Quantum toroidal algebras and motivic Hall algebras of elliptic singular fibers

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§0 Contents

§1 Quantum toroidal algebra of type A

§1.1 Definition

$d, q \in \mathbb{C}$: generic complex parameters, i.e.,

$$q_1 := dq^{-1}, \quad q_2 := q^2, \quad q_3 := d^{-1}q^{-1}$$

satisfies $q_1^{n_1} q_2^{n_2} q_3^{n_3} = 1$ for $n_1, n_2, n_3 \in \mathbb{Z}$ only if $n_1 = n_2 = n_3$.

$\mathcal{E}_n$: the quantum toroidal algebra of type A ($n \geq 1$).
Generators:

$$E_{i,k}, F_{i,k}, H_{i,r}, K_i^{\pm 1}, q^{\pm c} \qquad (i \in \mathbb{Z}/n\mathbb{Z},\ k \in \mathbb{Z},\ r \in \mathbb{Z}/\{0\}).$$

Currents:

$$E_i(z) := \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) := \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k},$$

$$K_i^{\pm}(z) := K_i^{\pm 1} \exp\Big(\pm(q - q^{-1}) \sum_{r=1}^{\infty} H_{i,\pm r} z^{\mp r}\Big).$$

Relations:

- $q^{\pm c}$ are central, $\quad q^c q^{-c} = q^{-c} q^c = 1, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1$
- $K_i^{\pm}(z) K_j^{\pm}(w) = K_j^{\pm}(w) K_i^{\pm}(z)$
- $\dfrac{g_{i,j}(q^{-c}z, w)}{g_{i,j}(q^c z, w)} K_i^{-}(z) K_j^{+}(w) = \dfrac{g_{j,i}(w, q^{-c}z)}{g_{j,i}(w, q^c z)} K_j^{+}(w) K_i^{-}(z)$
- $d_{i,j} g_{i,j}(z, w) K_i^{\pm}(q^{(1\mp 1)c/2} z) E_j(w) + g_{j,i}(w, z) E_j(w) K_i^{\pm}(q^{(1\mp 1)c/2} z) = 0$

  $d_{j,i} g_{j,i}(w, z) K_i^{\pm}(q^{(1\pm 1)c/2} z) F_j(w) + g_{i,j}(z, w) F_j(w) K_i^{\pm}(q^{(1\pm 1)c/2} z) = 0$
- $[E_i(z), F_j(w)] = \dfrac{\delta_{i,j}}{q - q^{-1}} (\delta(q^c w/z) K_i^{+}(z) - \delta(q^c z/w) K_i^{-}(w))$
- $d_{i,j} g_{i,j}(z, w) E_i(z) E_j(w) + g_{j,i}(w, z) E_j(w) E_i(z) = 0$

  $d_{j,i} g_{j,i}(w, z) F_i(z) F_j(w) + g_{i,j}(z, w) F_j(w) F_i(z) = 0$
- Serre-like relation

Structure constant/function:

$$\mathbf{d_{i,j}} := \begin{cases} \mathbf{d^{\mp 1}} & (\mathbf{i \equiv j \mp 1,\ n \geq 3}), \\ \mathbf{-1} & (\mathbf{i \not\equiv j,\ n = 2}), \\ \mathbf{1} & (\text{otherwise}). \end{cases}$$

and

$$\mathbf{n \geq 3 : g_{i,j}(z, w)} := \begin{cases} \mathbf{z - q_2 w} & (\mathbf{i \equiv j \mod n}), \\ \mathbf{z - q_1 w} & (\mathbf{i \equiv j - 1}), \\ \mathbf{z - q_3 w} & (\mathbf{i \equiv j + 1}), \\ \mathbf{z - w} & (\mathbf{i \not\equiv j, j + 1}), \end{cases}$$

$$\mathbf{n = 2 : g_{i,j}(z, w)} := \begin{cases} \mathbf{z - q_2 w} & (\mathbf{i \equiv j}), \\ \mathbf{(z - q_1 w)(z - q_3 w)} & (\mathbf{i \not\equiv j}), \end{cases}$$

$$\mathbf{n = 1 : g_{0,0}(z, w) := (z - q_1 w)(z - q_2 w)(z - q_3 w)}$$

§1.2 Grading and coproduct

- $\mathcal{E}_n$ is $\mathbb{Z}^n \times \mathbb{Z}$-graded by the degree assignment

$$\deg E_{i,k} = (1_i, k), \quad \deg F_{i,k} = (-1_i, k), \quad \deg H_{i,r} = (0, r),$$
$$\deg K_i = \deg q^c = (0, 0),$$

where $1_i = (0, \dots, \overset{i-th}{1}, \dots, 0) \in \mathbb{Z}^n$.

- The algebra $\mathcal{E}_n$ has also a formal coproduct (Drinfeld coproduct)

$$\Delta E_i(z) = E_i(z) \otimes 1 + K_i^-(C_1 z) \otimes E_i(C_1 z),$$
$$\Delta F_i(z) = F_i(C_2 z) \otimes K_i^+(C_2 z) + 1 \otimes F_i(z),$$
$$\Delta K_i^+(z) = K_i^+(z) \otimes K_i^+(C_1^{-1} z),$$
$$\Delta K_i^-(z) = K_i^-(C_2^{-1} z) \otimes K_i^-(z),$$
$$\Delta q^c = q^c \otimes q^c$$

with $C_1 := q^c \otimes 1$ and $C_2 := 1 \otimes q^c$.

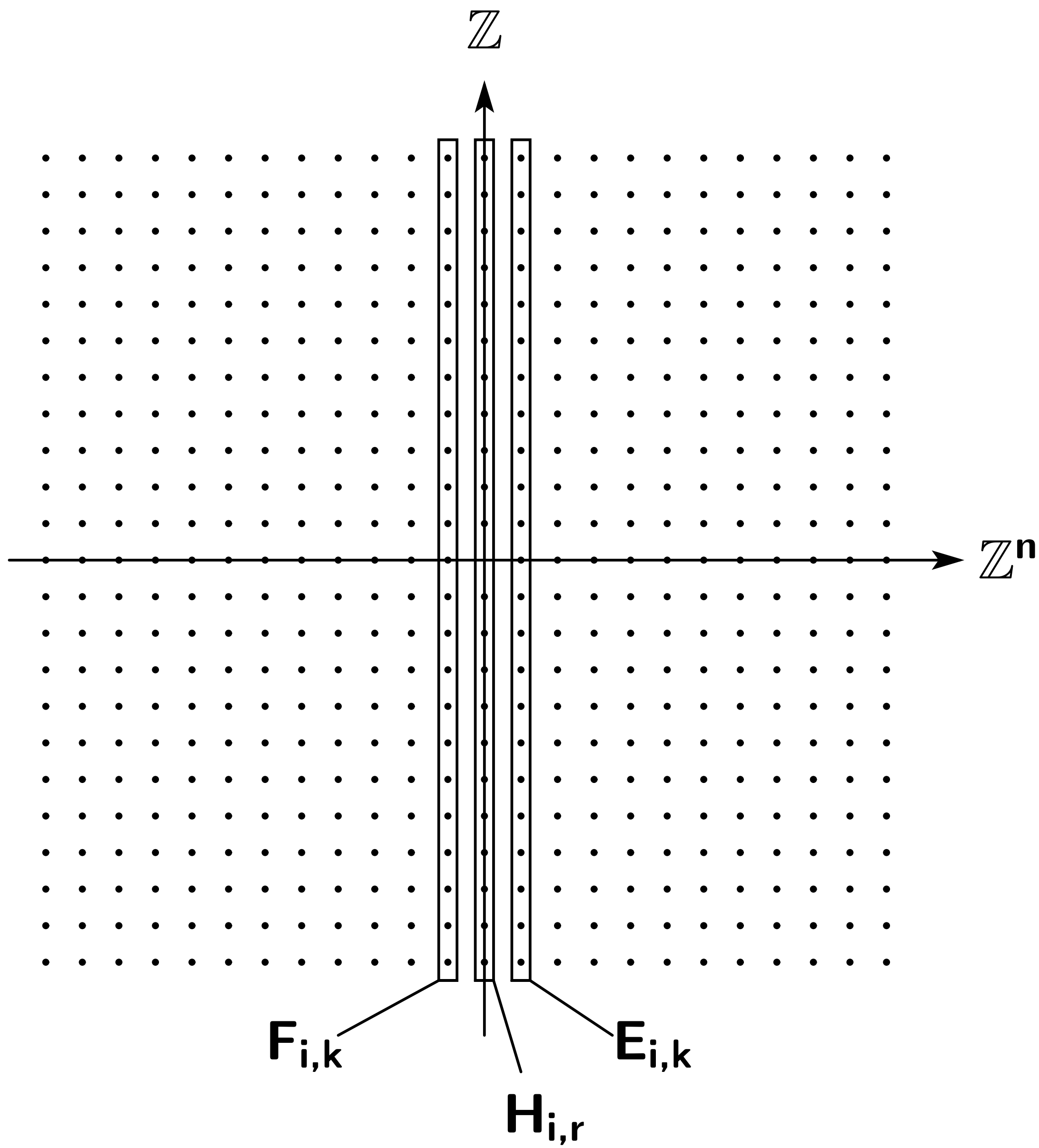


Figure 1: Grading on $\mathcal{E}_n$

Remark 1.1. 1. This definition is given in B.Feigin-Jimbo-Miwa-Mukhin (arXiv:1309.2147),

2. $\mathcal{E}_n$ with $n \geq 3$ is introduced in Ginzburg-Kapranov-Vasserot (1995).

3. $\mathcal{E}_1$ has several names.

   (a) It was introduced in Burban-Schiffmann (2012) with the name (Drinfeld double of) elliptic Hall algebra.

   (b) Miki (2007) named (q, γ)-analog of $W_{1+\infty}$,

   (c) B.Feigin-Hashizume-Hoshino-Shiraishi-Y (2009) named Ding-Iohara(-Miki) algebra.

   (d) B.Feigin-E.Feigin-Jimbo-Miwa-Mukhin (2010) named quantum continuous gl_∞.

§1.3 Horizontal and vertical subalgebras

- $\mathcal{E}_n$ $(n \geq 2)$ has two subalgebras, both are isomorphic to $U_q(\widehat{sl}_n)$.
- The horizontal embedding $h : U_q(\widehat{sl}_n) \hookrightarrow \mathcal{E}_n$

$$e_i \mapsto E_{i,0}, \quad f_i \mapsto F_{i,0}, \quad t_i \mapsto K_{i,0} \quad (0 \leq i \leq n-1)$$

with Chevalley generators of $U_q(\widehat{sl}_n)$

$$e_i,\ f_i,\ t_i^{\pm 1} \quad (0 \leq i \leq n-1).$$

- The vertical embedding $v : U_q(\widehat{sl}_n) \hookrightarrow \mathcal{E}_n$ is given by

$x^+_{i,k} \mapsto d^{ik} E_{i,k},\ x^-_{i,k} \mapsto d^{ik} F_{i,k},\ k_i \mapsto K_i,\ h_{i,r} \mapsto d^{ir} H_{i,r}\ (1 \leq i \leq n-1),$
$q^c \mapsto q^c$

with Drinfeld generators

$$x^{\pm}_{i,k},\ h_{i,r},\ k_i^{\pm 1},\ q^{\pm c} \quad (1 \leq i \leq n-1,\ k \in \mathbb{Z},\ r \in \mathbb{Z} \setminus \{0\}).$$

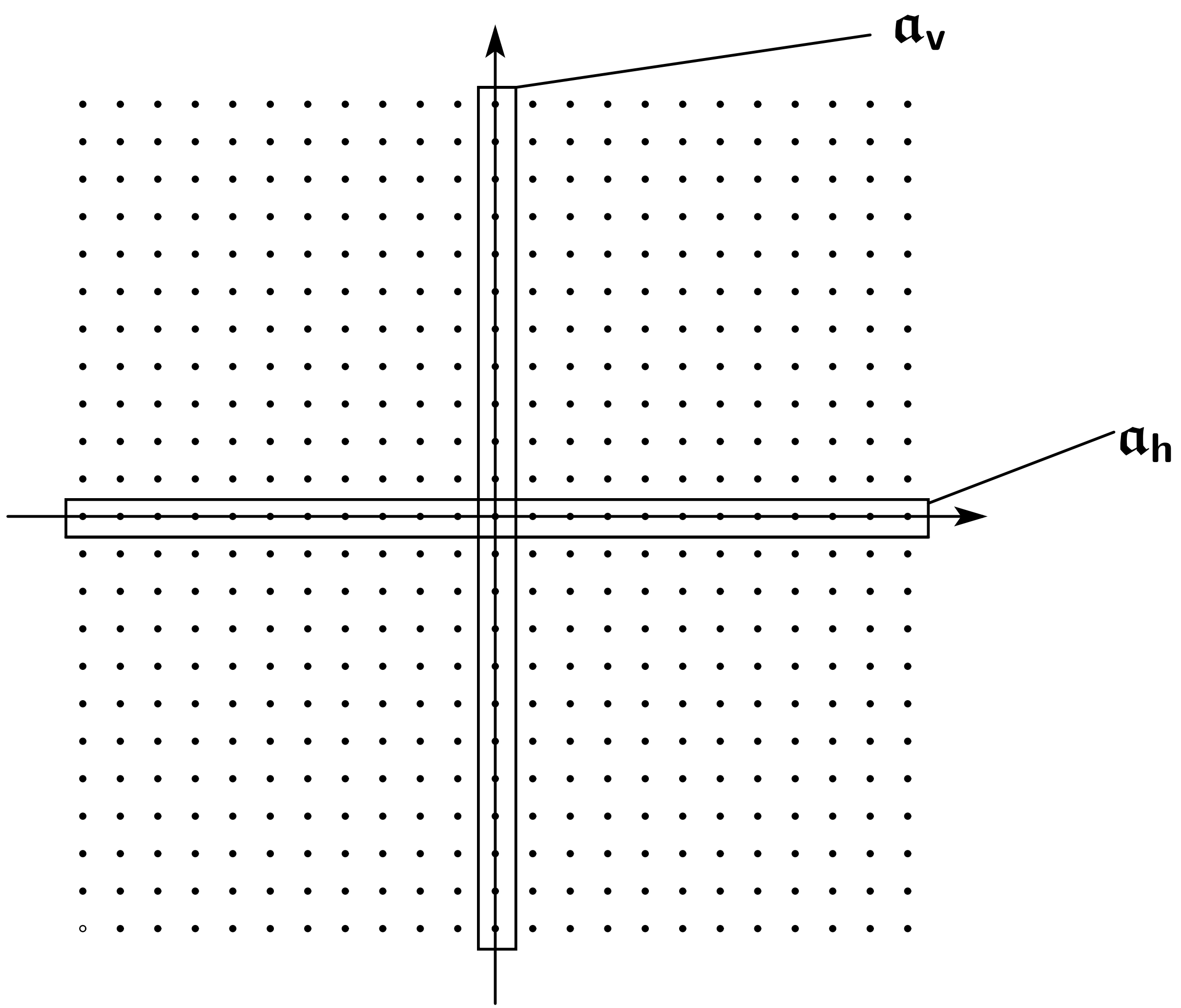


Figure 2: The two Heisenberg subalgebras of $\mathcal{E}_1$

Remark 1.2. **For $n = 1$, there are two subalgebras of $\mathcal{E}_1$ which are isomorphic to some Heisenberg algebra.**

§1.4 Automorphism

∘ Miki (1999,2001) gave an algebra automorphism θ_n of $\mathcal{E}_n$ $(n \geq 3)$ s.t.

$$\theta_n \circ v = h, \quad \theta_n \circ h = v \circ \tau \circ \sigma,$$

with σ, τ anti-automorphisms of $U_q(\widehat{sl}_n)$

$$\sigma : e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad t_i \mapsto t_i^{-1}$$
$$\tau : x^{\pm}_{i,k} \mapsto x^{\pm}_{i,-k}, \quad h_{i,r} \mapsto -q^{rc} h_{i,-r}, \quad k_i \mapsto k_i^{-1}, \quad q^c \mapsto q^c.$$

∘ $n = 2$ case is stated in B.Feigin-Jimbo-Miwa-Mukhin (2013).

∘ For $\mathcal{E}_1$, Burban-Schiffmann constructed θ_1 s.t.

$$E_{0,0} \mapsto -q^c H_{0,-1}, \quad F_{0,0} \mapsto a q^c H_{0,1},$$
$$H_{0,1} \mapsto E_{0,0}, \quad H_{0,-1} \mapsto -a F_{0,0},$$
$$q^c \mapsto K_0, \quad K_0 \mapsto q^{-c}$$

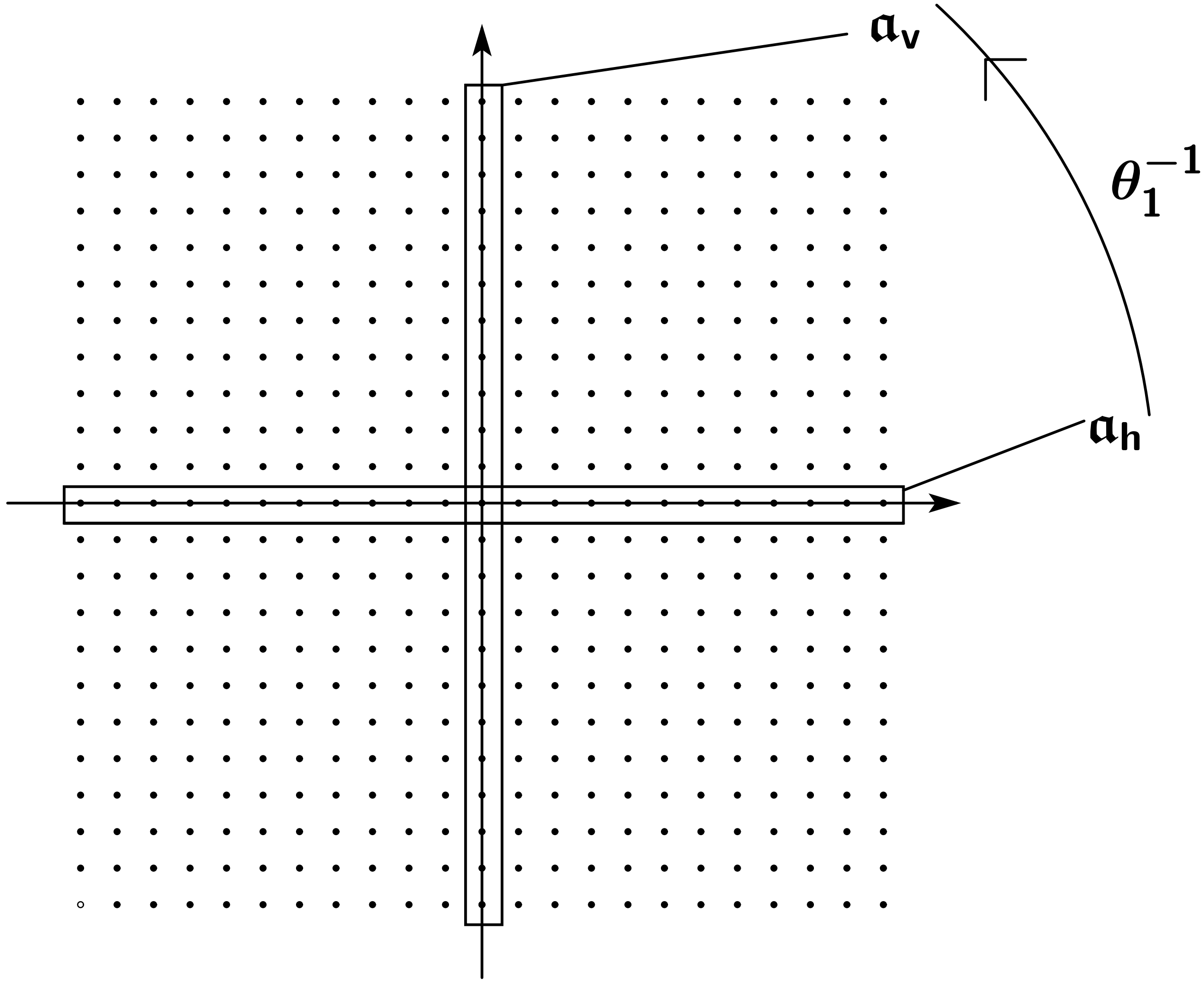


Figure 3: The automorphism θ_1

§1.5 Motivation of this study

• Miki's construction of θ_n is purely algebraic.
The construction of θ_1 by Burban-Schiffmann utilizes the Hall algebra of coherent sheaves on an elliptic curve C defined over $\mathbb{F}_q$ and Foruier-Mukai transform on $D^b \mathrm{Coh}(C)$.

• I want to construct θ_n in the spirit of Burban and Schiffmann.
The algebra $\mathcal{E}_n$ will be identified with the Hall algebra of a cycle of $\mathbb{P}^1$'s which appears in the singular fiber of elliptic fibration $S \to B$.
θ_n is induced from the relative Fourier-Mukai transform on $D^b \mathrm{Coh}(S)$.

• Drawback: the relative Fourier-Mukai transforms are defined over $\mathbb{C}$, and cumbersome to check the reduction to $\mathbb{F}_q$.
Solution: use motivic Hall algebras

• My construction also works for quantum toroidal algebras of type ADE (introduced by Ginzburg-Kapranov-Vasserot).

§2 Quick review of ordinary Hall algebras

§2.1 Hall algebra $\mathcal{H}(\mathcal{A})$

$\mathcal{A}$: Abelian category such that

(a) essentially small and every set of morphisms are finite.
(b) $\mathcal{A}$ is $\mathbb{F}_q$-linear
(c) $\mathcal{A}$ has enough projective objects and the global dimension is finite

Definition 2.1. (Ringel-Hall algebra).

1. For $A, B, C \in \mathrm{Iso}(\mathcal{A})$,

$$\mathrm{Ext}^1_{\mathcal{A}}(A, C)_B := \{0 \to C \to B \to A \to 0 \mid \text{exact sequences in } \mathcal{A}\}/\sim .$$

2. $\mathcal{H}(\mathcal{A})$ is the associative $\mathbb{C}$-algebra with unit $[0]$ given by
   (i) $\langle [A] \mid A \in \mathrm{Iso}(\mathcal{A})\rangle_{\mathrm{lin}}$ as a $\mathbb{C}$-linear space.
   (ii) The multiplication $\diamond$ given by

$$[A]\diamond[C] := \sum_{B\in \mathrm{Iso}(\mathcal{A})} \frac{|\mathrm{Ext}^1_{\mathcal{A}}(A, C)_B|}{|\mathrm{Hom}_{\mathcal{A}}(A, C)|} \cdot [B].$$

§2.2 Twisted and extended Hall algebras

$K(\mathcal{A})$: Grothendieck group of $\mathcal{A}$,

$\overline{M}$: the class of $M \in \mathcal{A}$ in $K(\mathcal{A})$.

Remark 2.2. 1. Euler form: bilinear form on $K(\mathcal{A})$ given by

$$\langle \alpha, \beta \rangle := \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^i \dim_{\mathbb{F}_q} \operatorname{Ext}^i_{\mathcal{A}}(A, B),$$

where $A, B \in \mathcal{A}$ are such that $\overline{A} = \alpha, \overline{B} = \beta$.

2. symmetric Euler form
$(\cdot, \cdot) : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}, \quad (\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$

Now fix $\sqrt{q}$.

Definition 2.3. 1. The twisted Hall algebra $\mathcal{H}_{tw}(\mathcal{A})$ is $\mathcal{H}(\mathcal{A})$ as $\mathbb{C}$-linear space with the multiplication $*$ given by

$$[A]*[C] := \sqrt{q}^{\langle \overline{A}, \overline{C} \rangle} \cdot [A] \diamond [C].$$

2. The extended Hall algebra $\mathcal{H}^{ext}(\mathcal{A})$ is the extension of $\mathcal{H}_{tw}(\mathcal{A})$ by $\{k_\alpha \mid \alpha \in K(\mathcal{A})\}$ with the relation

$$k_\alpha * k_\beta = k_{\alpha+\beta}, \quad k_\alpha * [B] = \sqrt{q}^{(\alpha, \overline{B})} \cdot [B] * k_\alpha.$$

Remark 2.4.

$$\{k_\alpha * [B] \mid \alpha \in K(\mathcal{A}),\ B \in \mathrm{Iso}(\mathcal{A})\}$$

gives a basis of $\mathcal{H}^{ext}(\mathcal{A})$.

§2.3 Bialgebra structure

Fact 2.5. (Green 1996)

1. The maps

$$\Delta([A]) := \sum_{B,C} \sqrt{q}^{\langle \overline{B}, \overline{C} \rangle} \frac{|\operatorname{Ext}_{\mathcal{A}}(B,C)_A|}{|\operatorname{Aut}_{\mathcal{A}}(A)|} [B] \otimes [C], \quad \epsilon([A]) := \delta_{A,0}.$$

$$\Delta(k_\alpha) := k_\alpha \otimes k_\alpha, \quad \epsilon(k_\alpha) := 1,$$

   make $\mathcal{H}^{\mathrm{ext}}(\mathcal{A})$ a formal coassociative coalgebra.

2. If $\mathcal{A}$ also satisfies

   (d) $\mathcal{A}$ is of global dimension 1

   then $\mathcal{H}^{\mathrm{ext}}(\mathcal{A})$ is a formal bialgebra.
   It also has the Hopf pairing

$$([A] * k_\alpha, [B] * k_\beta) := \frac{\delta_{A,B}(\alpha, \beta)}{|\operatorname{Aut}_{\mathcal{A}}(A)|}$$

§2.4 Examples of Hall bialgebras

1. (Ringel) Q: finite quiver without edge loop,
   $\mathcal{A} = \operatorname{Rep} Q$: category of finite-dim. representations of Q over $\mathbb{F}_q$
   C_Q: (symmetric) Cartan matrix associated to Q.
   $(C_Q)_{ij} := 2\delta_{i,j} - a_{ij} - a_{ji}, \quad a_{ij} := |\{i \to j\}|$.
   $U_t(\mathfrak{g}_Q)$: quantum enveloping algebra of $\mathfrak{g}_Q$ determined by C_Q.
   $U_t(\mathfrak{n}), U_t(\mathfrak{b})$: nilpotent/Borel subalgebras.
   Then we have embeddings

   $$U_{t=\sqrt{q}}(\mathfrak{n}) \hookrightarrow \mathcal{H}_{tw}(\mathcal{A}), \quad U_{t=\sqrt{q}}(\mathfrak{b}) \hookrightarrow \mathcal{H}^{ext}(\mathcal{A}).$$

   If Q is Dynkin diagram of type ADE, then these are isomorphisms.
2. (Kapranov) $\mathcal{A} = \operatorname{Coh} \mathbb{P}^1$ (over $\mathbb{F}_q$), then

   $$U_{t=\sqrt{q}}(L\mathfrak{b}) \hookrightarrow \mathcal{H}^{ext}(\mathcal{A})$$

   Here $\mathfrak{b}$ is the Borel subalgebra of sl_2.

§3 Grothendieck ring of stacks

§3.1 The Grothendieck ring of algebraic varieties

§3.2 Motivic zeta function

§3.3 Grothendieck ring of stacks

§3.4 Relative Grothendieck ring

§3.1 The Grothendieck ring of algebraic varieties

Definition 3.1. **K(Var/k)**: the Grothendieck ring of varieties over a field k

the quotient of the free abelian group on isomorphism classes of varieties over k, by the relations

$$[X] = [Y] + [X \setminus Y], \quad Y : \text{closed subvariety of the variety } X.$$

with the product

$$[X] \cdot [Y] = [(X \times Y)_{\mathrm{red}}].$$

∘ Denote the class of the affine line by

$$\mathbb{L} := [\mathbb{A}^1] \in K(\mathrm{Var}/k).$$

Examples.

1. For the d-th general linear group GL_d,

$$[GL_d] = \mathbb{L}^{d(d-1)/2} \prod_{k=1}^{d} (\mathbb{L}^d - 1) = \mathbb{L}^{d(d-1)/2} (\mathbb{L} - 1)^d [d]_{\mathbb{L}}!,$$

   where $[n]_v := 1 + v + \cdots + v^{n-1}$ and $[n]_v! := [1]_v \cdot [2]_v \cdot \cdots \cdot [n]_v$ for $n \in \mathbb{Z}_{\geq 0}$ and v an indeterminate.

2. For the Grassmann variety $Gr(d, n)$ with $0 \leq d \leq n$,

$$[Gr(d, n)] = \begin{bmatrix} n \\ d \end{bmatrix}_{\mathbb{L}} := \frac{[n]_{\mathbb{L}}}{[d]_{\mathbb{L}}[n-d]_{\mathbb{L}}}.$$

§3.2 Motivic zeta function

For simplicity, assume $k = \mathbb{C}$.

Definition 3.2. (Kapranov 2000)
For a quasi-projective variety X over a field $\mathbb{C}$, the motivic zeta function of X is defined to be

$$Z_{mot}(X;t) := \sum_{n\geq 0} [\mathrm{Sym}^n(X)] t^n \in 1 + t \cdot K(\mathrm{Var}/\mathbb{C})[[t]].$$

Fact 3.3. (Kapranov 2000)
X_g: a smooth, geometrically connected, projective curve of genus g.

$$Z_{mot}(X_g;t) = \frac{f(t)}{(1-t)(1-\mathbb{L}t)}$$

where f is a polynomial of degree 2g with coefficients in $K(\mathrm{Var}/\mathbb{C})$.

§3.3 Grothendieck ring of stacks

By the word 'stack', we mean an Artin stack which is locally of finite type over a field k.

Given a scheme S over k and a stack X, we denote by X(S) the groupoid of S-valued points of X.

Definition 3.4. 1. A stack X locally of finite type over k is said to have affine stabilizers if for every k-valued point $x \in X(k)$ the group $\mathrm{Iso}_k(x, x)$ of isomorphisms is affine.

2. A morphism $f : X \to Y$ in the category St/k will be called a geometric bijection if it is representable and the induced functor on groupoids of k-valued points $f(k) : X(k) \to Y(k)$ is an equivalence of categories.
3. A morphism of stacks $f : X \to Y$ is a Zariski fibration if its pullback to any scheme is a Zariski fibration of schemes.

Definition 3.5. (Joyce 2006).
$K(St/k)$: the free abelian group spanned by isomorphism classes of stacks of finite type over k with affine stabilizers, modulo relations

(a) $[X_1 \sqcup X_2] = [X_1] + [X_2]$ for every pair of stacks X_1 and X_2,
(b) $[X] = [Y]$ for every geometric bijection $f : X \to Y$,
(c) $[X_1] = [X_2]$ for every pair of Zariski fibrations $f_i : X_i \to Y$ with the same fibres.

∘ Fibre product of stacks over k makes $K(St/k)$ a commutative ring.

∘ By considering a variety as a stack, we have a ring homomorphism

$$K(Var/k) \longrightarrow K(St/k) \tag{1}$$

Fact 3.6. (Bridgeland 2012).
The homomorphism (1) induces an isomorphism of commutative rings

$$Q : K(Var/\mathbb{C})\big[[GL_d]^{-1} \mid d \in \mathbb{Z}_{\geq 1}\big] \xrightarrow{\sim} K(St/\mathbb{C}).$$

§3.4 Relative Grothendieck ring

S: stack which is locally of finite type over k and has affine stabilizers.

There is a 2-category of stacks over S.

Definition 3.7. **St/S: the full subcategory of the above 2-category consisting of objects $f : X \to S$ for which X is of finite type over k and have affine stabilizers.**

Remark 3.8. **For our purpose, take $S = \mathcal{M}$: moduli stack of coherent sheaves over a projective variety.**

Definition 3.9. (Relative Grothendieck group)
$K(St/S)$: the free abelian group spanned by isomorphism classes of objects of St/S modulo relations

(a) for every pair of objects $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$, a relation

$$[X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} S] = [X_1 \xrightarrow{f_1} S] + [X_2 \xrightarrow{f_2} S]$$

(b) for every commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad g \quad} & X_2 \\ & {}_{f_1}\searrow \quad \swarrow_{f_2} & \\ & S & \end{array}$$

with g a geometric bijection, a relation

$$[X_1 \xrightarrow{f_1} S] = [X_2 \xrightarrow{f_2} S]$$

(c) for every pair of Zariski fibrations $h_1 : X_1 \to Y$ and $h_2 : X_2 \to Y$ with the same fibers, and every morphism $g : Y \to S$, a relation

$$[X_1 \xrightarrow{g \circ h_1} S] = [X_1 \xrightarrow{g \circ h_2} S].$$

- The group $K(St/S)$ is a $K(St/k)$-module by

$$[X] \cdot [f : Y \longrightarrow S] = [f \circ p_2 : X \times Y \longrightarrow S].$$

Fact 3.10. **(Bridgeland).**
Assume that all stacks appearing have affine stabilizers.

1. **A morphism of stacks $a : S \to T$ induces a push-forward morphism of $K(St/k)$-modules**

$$a_* : K(St/S) \longrightarrow K(St/T), \quad [f : X \longrightarrow S] \longmapsto [a \circ f : X \longrightarrow T].$$

2. **A morphism of stacks $a : S \to T$ of finite type induces a pullback morphism of $K(St/k)$-modules**

$$a^* : \; K(St/T) \longrightarrow K(St/S), \quad [Y \xrightarrow{g} T] \longmapsto [X \xrightarrow{f} S],$$

where f is given by

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ S & \xrightarrow[a]{} & T \end{array}$$

3 For a pair of stacks (S_1, S_2), we have a **Künneth morphism**

$$K : K(St/S_1) \otimes K(St/S_2) \longrightarrow K(St/S_1 \times S_2)$$

of $K(St/k)$-modules given by

$$[X_1 \xrightarrow{f_1} S_1] \otimes [X_2 \xrightarrow{f_2} S_2] \longmapsto [X_1 \times X_2 \xrightarrow{f_1 \times f_2} S_1 \times S_2].$$

§4 Motivic Hall algebra

X: a smooth projective variety over $k = \mathbb{C}$

§4.1 Moduli of flags

∘ $\mathcal{M}^{(n)} = \mathcal{M}^{(n)}(X)$: moduli stack of n-flags of coherent sheaves on X.

The objects of $\mathcal{M}^{(n)}$ over a scheme S are chains of monomorphisms of coherent sheaves on $S \times X$ of the form

$$E_\bullet := (0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = E)$$

such that each factor $F_i := E_i/E_{i-1}$ is S-flat.

∘ $\mathcal{M} := \mathcal{M}^{(1)}$.

Definition 4.1. 1. There are morphisms of stacks

$$a_i : \mathcal{M}^{(n)} \longrightarrow \mathcal{M}$$

for $1 \leq i \leq n$, sending a flag $E_\bullet$ to its i-th factor $F_i = E_i/E_{i-1}$.

2. There is another morphism

$$b : \mathcal{M}^{(n)} \longrightarrow \mathcal{M}$$

sending a flag $E_\bullet$ to the maximum sheaf $E_n = E$.

§4.2 Definition of motivic Hall algebra

Recall $\mathcal{M} = \mathcal{M}(X)$: moduli stack of coherent sheaves on the fixed X.

Definition 4.2. (Joyce 2006, Bridgeland 2012).

1. As a K(St/k)-module, set

$$H(X) := K(St/\mathcal{M}).$$

2. Introduce $m : H(X) \otimes H(X) \longrightarrow H(X)$ of K(St/k)-modules by

$$m : H(X) \otimes H(X) = K(St/\mathcal{M}) \otimes K(St/\mathcal{M}) \xrightarrow{K} K(St/\mathcal{M} \times \mathcal{M}) \xrightarrow{(a_1,a_2)^*} K(St/\mathcal{M}^{(2)}) \xrightarrow{b_*} K(St/\mathcal{M}) = H(X)$$

where

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow{\scriptstyle (a_1,a_2)} & & \\ \mathcal{M} \times \mathcal{M} & & \end{array} \qquad \begin{array}{ccc} E_\bullet = (E_1 \hookrightarrow E) & \longmapsto & E\,. \\ \downarrow & & \\ (E_1, E/E_1) & & \end{array}$$

∘ **Denote m by ⋄ as a binary operator.**

Remark 4.3. ⋄ **can be rewritten as**

$$[X_1 \xrightarrow{f_1} \mathcal{M}] \diamond [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{b \circ h} \mathcal{M}],$$

where Z and h are defined by the following Cartesian square.

$$\begin{array}{ccccc} Z & \xrightarrow{\quad h \quad} & \mathcal{M}^{(2)} & \xrightarrow{b} & \mathcal{M} \\ \downarrow & & \downarrow{\scriptstyle (a_1,a_2)} & & \\ X_1 \times X_2 & \xrightarrow[f_1 \times f_2]{} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

Fact 4.4. (Joyce, Bridgeland)
$(\mathsf{H}(\mathsf{X}), \diamond)$ is an associative unital algebra over $\mathsf{K}(\mathsf{St}/\mathsf{k})$.
The unit element is $1 := [\mathcal{M}_0 \hookrightarrow \mathcal{M}]$, where $\mathcal{M}_0$ is the stack of zero objects.

Proposition 4.5. 1. One can extend the algebra $\mathsf{H}(\mathsf{X})$ by $\mathsf{K}(\mathsf{Coh}(\mathsf{X}))$ as in the ordinary Hall algebra. The extended one will be denoted by $(\mathsf{H}^{\mathrm{ext}}(\mathsf{X}), *)$.
2. If X is of dimension 1, then $\mathsf{H}^{\mathrm{ext}}(\mathsf{X})$ has a structure of formal bialgebra with a Hopf pairing.

§4.4 Case of projective line

Reorganize Kapranov's result on $\mathcal{H}^{\mathrm{ext}}(\mathrm{Coh}(\mathbb{P}^1))$ in terms of $\mathrm{H}^{\mathrm{ext}}(\mathbb{P}^1)$.

Fact 4.6. Indecomposable objects of the category $\mathrm{Coh}(\mathbb{P}^1)$ are

1. line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$ $(n \in \mathbb{Z})$,
2. torsion sheaves $T_{x,j} := \mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x^j$, where x is a closet point of $\mathbb{P}^1$ and $j \in \mathbb{Z}_{\geq 1}$.

Corollary 4.7. The moduli stack $\mathcal{M}_{\mathrm{tor}}$ of torsion sheaves on $\mathbb{P}^1$ has the decomposition

$$\mathcal{M}_{\mathrm{tor}} = \bigsqcup_{j \in \mathbb{Z}_{\geq 1}} \mathcal{M}_{\mathrm{tor},j},$$

where $\mathcal{M}_{\mathrm{tor},x,j}$ is the moduli stack of torsion sheaves of length j.

For a coherent sheaf E on $\mathbb{P}^1$, denote by
$\overline{E} := (\mathrm{rk}(E), \deg(E)) \in K(\mathrm{Coh}(\mathbb{P}^1)) \cong \mathbb{Z}^2$ the rank and degree of E.

Definition 4.8. 1. For any positive integer $j \in \mathbb{Z}_{\geq 1}$, define the element

$$\mathbf{1}_{(0,j)} := [\mathcal{M}_{\mathrm{tor},j} \hookrightarrow \mathcal{M}] \quad \in H(\mathbb{P}^1).$$

2. Set the family $\{T_j\}_{j \geq 1} \subset H(\mathbb{P}^1)$ by

$$1 + \sum_{j=1}^{\infty} \mathbf{1}_{(0,j)} t^j = \exp\Big(\sum_{j=1}^{\infty} \frac{T_j}{[j]_{\mathbb{L}}} t^j\Big).$$

3. Set $\mathbf{1}_{(0,0)} = T_0 := [\mathcal{M}_0 \hookrightarrow \mathcal{M}] = 1.$

Proposition 4.9. 1. Each one of the families $\{1_{(0,j)}\}_{j\geq 1}$ and $\{T_j\}_{j\geq 1}$ generates the same subalgebra $U(\mathbb{P}^1)_{tor}$ of $H(\mathbb{P}^1)$.

2. For any $i, j \in \mathbb{Z}_{\geq 1}$ we have

$$\Delta(T_j) = T_j \otimes 1 + K_{(0,j)} \otimes T_j.$$

Definition 4.10. 1. Denote by $\mathcal{M}_{1,n}$ the stack of line bundles with degree n.

2. The **composition algebra $U(\mathbb{P}^1)$** is the subalgebra of the Hall algebra $H^{ext}(\mathbb{P}^1)$ generated by the elements

$$L_n := [\mathcal{M}_{1,n} \hookrightarrow \mathcal{M}], \quad T_j, \quad C = K_{(0,1)}, \quad K = K_{(1,0)}$$

with $n \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 1}$.

Definition 4.11. Denote by $DU(\mathbb{P}^1)$ the reduced Drinfeld double of the bialgebra $U(\mathbb{P}^1)$.

◦ **Triangular decomposition:**

$$DU(\mathbb{P}^1) \cong U(\mathbb{P}^1)^+ \otimes K(St/\mathbb{C})[K] \otimes U(\mathbb{P}^1)^-.$$

Proposition 4.12. $U_{\sqrt{\mathbb{L}}}(\widehat{sl}_2)$ is isomorphic to $DU(\mathbb{P}^1)$ with

$$\begin{aligned}
&x^+_{1,n} \longmapsto L^+_n, \quad x^-_{1,n} \longmapsto L^-_n \quad (n \in \mathbb{Z}),\\
&h_{1,r} \longmapsto T^+_r, \quad h_{1,-r} \longmapsto -T^-_r \quad (r \in \mathbb{Z}_{>0}),\\
&k_1 \longmapsto K, \quad q^c \longmapsto C.
\end{aligned}$$

Remark 4.13. 1. To check the relation, one can use the motivic zeta function.
The proof of isomorphism uses the coproduct

$$\Delta L_n = L_n \otimes 1 + KC^n \otimes L_n + \sum_{r=1}^{\infty} \Theta_r KC^{n-r} \otimes L_{n-r}.$$

$$1 + \sum_{j=1}^{\infty} \Theta_j t^j = \exp\Big((\mathbb{L}^{-1} - \mathbb{L}) \sum_{r=1}^{\infty} T_r t^r\Big).$$

and the Hopf pairing on $H^{ext}(\mathbb{P}^1)$

$$(T_i, T_j) = \delta_{i,j} \frac{[2j]_{\mathbb{L}}}{j(\mathbb{L}^{-1} - \mathbb{L})}, \quad (\Theta_r, T_r) = \frac{[2r]_{\mathbb{L}}}{r}.$$

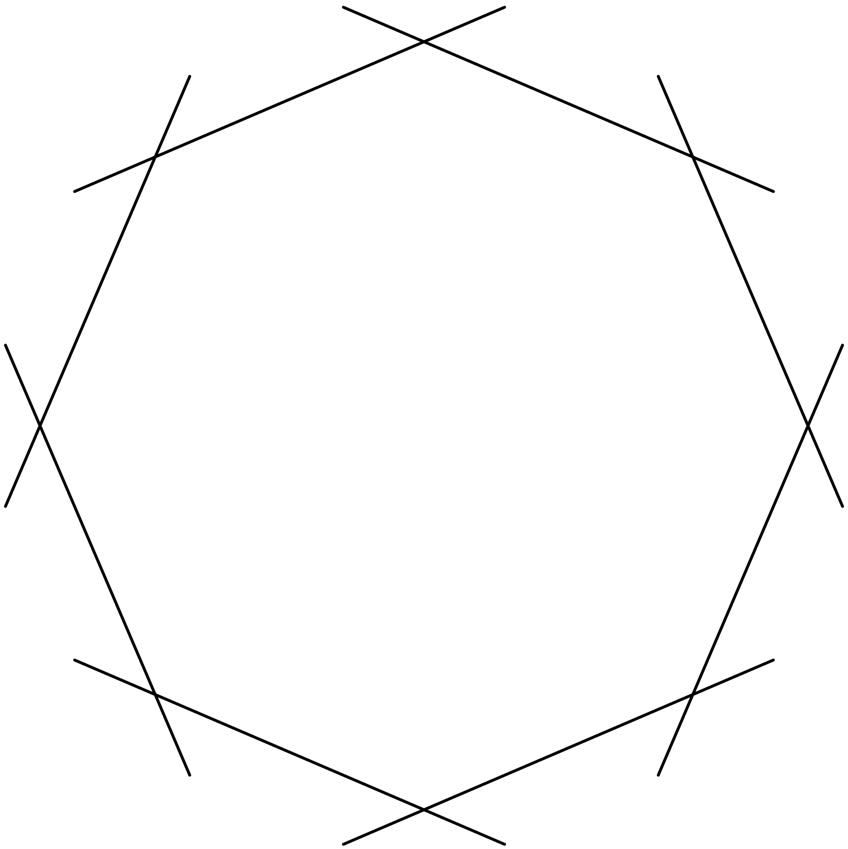
2. (Burban-Schiffmann) In the case of the elliptic curve C, DU(C) is isomorphic to $\mathcal{E}_1$.

§5 Hall algebras of cycles of projective lines

The formalism of motivic Hall algebra can be extended to the projective reduced connected curves.

§5.1 The algebraic structure

C_n: n-cycles of projective lines with numbering $1, 2, \dots, n$ ($n \geq 3$).



∘ A vector bundles on C_n of rank r is given by a collection of bundles $V^{(i)}$ of rank r on the i-th component of C_n with gluing data.

Definition 5.1. 1. $\mathcal{M}_{(i),1,n}$: the stack of coherent sheaves on C_n with $V^{(i)} \cong \mathcal{O}(n)$ and $V^{(j)} \cong \mathcal{O}$ for $j \neq i$.

2. $\mathcal{M}_{(i),0,r}$: the stack of torsion sheaves with support on the i-th component with length r.

3. The composition algebra $U(C_n) \subset H^{ext}(\mathbb{P}^1)$ generated by

$$L_{i,n} := [\mathcal{M}_{(i),1,n} \hookrightarrow \mathcal{M}], \quad T_{i,r} := [\mathcal{M}_{(i),0,r} \hookrightarrow \mathcal{M}], \quad K_\alpha$$

with $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 1}$ and $\alpha \in K(Coh(C_n))$.

Theorem 5.2. For $n \in \mathbb{Z}_{\geq 1}$, the quantum toroidal algebra $\mathcal{E}_n$ is isomorphic to the reduced Drinfeld double $DU(C_n)$ with

$$E_{i,k} \mapsto L^+_{i,k}, \quad F_{i,k} \mapsto L^-_{i,k} \quad (n \in \mathbb{Z}), \quad H_{i,\pm r} \mapsto \pm T^{\pm}_{i,r} \quad (r \in \mathbb{Z}_{>0}).$$

§5.2 The automorphism θ_n

Proposition 5.3. 1. For a curve X, an auto-equivalence Φ of $D^b \mathrm{Coh}(X)$ induces an automorphism Φ^H of $DH^{ext}(X)$.

2. Φ^H preserves the reduced Drinfeld double DU(X) of the composition algebra.

Remark 5.4. For the ordinary Hall algebra $\mathcal{H}^{ext}(\mathcal{A})$, the result 1 was shown by Cramer (2010).

Theorem 5.5. If $X = C_n$, then consider the relatively minimal elliptic surface $Y \to S$ with C_n its singular fiber.

Let Φ be the relative Fourier-Mukai transform on $D^b(Coh(Y))$ with the universal sheaf of the compactified relative Jacobian as its kernel.

Then the restriction of Φ to the singular fiber induces the algebra automorphism θ_n on $DU(C_n) = \mathcal{E}_n$.

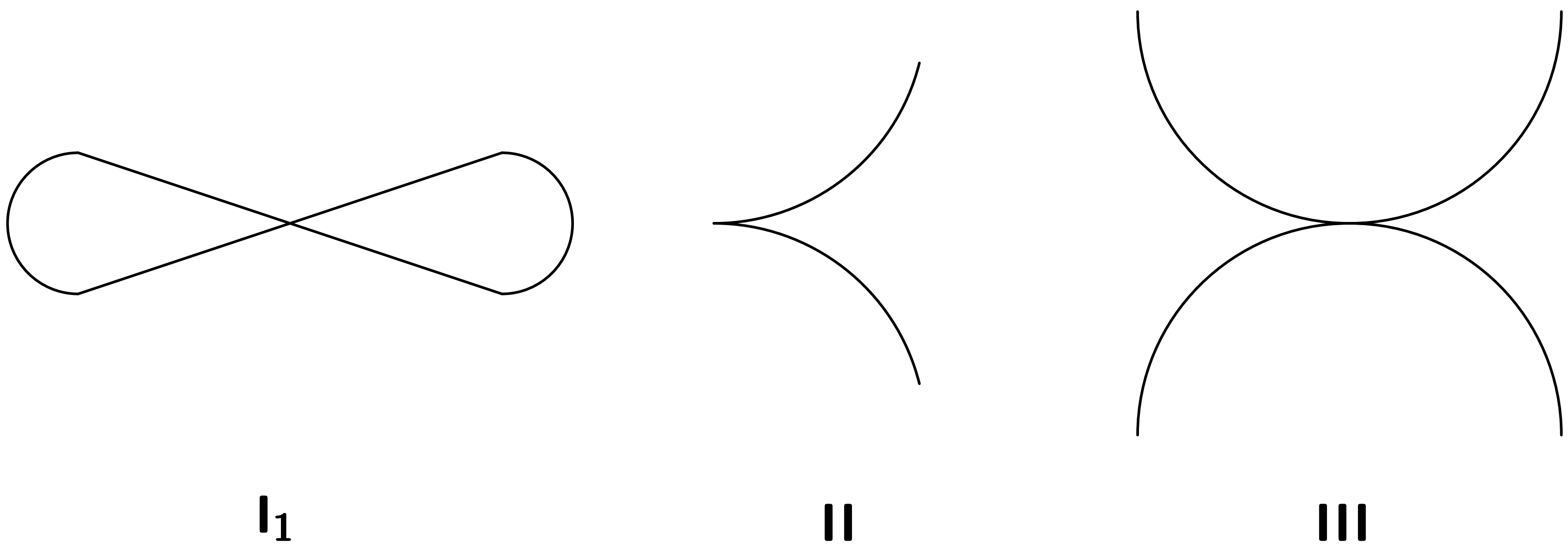
Remark 5.6. 1. Such $Y \to S$ exists by Koraira's classification (Kodaira symbol $I_n = \widetilde{A}_{n-1}$).
2. The relative Fourier-Mukai transform Φ was constructed by Bridgeland (1998).

§6 Quantum toroidal algebra of type affine ADE

Y $\to$ S: relatively minimal elliptic surface over $\mathbb{C}$.

C: the singular fiber with Kodaira symbol I_1 (rational with a node), II (rational with a cusp), III $= \widetilde{A}_1$, $I_n^* = \widetilde{D}_n$, II$^* = \widetilde{E}_8$, III$^* = \widetilde{E}_7$, IV$^* = \widetilde{E}_6$.

Proposition 6.1. **(Ginzburg-Kapranov-Vasserot)**
The composition subalgebra $DU(C) \subset H^{ext}(C)$ is isomorphic to the quantum toroidal algebra of type ADE according as the intersection matrix of C.



Theorem 6.2. DU(C) has two subalgebras which are isomorphic to Heisenberg algebras. These are mapped to each other under automorphisms of DU(C).

Expectations

DU(C) has two subalgebras which are isomorphic to some quantum affine algebras.