Deformed conformal block

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§1 Quick review of the usual conformal block

§1.1 Definition

$\mathsf{M}_{*,n}$: the moduli stack of stable smooth (not necessarily connected) n-pointed curves.
$\overline{\mathsf{M}}_{*,n}$: the Deligne-Mumford compactification.
$\mathsf{M}_{*,A}$, $\overline{\mathsf{M}}_{*,A}$: similar notations for a finite set A.

$\mathcal{C}$: a semi-simple abelian category, linear over $\mathbb{C}$.
Assume $\mathcal{C}$ has a distinguished object $1 \in \mathcal{C}$
and a symmetric object $\mathsf{R} \in \text{ind-}\mathcal{C}^{\boxtimes 2}$
(i.e. it has $\sigma : \mathsf{R}^{\mathsf{op}} \xrightarrow{\sim} \mathsf{R}$ s.t. $\sigma^{\mathsf{op}}\sigma = \mathrm{id}$).

Definition

A complex analytic modular functor is the data (1) – (4) satisfying some functoriality and compatibility conditions.

(1) The bundle of conformal block.
For any finite set A and $V \in \mathcal{C}^{\boxtimes A}$, a finite dim. vector bundle $\langle V \rangle$ on $\overline{M}_{*,A}$ with flat connection with regular singularities.
The fiber at $C \in \overline{M}_{*,A}$ is denoted by $\langle V \rangle_C$,

(2) An isom. $\langle V_1 \rangle_{C_1} \otimes \langle V_2 \rangle_{C_2} \xrightarrow{\sim} \langle V_1 \otimes V_2 \rangle_{C_1 \sqcup C_2}$

(3) The gluing isomorphism.
For $\alpha \neq \beta \in A$ and for $V \in C^{\boxtimes A'}$ with $A' := A \setminus \{\alpha, \beta\}$, an isom. $\langle V \boxtimes R \rangle \xrightarrow{\sim} G^*_{\alpha,\beta} \langle V \rangle$ over $M_{*,A}$.

(4) The vacuum propagation.
For $\alpha \in A$, an isomorphism $\langle V \boxtimes 1 \rangle \xrightarrow{\sim} F^*_{\alpha} \langle V \rangle$.

In (3) $G_{\alpha,\beta} : M_{*,A} \to N(D^0) \to M_{*,A'}$ is the clutching morphism, and in (4) $F_\alpha : M_{*,A} \to M_{*,A \setminus \{\alpha\}}$ is the morphism forgetting the point α.

Fact

A complex analytic MF is equivalent to a ($\mathcal{C}$-extended) topological 2-dim. MF.

Outline of the proof:
By the Riemann-Hilbert correspondence, $\langle \mathbf{V} \rangle$ gives a local system on $\mathbf{M}_{*,\mathbf{A}}$.
The topological MF is a collection of functors $\tau(\Sigma) : \mathcal{C}^{\boxtimes \pi_0(\partial\Sigma)} \to \mathcal{V}\mathrm{ec}^{\mathrm{fin}}$ for an oriented surface Σ with a specified point on each component of the boundary.
The local system determines $\tau(\Sigma)$ and vice versa.

Remark

One can define an infinite rank analogue of complex MF by replacing "vector bundles with ..." by D-modules.

§1.2 MF with central charge

Definition

A MF with central charge $c \in \mathbb{C}$ is the data (1'),(2)–(4) with the same condition.
(1') A coherent $\mathcal{D}_{Q^c}$-module $\langle V \rangle$ for any A and $V \in \mathcal{C}^{\boxtimes A}$.

For a flat family $\pi : C_S \to S$, $Q = Q_S := \det R^\bullet \pi_* \mathcal{O}_{C_S}$.
$\mathcal{D}_{Q^c}$ is the sheaf of twisted differential operators, or
$\mathcal{D}_{Q^c} := U(\mathcal{A}_{Q^c})/(f - \psi_c(f))$.
$\mathcal{A}_Q$ is the sheaf of Lie algebra of 1st. order differential operators.

$$0 \to \mathcal{O}_S \xrightarrow{\psi} \mathcal{A}_Q \to \Theta_S \to 0.$$

The sheaf $\mathcal{A}_{Q^c}$ is similarly defined but

$$0 \to \mathcal{O}_S \xrightarrow{\psi_c} \mathcal{A}_{Q^c} \to \Theta_S \to 0$$

with $\psi_c := \psi \exp(2\pi\sqrt{-1}c)$.

§1.3 Examples from CFT - WZNW models

$\mathfrak{g}$: fin. dim. simple Lie algebra.
$\widehat{\mathfrak{g}}$: the associated affine Lie algebra.
$\mathcal{C} := \mathcal{O}_k^{\mathrm{int}}$: cat. of integrable modules of level $k \in \mathbb{Z}_{\geq 0}$ of $\widehat{\mathfrak{g}}$.
$1 :=$ triv, $R := \sum_i V_i \otimes V_i^*$ (V_i :simple modules).

Definition

For a marked stable curve $C = (C, p)$ and $V = V_1 \otimes \cdots \otimes V_n$,

$$\tau(C, V) := V/\mathfrak{g}(C - p)V,$$

called the space of coinvariants.

$\mathfrak{g}(C - p) := \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}(C - p)$ acts on V via

$$\oplus_i \gamma_i : \mathfrak{g}(C - p) \to \mathfrak{g}((t))^n$$

given by the Laurent expansion around each point p_i

Fact

$\langle V \rangle_C := \tau(C, DV_1 \otimes \cdots \otimes DV_n)^*$ gives rise to a MF with central charge $c = k \dim \mathfrak{g}/(k + h^\vee)$.

D is the duality functor in $\mathcal{O}_k^{\mathrm{int}}$.

Sketch of the construction of projective flat connection.

Consider a family $C_S = C \times S \to S$ with a fixed curve $C = (c, p)$ with one marked point,
and S is the set of formal local parameters at the point p.
$V_S := \mathcal{O}_S \otimes V$, $\tau_S := V_S/\mathfrak{g}(C_S - p(S))V_S$.
Enough to construct Θ_S-action on τ_S compatible with $\mathcal{O}_S$-module.
One can act Θ_S on V_S by the Sugawara construction.
Then check the action preserves $\mathfrak{g}(C_S - p(S))V_S$, so descends to τ_S.

§1.4 Hidden role of Virasoro algebra

Sugawara construction means there is an injection

$$\mathrm{Vir}_c \hookrightarrow \widehat{U}(\widehat{\mathfrak{g}})_k$$

of Virasoro Lie algebra Vir_c with central charge $c = k \dim \mathfrak{g}/(k + h^\vee)$.

Then Vir_c acts on V, so that its subalgebra

$$\mathrm{Vir}_+ := t\mathbb{C}[[t]]\partial_t \subset \mathrm{Vir}_c, \quad t^{n+1}\partial_t \longmapsto -L_n$$

acts on V.

On the other hand, $K_0 := \mathrm{Aut}\,\mathbb{C}[[t]]$ acts on $\mathcal{O}_S$ by the change of parameters, and $\mathrm{Lie}(K_0) \simeq \mathrm{Vir}_+$.

Using these two actions, one can construct a Θ_S-action on V_S.

Remark

The same construction works for rational VOA.

§2 Deformed conformal blocks

§2.1 Deformed Virasoro algebra

Definition (Shiraishi-Kubo-Awata-Odake, 1996)

$\mathrm{Vir}_{q,t}$ with $q, t \in \mathbb{C}$ is the assoc. algebra over $\mathbb{C}$ generated by T_n ($n \in \mathbb{Z}$) and 1 with the relation

$$[T_n, T_m] = -\sum_{k=1}^{\infty} f_k (T_{n-k} T_{m+k} - T_{m-k} T_{n+k}) - \frac{(1-q)(1-t^{-1})}{1-q/t} \left((q/t)^n - (q/t)^{-n}\right) \delta_{n+m,0}$$

$$\text{where } \sum_{k=0}^{\infty} f_k z^k = \exp\Big[\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{1+(q/t)^n} \frac{z^n}{n}\Big].$$

Remark

The expansion under the limit $q = e^{\hbar\epsilon_1}$, $t = e^{\hbar\epsilon_2}$, $\hbar \to 0$ gives

$$\sum T_n z^{-n} = 2 + \beta\hbar^2 \left(\sum L_n z^{-n} + \frac{(1-\beta)^2}{4\beta}\right) + O(\hbar^4), \quad \beta = \epsilon_2/\epsilon_1$$

and L_n satisfying the defining relation of Vir_c with $c = 1 - 6(1-\beta)^2/\beta$.

There are several reasons to consider it as a correct deformation.
∘ Representation theory looks quite similar.
∘ Connection to Macdonald symmetric functions, similar as Jack symmetric functions appearing in Vir.
∘ Connection to solvable integrable lattices.
∘ There is a family of deformed W-algebras including $Vir_{q,t}$, which can be considered as a deformation of representation ring of the quantum affine algebras.
∘ K-theoretic AGT correspondence.

§2.2 Deformed conformal blocks

We want to introduce a deformation of MF by replacing Vir_c (appearing implicitly) by $Vir_{q,t}$.
It seems that one should replace D-module by holonomic difference system.
$\Longrightarrow$ need to specify the asymptotic expansion.
One should modify $M_{*,A}$ correctly.

Definition

For a curve (C, p) with n-marked points,
an admissible graph is an oriented trivalent graph G on C with the marked points as its leaves
together with length on each internal edge,
and 3 non-zero vectors $v_i, v_j, v_k \in \mathbb{Z}^2$ at each inner vertex such that

$$v_i + v_j + v_k = 0, \quad v_i \wedge v_j = 1.$$

(Calabi-Yau and smoothness condition.)

$\mathcal{T}_{*,n}$: the groupoid of stable smooth curves with n-marked points and having admissible graphs modulo isotopy fixing the points.
$\mathcal{C}$: a ribbon category linear over $\mathbb{C}$.

Definition

A deformed MF for $\mathcal{C}$ is a tower of functors $\sqcup_n \mathcal{T}_{*,n} \to \mathrm{Fun(C)}$.

Fun(C) is the tower of groupoids whose objects are all the pairs (A, F) of a finite set A and a functor $F : \mathcal{C}^{\boxtimes A} \to \mathcal{V}\mathrm{ec}$.

Theorem

A deformed MF gives rises to a locally free sheaf $\langle V \rangle_C$ for any finite set A and $V \in \mathcal{C}^{\boxtimes A}$ which is equipped with a holonomic difference system.

The resulting object is called the deformed conformal block.

Key point of the proof.
The holonomic difference system comes from the braided tensor structure (or Yang-Baxter relation) of $\mathcal{C}$.

§2.3 Example from quantum algebra

$\mathsf{U} = \mathsf{U}_{tor}(\mathfrak{gl}_1)$: the quantum toroidal algebra for $\mathfrak{gl}_1$ with parameters $q, t \in \mathbb{C}$.
It is a (topological) Hopf algebra and the category of representations has a structure of ribbon category.

U has the 'level 0' module $\mathcal{F}_u^{(0)}$ [Feigin-Tsymbaliuk]
and the 'level 1' $\mathcal{F}_v^{(1)}$ [Feigin-Hashizume-Hoshino-Shiraihi-Y].

There are two trivalent intertwiners [Awata-Feigin-Shirahi]

$$\Phi : \mathcal{F}_v^{(0)} \otimes \mathcal{F}_u^{(1)} \to \mathcal{F}_w^{(1)}, \quad a\Phi = \Phi\Delta_U(a)$$

and

$$\Phi^* : \mathcal{F}_w^{(1)} \to \mathcal{F}_v^{(1)} \otimes \mathcal{F}_u^{(0)}, \quad \Phi^* a = \Delta_U(a)\Phi^*.$$

Since $\mathcal{F}_u^{(0)}$ has Macdonald symmetric functions $P_\lambda = P_\lambda(q,t)$ as linear basis, we can define Φ_λ by

$$\Phi_\lambda := \Phi(P_\lambda \otimes -) : \mathcal{F}_u^{(1)} \to \mathcal{F}_w^{(1)}$$

and similarly $\Phi_\lambda^* : \mathcal{F}_w^{(1)} \to \mathcal{F}_u^{(1)}$ by

$$\Phi(-) = \sum_\lambda \Phi_\lambda^*(-) \otimes Q_\lambda.$$

Fact

Let G be an admissible graph on $\mathbb{P}^1$ with $\{Q_k\}$ the lengths of the internal edges.
Then putting Φ_λ or Φ_λ^* on each internal vertex,
and putting the gluing factor $\propto Q_k^{|\lambda_k|}$,
we have a well-defined intertwines of U-modules
$\mathcal{F}_{u_1}^{v_1} \otimes \cdots \otimes \mathcal{F}_{u_m}^{v_m} \to \mathcal{F}_{u_1'}^{v_1'} \otimes \cdots \otimes \mathcal{F}_{u_n'}^{v_n'}$.

Remark

The gluing factor was discovered by Taki in the context of refined topological vertex.

Theorem

The above construction gives rise to a deformed MF for the category of highest weight level 1 U-modules.

Remark

Awata-Feigin-Shiraishi showed that a combination of Φ_λ and Φ^*_λ reproduces the Iqbal-Kozcaz-Vafa refined topological vertex $C^{\lambda}_{\mu,\nu}(q,t)$.
Actually the reproduced one is expressed by $P_\lambda(q,t)$ discovered by Awata-Kanno.

§2.4 Correct deformation?

Conjecture

In the deformed MF for U, take the limit $Q_k = e^{1/\hbar} \to \infty$. Then the resulting limit of deformed conformal blocks recovers the conformal blocks of Liouville theory (Virasoro CFT with generic central charge and highest weight).

This is a theorem for specialized spectral parameters u_i for the sake of (cohomological and K-theoretic) AGT relation. Namely, we know that the deformed conformal blocks coincides with the K-theoretic Nekrasov partition functions, and its limit is the cohomological Nekrasov partition functions, which is the irregular conformal blocks by cohomological AGT relation.

The role of $\mathrm{Vir}_{q,t}$ is a little bit obscure.

Fact (Awata-Feigin-Hoshino-Kanai-Shiraishi-Y)

There is an action of $\mathrm{Vir}_{q,t}$ in a two-component tensor $\mathcal{F}^{(1)}_{u_1} \otimes \mathcal{F}^{(1)}_{u_2}$ of level 1 U-modules.

More generally, one can show that n-component tensor has a structure of $W_{q,t}(\mathfrak{sl}_n)$-module.

We proposed a definition of deformed VOA using a modification of Beilinson-Drinfeld chiral algebra in a previous work (arXiv:1402.2943).

Conjecture

Any deformed VOA gives rises to a deformed MF.

Remark

At present we don't know what is the correct definition of rationality of deformed VOA.

§3.1 Comment on geometric engineering

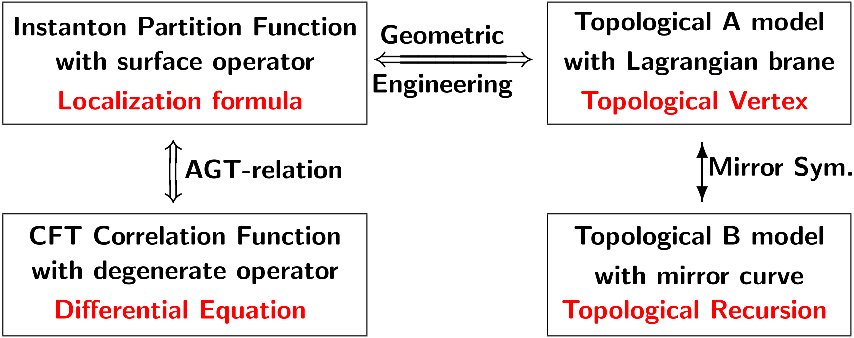

Figure: Citation from Awata-Fuji-Kanno-Manabe-Yamada (arXiv:1008.0574).

§3.2 Summary

Today we proposed a definition of deformed conformal blocks, and showed that it has an example in the quantum toroidal algebra and it seems to be a correct deformation.

There are still lots of things to do.
More examples should be discovered, in particular for the quantum affine algebras or toroidal algebras.
The degeneration limit should also be studied in careful, concerning the asymptotic behavior of conformal blocks.