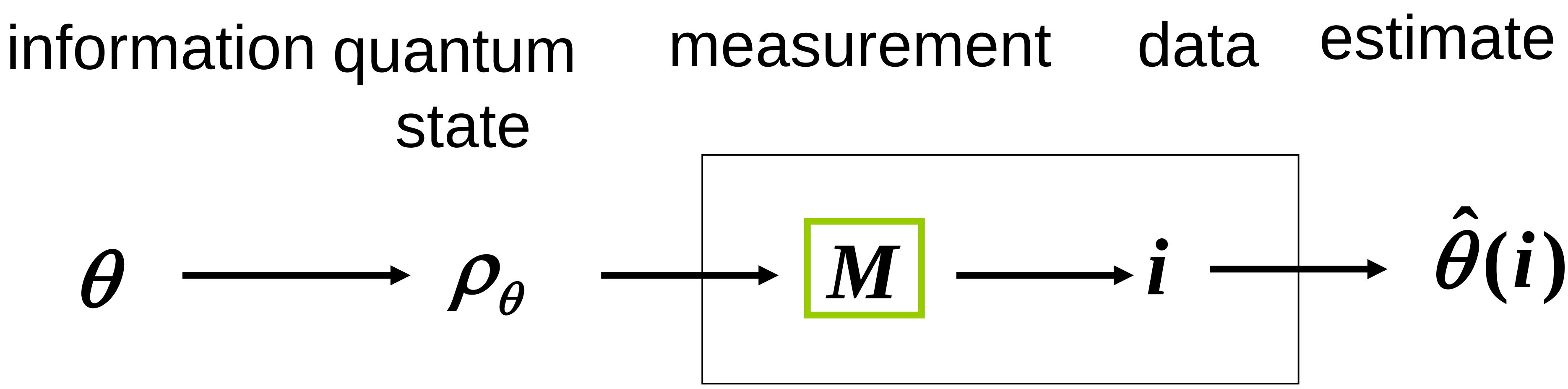
- Summary of estimation in group covariant family
- Estimation of group action with restriction for support of representation (input state)
- Estimation of group action with average energy restriction
- Practical construction of asymptotically optimal estimator
- Application to uncertainty relation (Robertson type)

Formulation of quantum state estimation

State family: $\{\rho_\theta \mid \theta \in \Theta\}$ on $\mathcal{H}$

Estimator: POVM $M = \{M_i\}_i$ on $\mathcal{H}$

Map $i \mapsto \hat{\theta}(i)$



M:Estimator

Group covariant state family

$\mathcal{H}$:Hilbert space

Holevo 1979

G :group

$\Theta = G/H$:Parameter space to be estimated

f :projective unitary representation

The set $\{\rho_\theta \mid \theta \in \Theta\}$ of densities is called covariant if

$$f(g)\rho_\theta f(g)^* = \rho_{g\theta}$$

Group covariant measurement

$\mathcal{H}$:Hilbert space

Holevo 1979

G :group $\Theta = G/H$:parameter space to be estimated

f :projective unitary representation

A POVM M taking values in Θ is called covariant if

$$f(g)M(B)f(g)^* = M(gB)$$

$\mathcal{M}(\Theta)$: Set of POVMs taking the values in Θ

$\mathcal{M}_{\mathbf{cov}}(\Theta)$: Set of covariant POVMs taking values in Θ

$M \in \mathcal{M}(\Theta)$ is included in $\mathcal{M}_{\mathbf{cov}}(\Theta)$

$$\Leftrightarrow \quad M(B) = M_T(B) := \int_B f(g) T f(g)^* \mu(dg)$$

Evaluation of error

$R(\theta,\hat{\theta})$:error function

The average error when the true is θ

$$\mathcal{D}_{R,\theta}(M) := \int_{\Theta} R(\theta,\hat{\theta}) \mathrm{Tr} M(d\hat{\theta})\rho_\theta$$

Bayesian error under the prior distribution ν on Θ :

$$\mathcal{D}_{R,\nu}(M) := \int_{\Theta} \mathcal{D}_{R,\theta}(M)\nu(d\theta)$$

Mini-max method: The worst case is optimized

$$\mathcal{D}_{R}(M) := \max_{\theta} \mathcal{D}_{R,\theta}(M)$$

$$R(\theta,\hat{\theta}) = R(g\theta, g\hat{\theta}), \quad M \in \mathcal{M}_{\mathrm{cov}}(\Theta)$$

$$\Rightarrow \quad \mathcal{D}_{R}(M) = \mathcal{D}_{R,\theta}(M) = \mathcal{D}_{R,\nu}(M)$$

Quantum Hunt-Stein Theorem

Assumption: $R(\theta, \hat{\theta}) = R(g\theta, g\hat{\theta})$

Invariant probability measure μ for Θ exists when Θ is compact

Then, we have the following:

Holevo 1979

$$\min_{M \in \mathcal{M}(\Theta)} \mathcal{D}_R(M) = \min_{M \in \mathcal{M}(\Theta)} \mathcal{D}_{R,\mu}(M)$$

$$= \min_{M \in \mathcal{M}_{\mathrm{cov}}(\Theta)} \mathcal{D}_R(M) = \min_{M \in \mathcal{M}_{\mathrm{cov}}(\Theta)} \mathcal{D}_{R,\mu}(M)$$

We have the following even when Θ is not compact.

$$\min_{M \in \mathcal{M}(\Theta)} \mathcal{D}_R(M) = \min_{M \in \mathcal{M}_{\mathrm{cov}}(\Theta)} \mathcal{D}_R(M)$$

Ozawa 1980

Bogomolov 1982

Estimation of group action

Given a projective unitary representation f of G on $\mathcal{H}$.

Input state | Unknown Unitary to be estimated | measurement | estimate

$$\rho \longrightarrow \boxed{f(g)} \longrightarrow M \longrightarrow \hat{g}$$

$\mathcal{E} = (\rho, M)$:Our operation

For example, $\theta \in G = (0, 2\pi],\qquad f(g) = U_\theta$

Estimation of group action

Given a projective unitary representation f of G on $\mathcal{H}$.

Input state	Unknown Unitary to be estimated	measurement	estimate
ρ	$\longrightarrow$ $f(g)$ $\longrightarrow$	M $\longrightarrow$	$\hat{g}$

$\mathcal{E} = (\rho, M)$:Our operation

$R(g,\hat{g}) = R(e, g^{-1}\hat{g}) = R(e, \hat{g}g^{-1})$:error function

Average error when the true parameter is g

$$\mathcal{D}_{R,g}(\mathcal{E}) := \int_G R(g,\hat{g}) \mathrm{Tr} M(d\hat{g}) f(g) \rho f(g)^*$$

$\mathcal{D}_{R,\nu}(M), \mathcal{D}_R(M)$ is similarly defined.

Extension of quantum Hunt-Stein theorem

Invariant probability measure μ exists for G when G is compact. Then, the following equations hold.

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho, M \in \mathcal{M}(\Theta)} \mathcal{D}_{R,\mu}(\rho, M)$$

$$= \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(\Theta)} \mathcal{D}_R(\rho, M)$$

$$= \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(\Theta)} \mathcal{D}_{R,\mu}(\rho, M)$$

The following relation holds even when G is not compact.

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(\Theta)} \mathcal{D}_R(\rho, M)$$

Fourier transform and inverse Fourier transform on group

$\hat{G}$: Set of irreducible unitary representation of G

$$L^2(\hat{G}) := \bigoplus_{\lambda \in \hat{G}} L^2(\mathcal{V}_\lambda)$$

$L^2(\mathcal{V}_\lambda)$: Set of HS operators on $\mathcal{V}_\lambda$

$\mathcal{F} : L^2(G) \to L^2(\hat{G})$: Fourier transform

$$(\mathcal{F}[\phi])_\lambda := \sqrt{d_\lambda} \int_G f_\lambda(g)^* \phi(g) \mu(dg)$$

$\mathcal{F}^{-1} : L^2(\hat{G}) \to L^2(G)$: Inverse Fourier transform

$$\mathcal{F}^{-1}[A](g) := \sum_{\lambda \in \hat{G}} \sqrt{d_\lambda} \mathrm{Tr} f_\lambda(g) A_\lambda$$

$$A = (A_\lambda) \in L^2(\hat{G})$$

Input system and inverse Fourier transform on group

$$\mathcal{H} = \bigoplus_{\lambda\in\hat{G}} \mathcal{U}_\lambda \otimes \mathbb{C}^{n_\lambda}$$

$\hat{G}$: Set of irreducible unitary representation of G

n_λ : Multiplicity of $\mathcal{U}_\lambda$

For any pure state $\varphi = \bigoplus_{\lambda\in\hat{G}} \varphi_\lambda$

φ_λ can be considered as an element of $L^2(\mathcal{U}_\lambda)$ by choosing a partial isometry from $\mathcal{U}_\lambda$ to $\mathbb{C}^{n_\lambda}$.

$\varphi = \bigoplus_{\lambda\in\hat{G}} \varphi_\lambda$ can be considered as

an element of $L^2(\hat{G}) := \bigoplus_{\lambda\in\hat{G}} L^2(\mathcal{U}_\lambda)$

$$\mathcal{F}^{-1}[\varphi](g) = \sum_{\lambda\in\hat{G}} \sqrt{d_\lambda}\, \mathrm{Tr} f_\lambda(g)\varphi_\lambda$$

Optimization via inverse Fourier transform

$$D_R(X) := \int_G R(e,\hat{g})\,|\mathcal{F}^{-1}[X](\hat{g}^{-1})|^2\,\mu(d\hat{g})$$

We consider constraint for support $\Lambda \subset \hat{G}$ of input state.

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho:\mathrm{pure}, M \in \mathcal{M}_{\mathrm{cov}}(\Theta)} \mathcal{D}_R(\rho, M)$$

$$= \min_{X \in L^2(\Lambda), \|X\|^2 = 1} D_R(X)$$

Example: $G = \mathbb{R}$

$|e_\lambda\rangle$:Base of 1-dim irreducible space with $\lambda \in \hat{G}$

$\mu_G(dg) = 1/\sqrt{2\pi}\, dg$:Measure on $G = \mathbb{R}$

$\mu_{\hat{G}}(d\lambda) = 1/\sqrt{2\pi}\, d\lambda$:Measure on $\hat{G} = \mathbb{R}$

$\varphi(\lambda) \in L^2(\hat{G}) = L^2(\mathbb{R})$:input state

Distribution of estimate $\hat{g}$: $\quad |\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 \dfrac{d\hat{g}}{\sqrt{2\pi}}$

Support constraint: $\mathrm{supp}[\varphi] \subset [-1,1]$

Minimizing under support constraint

$$\min_\varphi \int_{\mathbb{R}} \hat{g}^2 \, |\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 \frac{d\hat{g}}{\sqrt{2\pi}} = \frac{\pi^2}{4}$$

Minimum is attained with $\varphi(\lambda) = (2\pi)^{1/4} \sin \dfrac{\pi(1+\lambda)}{2}$

Example: $G = \mathrm{U}(1)$

$$G = U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \qquad \hat{G} = \mathbb{Z}$$

$$\varphi(\lambda) \in L^2(\hat{G}) = L^2(\mathbb{Z})$$:input state

$$\sum_{\lambda \in \mathbb{Z}} |\varphi(\lambda)|^2 = 1$$

Distribution of estimate $\hat{\theta}$ $\qquad |\mathcal{F}^{-1}[\varphi](-\hat{\theta})|^2 \, d\hat{\theta}$

Minimizing average error with restriction for support

$$\min_{\varphi \in L^2(\mathbb{Z} \cap [-n,n])} \int_0^{2\pi} 2\sin^2 \frac{\hat{\theta} - \theta}{2} |\mathcal{F}^{-1}[\varphi](-\hat{\theta})|^2 \, d\hat{\theta}$$

$$= 1 - \cos \frac{\pi}{2n+2}$$

Buzek et al 1999

Minimum is attained by $\displaystyle \varphi(\lambda) = \sin \frac{\pi(\lambda + n + 1)}{2n+2}$

Factor system of projective unitary representation

Chiribella 2011

Factor system

$$e^{i\theta(g,g')} := f(g) f(g') f(gg')^{-1}$$

$$\mathcal{L} := \{e^{i\theta(g,g')}\}_{g,g'}$$

$\hat{G}[\mathcal{L}]$: Set of projective irreducible representation with the factor system $\mathcal{L}$

$$D_R(X) := \int_G R(e,\hat{g}) |\mathcal{F}_{\mathcal{L}}^{-1}[X](\hat{g}^{-1})|^2 \mu(dg)$$

We consider constraint for support of input state. $\Lambda \subset \hat{G}[\mathcal{L}]$

$$\min_{\rho, M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) = \min_{\rho:\text{pure}, M \in \mathcal{M}_{\text{cov}}(\Theta)} \mathcal{D}_R(\rho, M)$$

$$= \min_{X \in L^2(\Lambda)} D_R(X).$$

Case of compact group

Chiribella *et al* 2005

$S := \{\lambda \in \hat{G} \mid n_\lambda \neq 0\}$ $\lambda^* \in \hat{G}$: adjoint representation

Assume that $n_\lambda \geq d_\lambda, \quad \lambda \in S$

$$R(g, \hat{g}) = -\sum_{\lambda \in \hat{G}} a_\lambda (\chi_\lambda(\hat{g} g^{-1}) + \chi_{\lambda^*}(\hat{g} g^{-1}))$$

$$a_\lambda = a_{\lambda^*} \geq 0$$

$X = (c_\lambda Y_\lambda)_{\lambda \in S}$ with $\mathrm{Tr} Y_\lambda^\dagger Y_\lambda = 1$

$$D_R(X) \geq -\sum_{\lambda, \lambda' \in S} c_\lambda c_{\lambda'} \sum_{\lambda'' \in \hat{G}} a_{\lambda''} (C^{\lambda''^*}_{\lambda, \lambda'^*} + C^{\lambda''}_{\lambda, \lambda'^*})$$

equality holds when Y_λ is maximal entangled state on $\mathcal{V}_\lambda$

$C^{\lambda''}_{\lambda, \lambda'^*}$: Littlewood-Richardson coefficient

Example: $G = \mathbf{SO(3)}$

Chiribella *et al* 2004
Bagan *et al* 2004, MH 2006

$\lambda \in \widehat{G}$ is identified with maximal weight.

$$R(g,\hat{g}) = |\mathrm{Tr}\, g^{\dagger}\hat{g}|^2 = 1 + \chi_1(g^{\dagger}\hat{g})$$: gate fidelity

$$C^{1}_{k,k'} = \begin{cases} \delta_{k,k'-1} + \delta_{k,k'} + \delta_{k,k'+1} & \text{if } k > 0 \\ \delta_{0,k'-1} & \text{if } k = 0 \end{cases}$$

$$\min_{|\phi\rangle \in \mathcal{H}_{\Lambda_n}} D_R(\phi) = 1 - \cos\frac{2\pi}{2n+3} \cong \frac{\pi^2}{2n^2}$$

$$\Lambda_n := \{0, 1, 2, \ldots, n\}$$

Example: $G = \mathbf{SO(3)}$

Chiribella *et al* 2004
Bagan *et al* 2004, MH 2006

projective representation

$\lambda \in \widehat{G}$ is identified with maximal weight.

$$R(g,\hat{g}) = |\mathrm{Tr}\, g^{\dagger}\hat{g}|^2 = 1 + \chi_1(g^{\dagger}\hat{g})$$: gate fidelity

$$\Rightarrow \quad C^{1}_{k+\frac{1}{2},k'+\frac{1}{2}} = \delta_{k,k'-1} + \delta_{k,k'} + \delta_{k,k'+1}$$

$$\Rightarrow \quad \min_{|\phi\rangle \in \mathcal{H}_{\Lambda_n}} D_R(\phi) = 1 - \cos\frac{\pi}{n+2} \cong \frac{\pi^2}{2n^2}$$

$$\Lambda_n := \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, n + \frac{1}{2}\}$$

Example: $G = \mathrm{SU}(2)$

$\lambda \in \widehat{G}$ is identified with maximal weight.

$$R(g,\hat{g}) = 1 - \chi_{\frac{1}{2}}(g^{\dagger}\hat{g})/2$$

$$\Rightarrow \quad C^{1/2}_{k/2,k'/2} = \delta_{k,k'-1} + \delta_{k,k'+1}$$

$$\Rightarrow \quad \min_{|\phi\rangle \in \mathcal{H}_{\Lambda_n}} D_R(\phi) = 1 - \cos\frac{\pi}{n+2} \cong \frac{\pi^2}{2n^2}$$

$$\Lambda_n := \{0, \frac{1}{2}, 1, \ldots, \frac{n}{2}\}$$

Energy constraint

Error

$$D_R(\rho, M) := \int_G R(g,\hat{g}) \mathrm{Tr} f(g) \rho f(g)^\dagger M(d\hat{g})$$

Energy constraint

$$\mathrm{Tr} \rho H \le E$$

$$D_R(X) := \int_G R(e,\hat{g}) |\mathcal{F}^{-1}[X](\hat{g}^{-1})|^2 \mu(dg)$$

Our target is

$$\min_{X \in L^2_{H,E}(\hat{G}), \|X\|^2 = 1} D_R(X)$$

$$L^2_{H,E}(\hat{G}) := \{ X \in L^2(\hat{G}) | \langle X | H | X \rangle \le E \}$$

Example: $G=\mathbb{R} \quad (\hat{G}=\mathbb{R})$

$$R(g,\hat{g})=(g-\hat{g})^2, \qquad f(g)\left|\lambda\right\rangle = e^{ig\lambda}\left|\lambda\right\rangle, \qquad H=Q^2$$

$$\min_{\rho\in\mathcal{S}(L^2(\mathbb{R}))}\ \min_{M\in\mathcal{M}_{\mathrm{cov}}(\mathbb{R})}\left\{D_R(\rho,M)\,|\,\mathrm{Tr}\rho Q^2\le E\right\}$$

$$=\min_{|\phi\rangle\in L^2(\mathbb{R})}\left\{\int_{\mathbb{R}}\hat{g}^2\,|\mathcal{F}^{-1}[\phi](\hat{g})|^2\frac{d\hat{g}}{\sqrt{2\pi}}\,\middle|\,\int_{\mathbb{R}}\lambda^2\,|\phi(\lambda)|^2\frac{d\lambda}{\sqrt{2\pi}}\le E\right\}$$

$$=\min_{|\phi\rangle\in L^2(\mathbb{R})}\left\{\left\langle\phi\right|Q^2\left|\phi\right\rangle\,\middle|\,\left\langle\phi\right|P^2\left|\phi\right\rangle\le E\right\}=\frac{1}{4E}$$

Minimum is attained with $\varphi(\lambda)=e^{-\frac{\lambda^2}{4E^2}}\Big/\sqrt{E}$

Mathieu Function (Preparation)

Periodic differential operator

$$P^2 + 2q\cos 2Q$$

Minimum eigenvalue	Eigen function	space
$a_0(q)$	$\mathrm{ce}_0(\theta, q)$	$L^2_{\mathrm{p,even}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$b_2(q)$	$\mathrm{se}_2(\theta, q)$	$L^2_{\mathrm{p,odd}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$a_1(q)$	$\mathrm{ce}_1(\theta, q)$	$L^2_{\mathrm{a,even}}((-\frac{\pi}{2}, \frac{\pi}{2}])$
$b_1(q)$	$\mathrm{se}_1(\theta, q)$	$L^2_{\mathrm{a,odd}}((-\frac{\pi}{2}, \frac{\pi}{2}])$

Estimation of U(1)

$$R(g,\hat{g}) = 1 - \cos(g - \hat{g}), \qquad f(g)|k\rangle = e^{ikg}|k\rangle,$$

$$H = \sum_k k^2 |k\rangle\langle k|$$

$$\min_{\rho \in \mathcal{S}(L^2(\hat{\mathrm{U}}(1)))} \min_{M \in \mathcal{M}_{\mathrm{cov}}(\mathrm{U}(1))} \left\{ D_R(\rho, M) \,|\, \mathrm{Tr}\rho H \le E \right\}$$

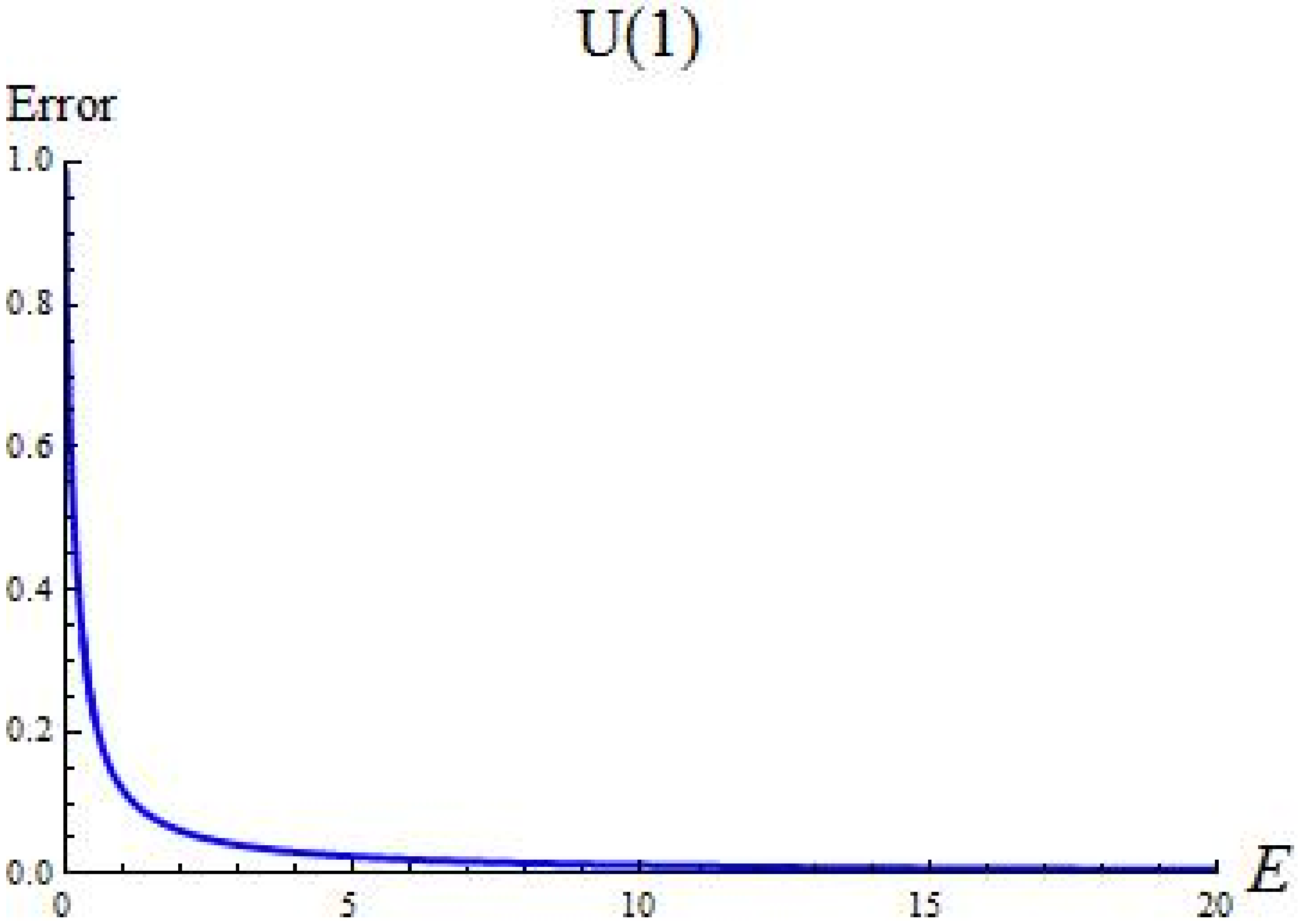
$$= \min_{|\phi\rangle \in L^2_{\mathrm{p,even}}((-\pi,\pi])} \left\{ \langle \phi | I - \cos Q | \phi \rangle \middle| \langle \phi | P^2 | \phi \rangle \le E \right\}$$

$$= \max_{s>0} \frac{s a_0(2/s)}{4} + 1 - sE$$

Optimal input is constructed by $\mathrm{ce}_0(\theta, q)$

$$\cong \begin{cases} \dfrac{1}{8E} - \dfrac{1}{128E^2} & \text{as } E \to \infty \\ 1 - \sqrt{2E} + \dfrac{7\sqrt{2}E^{\frac{3}{2}}}{16} & \text{as } E \to 0 \end{cases}$$

Graphs



Another energy constraint for $G = U(1)$

$$\tau(E) := \min \int_0^{2\pi} 2\sin^2 \frac{\hat{\theta}-\theta}{2} |\mathcal{F}^{-1}[\varphi](-\hat{\theta})|^2 \, d\hat{\theta}$$

under energy constraint: $\sum_{\lambda=0}^{\infty} n |\varphi(n)|^2 = \langle \varphi | \hat{n} | \varphi \rangle \leq E$

Assume $|\varphi\rangle = \sum_{n=0}^{\infty} f(\frac{n}{R}) \frac{1}{R} |2n\rangle \qquad f \in L^2(\mathbb{R})$

Energy: $\langle \varphi | \hat{n} | \varphi \rangle = 2R \langle f | Q | f \rangle$

Error:

$$\int_0^{2\pi} 2\sin^2 \frac{\hat{\theta}-\theta}{2} |\mathcal{F}^{-1}[\varphi](-\hat{\theta})|^2 \, d\hat{\theta} = \frac{|f(0)|^2}{2R} + \frac{\langle f | P^2 | f \rangle}{2R^2}$$

Another energy constraint for $G = U(1)$

$$\tau(E) := \min \int_0^{2\pi} 2\sin^2 \frac{\hat{\theta}-\theta}{2} |\mathcal{F}^{-1}[\varphi](-\hat{\theta})|^2 \, d\hat{\theta}$$

under energy constraint: $\sum_{\lambda=0}^{\infty} n|\varphi(n)|^2 = \langle \varphi | \hat{n} | \varphi \rangle \le E$

Assume $|\varphi\rangle = \sum_{n=0}^{\infty} f(\frac{n}{R}) \frac{1}{R} |2n\rangle \qquad f \in L^2(\mathbb{R})$

$$E^2 \tau(E) = 2 \min_{f: f(0)=0} \langle f | Q | f \rangle^2 \langle f | P^2 | f \rangle \le \frac{1}{8}$$

Suboptimal input state with Heisenberg scaling is realized by two-mode squeezed vacuum state.

MH et al 2019

Estimation of SU(2)

$$R(g,\hat{g}) = 1 - \frac{1}{2}\chi_{\frac{1}{2}}(\hat{g}g^{-1}), \qquad H = \bigoplus_{k=0}^{\infty} \frac{k}{2}\left(\frac{k}{2}+1\right) I_{\frac{k}{2}}$$

Reduce $L^2(\mathrm{S\hat{U}(2)})$ to $L^2_{\mathrm{p,odd}}((-\pi,\pi])$

$$\min_{\rho \in \mathcal{S}(L^2(\mathrm{S\hat{U}}(2)))} \min_{M \in \mathcal{M}_{\mathrm{cov}}(\mathrm{SU}(2))} \left\{ D_R(\rho, M) \,\middle|\, \mathrm{Tr}\rho H \le E \right\}$$

$$= \min_{|\phi\rangle \in L^2_{\mathrm{p,odd}}((-\pi,\pi])} \left\{ \left\langle \phi \middle| I - \cos\frac{Q}{2} \middle| \phi \right\rangle \,\middle|\, \left\langle \phi \middle| P^2 \middle| \phi \right\rangle \le E + \frac{1}{4} \right\}$$

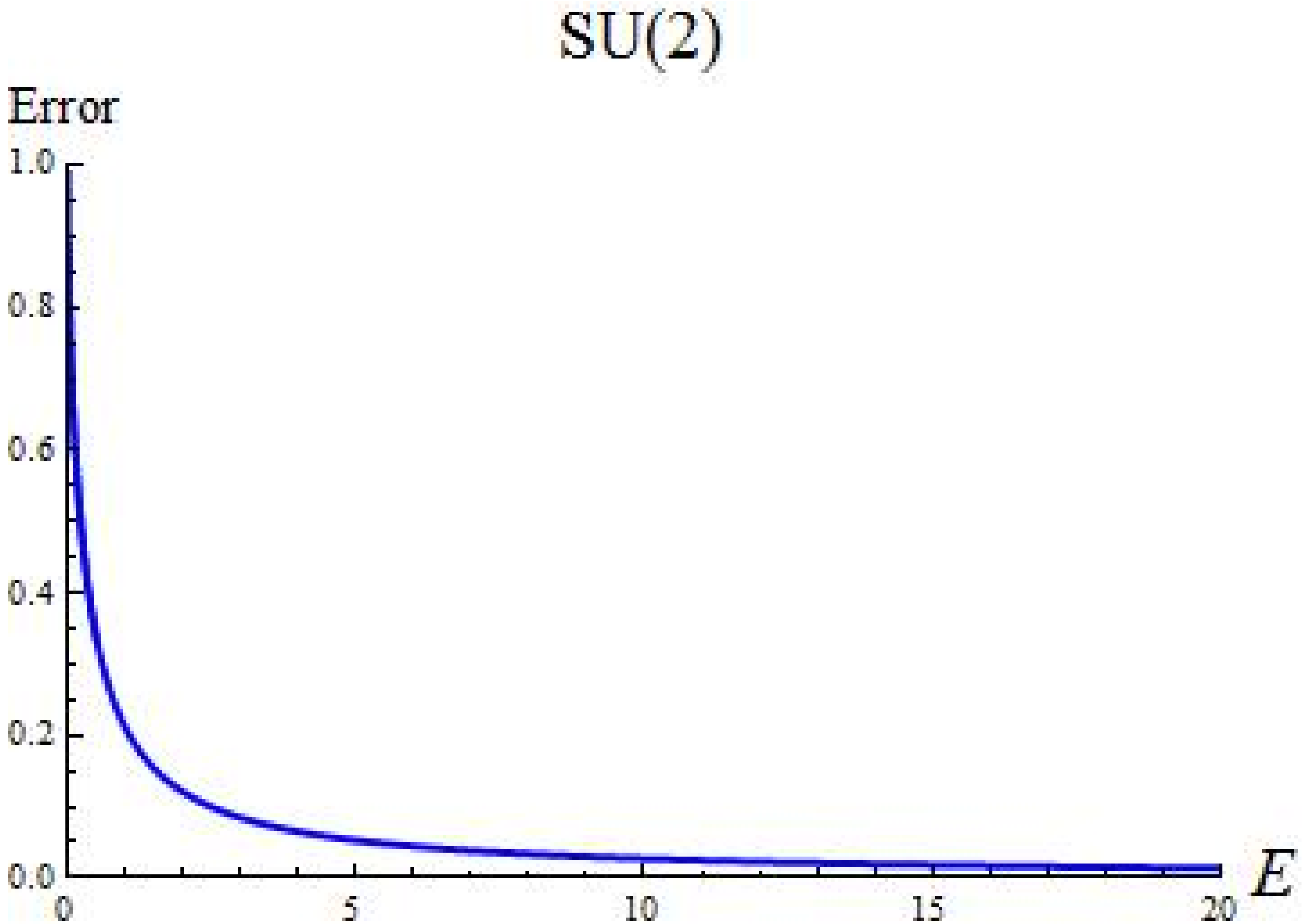
$$= \max_{s>0} \frac{s b_2(8/s)}{4} + 1 - s\left(E + \frac{1}{4}\right)$$

$$\cong \begin{cases} \dfrac{9}{32E} - \dfrac{7 \bullet 3^3}{2^{11}E^2} & \text{as } E \to \infty \\[2ex] 1 - \dfrac{2}{\sqrt{3}}\sqrt{E} + \dfrac{5E^{\frac{3}{2}}}{6\sqrt{3}} & \text{as } E \to 0 \end{cases}$$

Optimal input is constructed by

$$\mathrm{se}_2(\theta, q)$$

Graphs



Estimation of SO(3)

$$R(g,\hat{g}) = \frac{1}{2}(3 - \chi_1(\hat{g}g^{-1})), \quad H = \bigoplus_{k=0}^{\infty} \frac{k}{2}\left(\frac{k}{2}+1\right) I_{\frac{k}{2}}$$

Reduce $L^2(\hat{\mathrm{SO}}(3))$ to $L^2_{\mathrm{a,odd}}((-\pi,\pi])$ or $L^2_{\mathrm{p,odd}}((-\pi,\pi])$

$$\min_{\rho\in\mathcal{S}(L^2(\hat{\mathrm{SO}}(3)))} \min_{M\in\mathcal{M}_{\mathrm{cov}}(\mathrm{SO}(3))} \left\{ D_R(\rho,M) \mid \mathrm{Tr}\rho H \le E \right\}$$

$$= \begin{cases} \min_{\phi\in L^2_{\mathrm{a,odd}}} \{\langle\phi|I-\cos Q|\phi\rangle \mid \langle\phi|P^2|\phi\rangle \le E+\frac{1}{4}\} & \text{Integer case} \\ \min_{\phi\in L^2_{\mathrm{p,odd}}} \{\langle\phi|I-\cos Q|\phi\rangle \mid \langle\phi|P^2|\phi\rangle \le E+\frac{1}{4}\} & \text{Half integer case} \end{cases}$$

Integer case

$$\min_{\phi \in L^2_{\mathrm{a,odd}}} \{ \langle \phi | I - \cos Q | \phi \rangle \,|\, \langle \phi | P^2 | \phi \rangle \le E + \frac{1}{4} \}$$

$$= \max_{s>0} \frac{s a_1(2/s)}{4} + 1 - s(E + \frac{1}{4})$$

$$= \begin{cases} \dfrac{9}{8E} - \dfrac{81}{128E^2} & E \to \infty \\ \dfrac{3}{2} - \dfrac{\sqrt{E}}{\sqrt{2}} - \dfrac{E}{4} & E \to 0 \end{cases}$$

Optimal input is constructed by $\mathrm{ce}_1(\theta, q)$

Half integer case

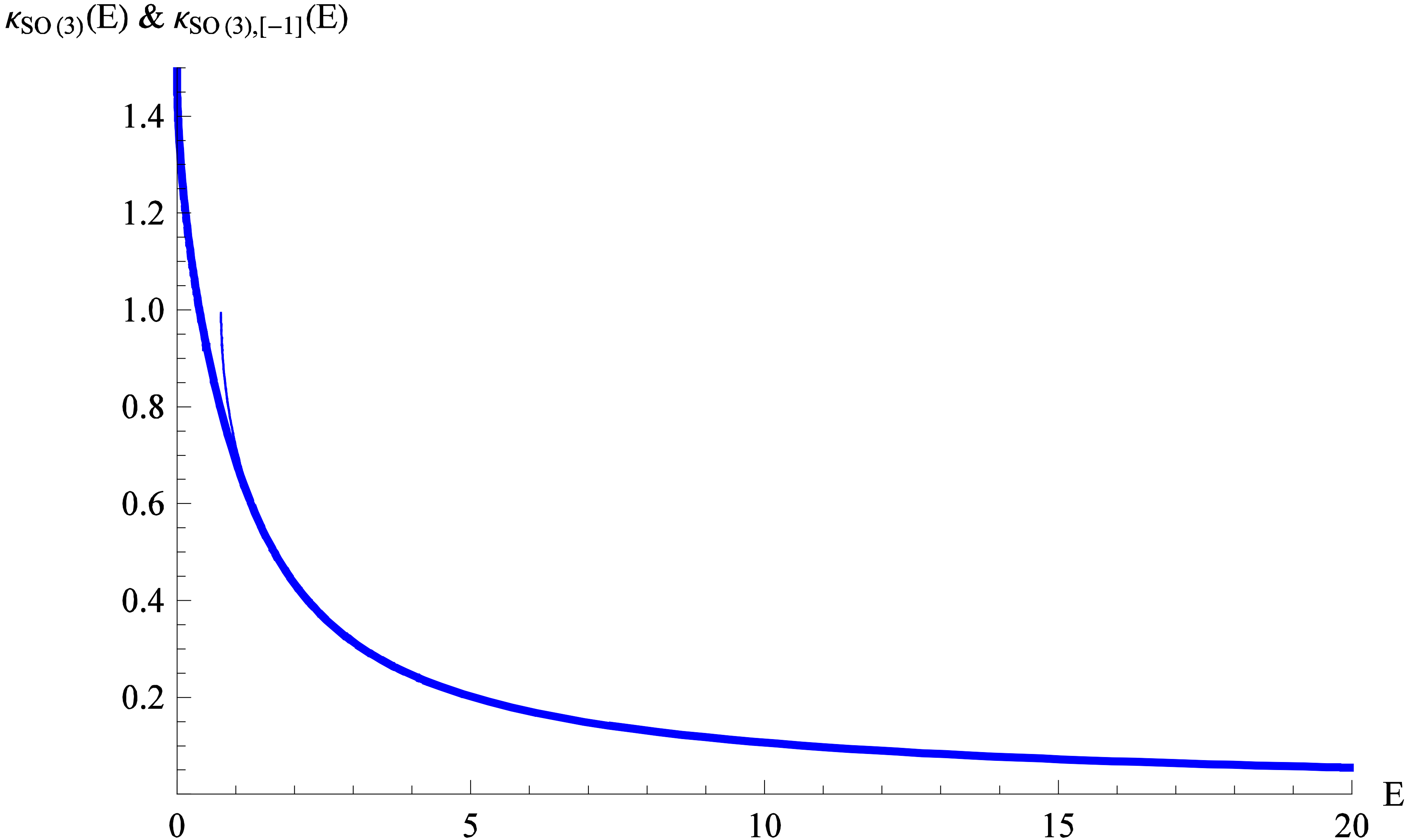
$$\min_{\phi \in L^2_{\mathrm{p,odd}}} \{ \langle \phi | I - \cos Q | \phi \rangle \,|\, \langle \phi | P^2 | \phi \rangle \le E + \frac{1}{4} \}$$

$$= \max_{s>0} \frac{s b_2(2/s)}{4} + 1 - s(E + \frac{1}{4})$$

$$= \begin{cases} \dfrac{9}{8E} - \dfrac{81}{128E^2} & E \to \infty \\ 1 - \dfrac{1}{\sqrt{3}}(E - \dfrac{3}{4})^{\frac{1}{2}} + \dfrac{5}{48\sqrt{3}}(E - \dfrac{3}{4})^{\frac{3}{2}} & E \to \dfrac{3}{4} \end{cases}$$

Optimal input is constructed by $\mathrm{se}_2(\theta, q)$

Graphs



Thick line expresses the projective case,
and Normal line expresses the representation case

Non-compact Example: $G = \mathbb{R}^2$

f : Heisenberg representation

$$|X\rangle \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$$

multiplicity

Minimize

$$\int_{\mathbb{R}^2} (x^2 + y^2) |\mathcal{F}^{-1}[X](\frac{x + yi}{\sqrt{2}})|^2 \, dxdy$$

under

$$\langle X | (Q^2 + P^2) \otimes I | X \rangle \leq E$$

Minimum value: $\dfrac{1}{2E}$

How to derive minimum

Fourier transform $\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$

$$\mathcal{F}^{-1}(Q \otimes I)\mathcal{F} = P_2 - \frac{1}{2}Q_1,\ \mathcal{F}^{-1}(P \otimes I)\mathcal{F} = -P_1 - \frac{1}{2}Q_2$$

Via $\phi = \mathcal{F}^{-1}[X]$, minimizing problem is equivalent with

Minimize $\langle \phi | Q_1{}^2 + Q_2{}^2 | \phi \rangle$

under $\langle \phi | (P_2 - \frac{1}{2}Q_1)^2 + (-P_1 - \frac{1}{2}Q_2)^2 | \phi \rangle \le E$

By choosing suitable coordinate, minimizing problem is equivalent with

Minimize $\langle \phi | Q_1{}^2 + Q_2{}^2 | \phi \rangle$

under $\langle \phi | P_1{}^2 + P_1{}^2 | \phi \rangle \le E$

Uncertainty relation → Minimum value $1/2E$

Practical realization of asymptotically optimal estimator

$$G = \mathrm{U}(1) \quad \mathcal{H} = \langle |k\rangle \rangle, \quad H = \sum_k k^2 |k\rangle\langle k|$$

Assume that ϕ satisfies $\sum_k k |\langle k|\phi\rangle|^2 = 0$

$\mathcal{F}[\phi]$ is even function

MLE

$$|\phi\rangle \to U \to \boxed{M} \to \theta_1$$

$$|\phi\rangle \to U \to \boxed{M} \to \theta_2$$

$$\vdots$$

$$|\phi\rangle \to U \to \boxed{M} \to \theta_n$$

$$\Rightarrow \quad \theta_1, \theta_2, \ldots, \theta_n \to \hat{\theta}^n$$

This method attains the optimal performance.

Practical realization of asymptotically optimal estimator

$G = \mathrm{SU}(2)$ $\qquad \mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_{\lambda}$

Assume that the support of ϕ contains both of integer rep. and half integer rep.

MLE

$$|\phi\rangle \to U \to \boxed{M} \to \theta_1$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_2$$
$$\vdots$$
$$|\phi\rangle \to U \to \boxed{M} \to \theta_n$$

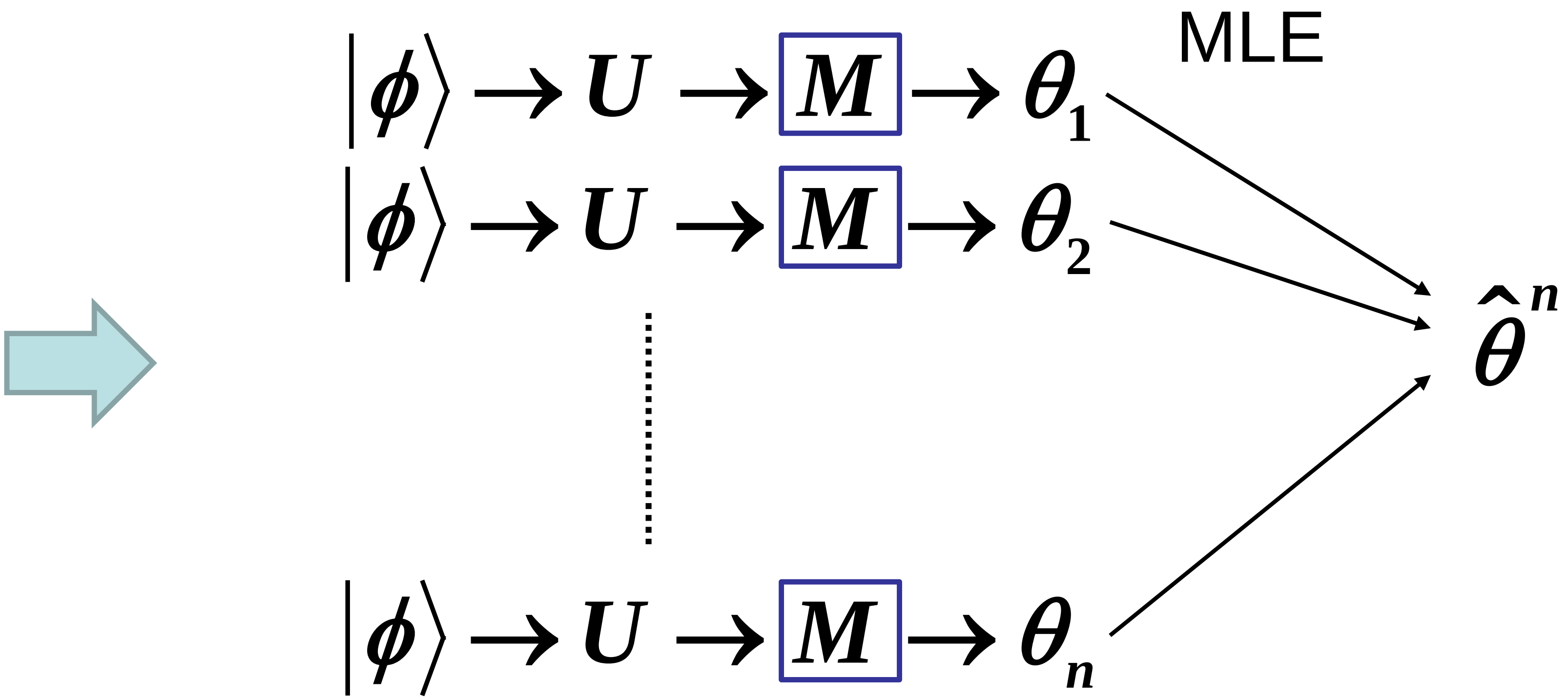
$$\to \hat{\theta}^n$$

This method attains the optimal performance.

Practical realization of asymptotically optimal estimator

$G = \mathbf{SO(3)}$ $\mathcal{H} = \bigoplus_{\lambda} \mathcal{H}_{\lambda}$

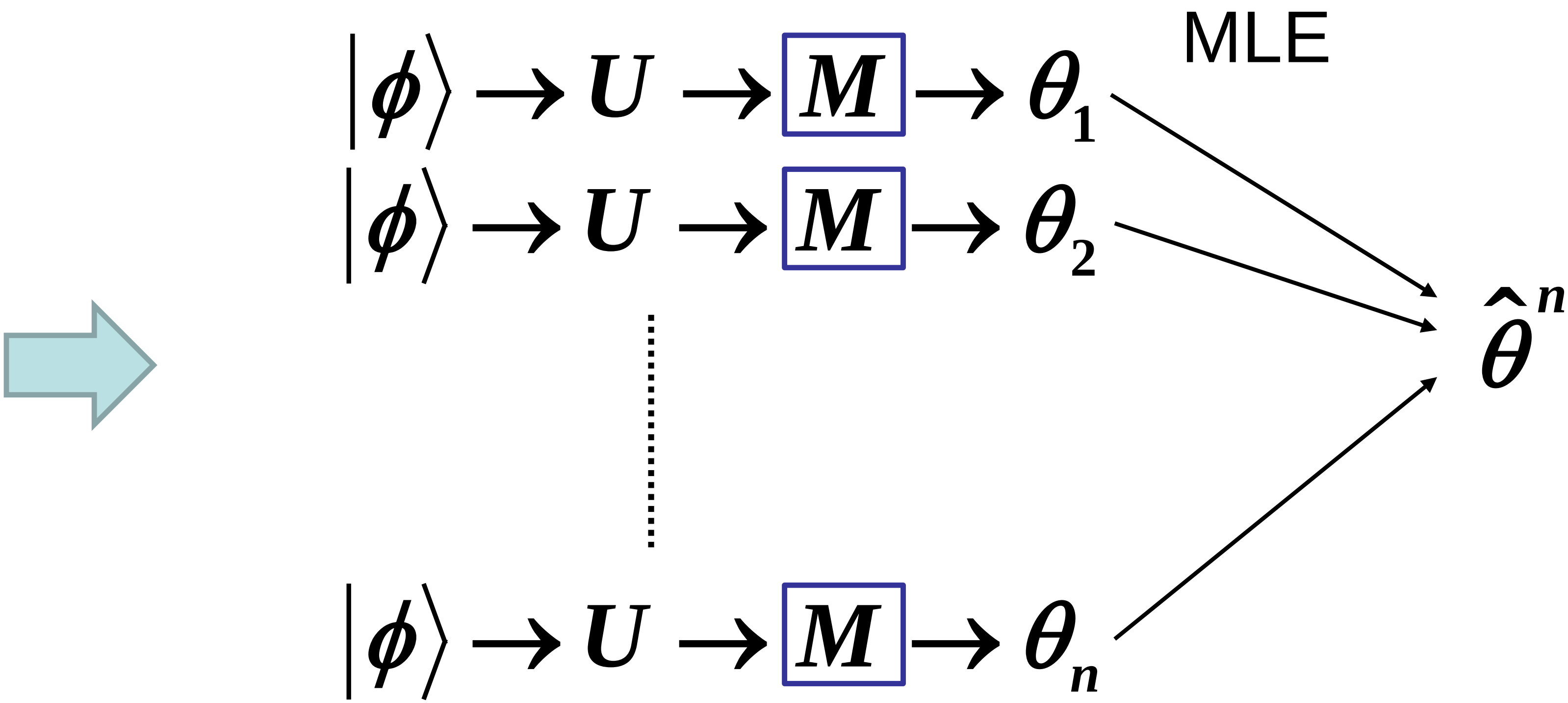
Assume that the support of ϕ contains only integer rep. or half integer rep.



This method attains the optimal performance.

Implication of these optimal estimators

When we consider the energy constraint, entangled input state and measurement with entangled basis do not enhance the quality of estimation.



This method attains the optimal performance.

Uncertainty relation on $L_p^2((-\pi,\pi])$

$$= L^2(\mathrm{U}(1)) = L^2(S^1)$$

$$\Delta_\phi^2(\cos Q, \sin Q) := \Delta_\phi^2 \cos Q + \Delta_\phi^2 \sin Q$$

$$\min_{\phi \in L_p^2((-\pi,\pi])} \left\{ \Delta_\phi^2(\cos Q, \sin Q) \mid \Delta_\phi^2 P \le E \right\}$$

$$= \max_{s>0} 1 - (sE - \frac{s a_0(2/s)}{4})^2$$

The minimum is realized by $\mathrm{ce}_0(\frac{\theta}{2}, -\frac{2}{s_E})$

$$s_E := \arg\max_{s>0} 1 - (sE - \frac{s a_0(\frac{2}{s})}{4})^2$$

Uncertainty relation on $L^2(\mathrm{SU}(2)) = L^2(S^3)$

$$g \mapsto (x_0(g), x_1(g), x_2(g), x_3(g)) \in S^3$$

$$\Delta_\phi^2 \vec{Q} := \sum_{j=0}^{3} \Delta_\phi^2 Q_j, \quad \Delta_\phi^2 \vec{P} := \sum_{j=1}^{3} \Delta_\phi^2 P_j$$

$$P_j \phi := \frac{d\phi(e^{it\sigma_{j/2}} g)}{dt}\Big|_{t=0}$$

$$\min_{\phi \in L^2(\mathrm{SU}(2))} \left\{ \Delta_\phi^2 \vec{Q} \mid \Delta_\phi^2 \vec{P} \le E \right\}$$

$$= \max_{s>0} 1 - (s(E + 1/4) - s b_2(\frac{8}{s}) / 16)^2$$

Function ϕ realizing the minimum is given by using

$$\mathrm{se}_2(\frac{\theta}{4}, -\frac{8}{s_E})$$

Uncertainty relation between $\Delta_{\phi,\max} Q$ and $\Delta_\phi P$ on $L^2(\mathbb{R})$

$$\min_{\phi \in L^2(\mathbb{R})} \left\{ \Delta_\phi^2 P \mid \Delta_{\phi,\max}^2 Q \le L \right\} = \min_{\phi \in L^2((-L,L])} \Delta_\phi^2 P = \frac{\pi^2}{4L^2}$$

where

$$\Delta_{\phi,\max} Q := \max_{\langle \phi | E_\lambda | \phi \rangle > 0} | \lambda - \langle \phi | Q | \phi \rangle |$$

$$Q = \int_{-\infty}^{\infty} \lambda E_\lambda d\lambda$$

Uncertainty relation between $\Delta_\phi Q$ and $\Delta_{\phi,\max} P$ on $L_p^2((-\pi,\pi])$

$$\Delta_\phi^2(\cos Q, \sin Q) := \Delta_\phi^2 \cos Q + \Delta_\phi^2 \sin Q$$

$$\min_{\phi \in L_p^2((-\pi,\pi])} \left\{ \Delta_\phi^2(\cos Q, \sin Q) \mid \Delta_{\phi,\max}^2 P \le E \right\}$$

$$= \sin^2 \frac{\pi}{2\lfloor E \rfloor + 2}$$

The minimum is realized by $C \sin \dfrac{\pi(\lambda + \lfloor E \rfloor + 1)}{2\lfloor E \rfloor + 2}$

Uncertainty relation between $\Delta_\phi^2 \vec{Q}$ and $\Delta_{\phi,\max}^2 \vec{P}$ on $L^2(\mathrm{SU}(2))$

$$\min_{\phi \in L^2(\mathrm{SU}(2))} \left\{ \Delta_\phi^2 \vec{Q} \mid \Delta_{\phi,\max}^2 \vec{P} \le E \right\}$$

$$= \sin^2 \frac{\pi}{\left\lfloor 2\left(\sqrt{E^2 + \frac{1}{4}} - \frac{1}{2}\right) \right\rfloor + 2}$$

Function ϕ realizing the minimum is given by using

$$\sum_{k=0}^{n} \sin \frac{(k+1)\pi}{n+2} \frac{\sin(k+1)\theta/2}{\sin\theta/2}$$

Conclusion

- We have proposed a method with Inverse Fourier transform as a unified approach for estimation of group action
- Using this method, we have derived the optimal estimator with energy constraint in several groups.
- We have shown that entanglement of input and output cannot improve under energy constraint.
- We have applied it to uncertainty relation.

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