Torus skein algebra and mirror symmetry

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§1 Introduction

- Turaev's skein algebra $\mathsf{Sk}(\Sigma)$ is a q-deformation of Goldman's Lie algebra.

○ Goldman's Lie algebra on an oriented surface Σ is the Lie algebra encoding the symplectic structures of character varieties on Σ.

- Recently, Morton and Samuelson discovered remarkable relationship between skein algebra for torus T and the Ringel-Hall algebra of elliptic curve E.

$$\mathsf{Sk}(\mathsf{T}) \simeq \mathsf{DHall}(\mathsf{E})$$

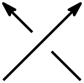
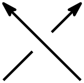
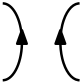
- In this talk I explain a new proof of Morton-Samuelson isomorphism based on a version of homological mirror symmetry of torus/elliptic curve.

Contents

§2.1 Definition of skein algebra

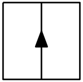
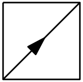
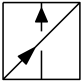
- $\mathsf{R} := \mathbb{Z}[s^{\pm 1}, v^{\pm 1}]$
- Skein module of oriented 3-fold M

$$S(M) := \left\langle \begin{array}{l} \text{isotopy classes of} \\ \text{framed oriented links in } M \end{array} \right\rangle_{\text{R-lin}} \Big/ (\text{sk}), (\text{fr})$$

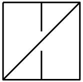
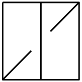
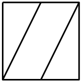
(sk)  $-$  $=$ $(s - s^{-1})$ 

- **Skein algebra** of oriented surface Σ
  - $\mathbf{Sk}(\Sigma) := S(\Sigma \times I)$ as R-module.
  - Multiplication is given by placing one copy of $\Sigma \times I$ on top of another $\Sigma \times I$.

- Example: $\Sigma =$ torus
  - $L_{0,1}$ $\quad$ $L_{1,1}$ $\quad$ $L_{1,1} \cdot L_{0,1}$

  - $L_{1,1} \cdot L_{0,1} = L_{0,1} \cdot L_{1,1} + (s - s^{-1})L_{1,2}$

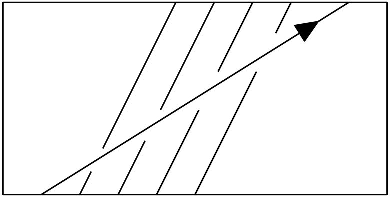
§2.2 Facts

Theorem [Turaev, 1990]

$$\mathrm{Sk(annulus)} \simeq \mathrm{R}[\mathrm{L_d} \mid \mathrm{d} \in \mathbb{Z} \setminus \{0\}]$$

For $\mathrm{d} > 0$, $\mathrm{L_d}$ is the class of the following loop. For $\mathrm{d} < 0$, $\mathrm{L_{-d}}$ is the class of the loops with the inverse orientation.

$\mathrm{L_5}$



Theorem [Turaev, 1989,1991]

Sk(Σ) is an s-deformation of the Goldman Lie algebra of Σ:

$$[\langle\alpha\rangle, \langle\beta\rangle]_{\mathrm{Goldman}} = \sum_{p\in\alpha\cap\beta} \pm\langle\alpha_p\beta\rangle.$$

Theorem [Morton-Samuelson, 2014]

Sk(torus) $\simeq$ a specialization of elliptic Hall algebra.

§3.1 Definition of Hall algebra

- $\mathcal{C}$: $\mathbb{F}_p$-linear finitary abelian category
  - Ringel-Hall algebra for $\mathcal{C}$
  - Hall($\mathcal{C}$) $:= \langle [M]; M \in \mathrm{Iso}(\mathcal{C})\rangle_{\mathbb{C}\text{-lin}}$ as R-module,
  - multiplication $[M]\cdot[N] = \sum_{L\in \mathrm{Iso}(\mathcal{C})} g^L_{M,N}[L]$,

    $g^L_{M,N} := \#\{N' \subset L \mid N' \simeq N,\ L/N' \simeq M\}$.

- If $\mathcal{C}$ is hereditary, then Hall($\mathcal{C}$) is a bialgebra with Hopf pairing $\langle [M],[N]\rangle := \delta_{M,N}/\#\mathrm{Aut}_{\mathcal{C}}(M)$.

- Drinfeld double of Hall($\mathcal{C}$) for hereditary $\mathcal{C}$:
  - DHall($\mathcal{C}$) $:= \mathrm{Hall}(\mathcal{C}) \otimes_{\mathbb{C}} \mathrm{Hall}(\mathcal{C})$
  - $(m \otimes 1)\cdot(1 \otimes n) = m \otimes n$
  - $\sum \langle m_{(2)}, n_{(1)}\rangle m_{(1)} \otimes n_{(2)} = \sum \langle m_{(1)}, n_{(2)}\rangle (1 \otimes n_{(1)})\cdot(m_{(2)} \otimes 1)$

§3.2 Elliptic Hall algebra

Theorem [Burban-Schiffmann, 2006]

$\mathcal{C} = \mathrm{Coh}(\text{elliptic curve } E/\mathbb{F}_p)$

$\mathrm{DHall}(\mathcal{C})$ has a basis $\{u_x \mid x \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$ with relations

(1) $[u_x, u_y] = 0$ if x, y parallel

(2) $[u_x, u_y] = \pm\theta_{x+y}/\alpha_1$ if $\Delta(0, x, x+y)$ contains no integral points.

$1 + \sum_{k \geq 1} \theta_{kz} w^k = \exp(\sum_{n \geq 1} \alpha_n u_{nz} w^n)$,

$\alpha_n := (1 - q^n)(1 - t^{-n})(1 - q^n t^{-n})/n$,

q, t^{-1}: Weil numbers of E. $q/t = p$,

$\zeta_E(w) := \exp(\sum \#E(\mathbb{F}_{p^n}) w^n/n) = \frac{(1-qw)(1-w/t)}{(1-w)(1-pw)}$.

Definition

$\mathbb{U}_{q,t}$: the algebra over $\mathbb{Z}[q^{\pm 1/2}, t^{\pm 1/2}]$ generated by $w_x \propto u_x$ with relations (1), (2).

§3.3 Morton-Samuelson isomorphism

Theorem [Morton-Samuelson (2017)]

$$\mathrm{Sk}(\mathrm{torus}) \xrightarrow{\sim} \mathbb{U}_{s^2,s^2} \otimes_{\mathbb{Z}[s^{\pm 1}]} \mathbb{Z}[s^{\pm 1}, v^{\pm 1}],$$

$$L_{r,d} \longmapsto w_{(r,d)} \text{ if } \gcd(r,d) = 1.$$

Remarks

(1) $\mathbb{U}_{q,t=q}$ is still well-defined since

$$\theta_x/\alpha_1 = ([\gcd(x)]_{q^{1/2}})^2 u_x.$$

(2) For $t = q$, $w_x = (q^{d/2} - q^{-d/2})u_x$, $d := \gcd(x)$.

§4.1 Outline of new proof

Morton and Samuelson showed the isomorphism "by hand".

However, as they mentioned in their paper, it is natural to invoke homological mirror symmetry for torus/elliptic curve:

$$\mathrm{D}\mathcal{F}\mathrm{uk}(\mathrm{T}) \simeq \mathrm{D}^{\mathrm{b}}\mathrm{Coh}(\mathrm{E}/\mathbb{C}), \quad \mathrm{L}_{1,\mathrm{d}} \leftrightarrow \mathcal{L}_{\mathrm{d}}.$$

One may guess that the Morton-Samuelson isomorphism comes from equivalence of categories.

One drawback: the algebra $\mathbb{U}_{s^2,s^2}$ corresponds to elliptic curve E with Weil numbers $q = t = s^2$.
E is defined over $\mathbb{F}_p$, $p = q/t$.
So in our situation E is "defined over $\mathbb{F}_1$".

Outline of our proof is

(B1) Build a category $\mathcal{B}$ using
monoidal Tate curve $\widehat{\mathbf{E}}$.
("category of coherent shaves over $\mathbf{E}/\mathbb{F}_1$")

(B2) Consider Hall($\mathcal{B}$) following Szczesny's definition.

(B3) Check DHall($\mathcal{B}$) $\simeq \mathbb{U}_{\mathbf{q},\mathbf{q}}$.

(A1) Build another $\mathcal{A}$ associated to tori
(Fukaya category of tori)

(A2) Consider Hall algebra Hall($\mathcal{A}$)

(A3) Check DHall($\mathcal{A}$) $\simeq$ Sk($\mathcal{A}$).

(HMS) Show $\mathcal{A} \simeq \mathcal{B}$.

- **Today only B-side is explained.**
- **A-side and HMS are established by taking $\mathbb{F}_1$-analogue of**

Theorem [Gross, Lekili-Perutz]

HMS holds for torus/elliptic curves over $\mathbb{Z}$.

§4.2 Quasi-exact category

- A: (multiplicative commutative) monoid with 0.
- ∘ A-module is a pointed set $(M, *)$ with A-action $\cdot$ such that $0 \cdot m = *$.
- ∘ Morphisms, submodules, images, kernels are as usual.
- ∘ Quotient $M/N := (M \setminus N) \sqcup \{*\}$.
- ∘ A-mod: the category of A-modules.

- A-mod is NOT abelian since in general $\mathrm{coim}(f) \to \mathrm{im}(f)$ is not an isomorphism.

- A-mod is not even additive, but has
  - ∘ zero object $\{*\}$
  - ∘ $M \oplus N := M \sqcup N/\sim$ with $\sim$ identifying base-points
  - ∘ product, finite limits and colimits ...

§4.3 Hall algebra for quasi-exact category

- Definition for module category of semigroup.
- morphism f in A-mod is called normal if $\mathrm{coim}(f) \xrightarrow{\sim} \mathrm{im}(f)$.
- $A\text{-mod}^n$: the subcategory of A-mod with the same objects and only normal morphisms.

Definition (Szczesny 2014)

$\mathrm{Hall}(A\text{-mod}^n)$ is the algebra counting exact sequences consisting of normal morphisms.

§4.4 Tate curve

- Recall the (usual) Tate curve $\widehat{E}_{\text{Tate}}$:

  $\rho_i := \mathbb{Q}_{\geq 0}(i,1) \subset \mathbb{Q}^2 \quad (i \in \mathbb{Z})$
  $\rho_i^\vee := \{x \in \mathbb{Q}^2 \mid \langle x, \rho_i \rangle \subset \mathbb{Q}_{\geq 0}\} = \{(x_1, x_2) \mid ix_1 + x_2 \geq 0\}$
  $\sigma_{i+1/2}^\vee := \rho_i^\vee \cap \rho_{i+1}^\vee$.

  For a commutative ring R, $U_{i+1/2} := \operatorname{Spec} R[\sigma_{i+1/2}^\vee \cap \mathbb{Z}^2]$
  These glue to give E over R.
  The map $\sigma_{i+1/2} \to \mathbb{Q}$, $(x,y) \mapsto y$ gives $E \to \operatorname{Spec} R[q]$.

  Tate curve is a formal thickening $\widehat{E}_{\text{Tate}} \to \operatorname{Spf} R[[q]]$.

Monoidal schemes / $\mathbb{F}_1$-schemes

- I use monoidal scheme analogue of Tate curve.
- ◦ I follow Deitmar for the definition of $\mathbb{F}_1$-schemes.
  (One can also use Connes and Consani's $\mathbb{F}_1$-schemes.)

- For a monoid A, an ideal $\mathfrak{a}$ is a subset such that $\mathfrak{a}A \subset \mathfrak{a}$.
- ◦ Prime ideals, localizations, ... are defined as usual.
- ◦ Affine monoidal scheme $\mathrm{Spec}^{\mathrm{mon}}(A)$ is the set of prime ideals of A with Zariski topology.

- A monoidal schemes is a topological space X with a sheaf $\mathcal{O}_X^{\mathrm{mon}}$ of monoids, locally isomorphic to some $\mathrm{Spec}^{\mathrm{mon}}(A)$.

- (coherent) $\mathcal{O}_X^{\mathrm{mon}}$-modules are defined as usual.

Category $\mathcal{B}$

- $\mathrm{Spf}^{\mathrm{mon}}\langle\langle q\rangle\rangle$: (formal) monoidal scheme of $\langle\langle q\rangle\rangle = q^{\mathbb{N}}$.
- $\widehat{E}$: monoidal Tate curve, defined by replacing rings in the construction of $\widehat{E}_{\mathrm{Tate}}$ by monoids.

◦ $\rho_i := \mathbb{Q}_{\geq 0}(i, 1) \subset \mathbb{Q}^2 \quad (i \in \mathbb{Z})$

◦ $\rho_i^\vee := \{x \in \mathbb{Q}^2 \mid \langle x, \rho_i\rangle \subset \mathbb{Q}_{\geq 0}\} = \{(x_1, x_2) \mid ix_1 + x_2 \geq 0\}$

◦ $\sigma_{i+1/2}^\vee := \rho_i^\vee \cap \rho_{i+1}^\vee$.

◦ $U_{i+1/2}^{\mathrm{mon}} := \mathrm{Spec}^{\mathrm{mon}}\langle \sigma_{i+1/2}^\vee \cap \mathbb{Z}^2\rangle$

Gluing and formal thickening gives $\widehat{E} \to \mathrm{Spf}^{\mathrm{mon}}\langle\langle q\rangle\rangle$.

Definition (Step (B1))

$\mathcal{B}$:= category of coherent $\mathcal{O}_{\widehat{E}}^{mon}$-modules and normal morphisms.

- One can apply Szczesny's construction of Hall algebra to $\mathcal{B}$ (Step (B2)).

Theorem (Step (B3))

$$\mathrm{DHall}(\mathcal{B}) \simeq \mathbb{U}_{q,q}$$

Speculations

- Higher genus analogue

∘ ∃? isomorphism

Sk(higher genus) ≃ Hall(Matrix Factorization).

∘ Should consider an $\mathbb{F}_1$-analogue of HMS for higher genus (Seidel-Efimov)

- Topological realization of Hall algebra extension

∘ Ding-Iohara-Miki algebra (=quantum toroidal $\mathfrak{gl}_1$) is a central extension of $\mathbb{U}_{q,t}$:

$$0 \to \mathbb{Z}[q^{\pm 1}, t^{\pm 1}][K_1, K_2] \to \mathrm{DIM} \to \mathbb{U}_{q,t} \to 0.$$

∘ $\mathbb{U}_{q,q}$ is realized as a diagrammatic algebra. DIM may have a topological realization related to character variety on torus.