Day3.

Thm. B. [Y.-Yoshioka (2012)] X: abel. surf. $/\mathbb{C}$, princ. pol. $NS(X)=\mathbb{Z}H$
$v\in H^{ev}(X,\mathbb{Z})_{alg}$.
If $\ell := \langle v^2\rangle/2$ is 1, 2 or 3, then
$$M_X^H(v) \xrightarrow{\sim} X\times \mathrm{Hilb}^\ell(X)$$ //

Rmk. Thm A; if $\# CL(\ell)=1$, then $M_X^H(v) \overset{\text{bir'l}}{\dashrightarrow} X\times \mathrm{Hilb}^\ell(X)$
$\# CL(2) = \# CL(3) = 1$. //

§1. Bridgeland's stability conditions
§2. Wall-chamber structures on the space of stab. cond.
§3. Outline of the proof of Thm. B.

§1.

a triangulated category. $\mathcal{D}$ is an additive cat.
with • auto-equiv. $[1]:\mathcal{D}\to\mathcal{D}$ (~~triang.~~ transl. funct.)
• class of diagrams $X\to Y\to Z\to X[1]$ (dist. triang.)
satisfying several conditions
$D(X)=D^b Coh(X)$ is an example.

Dfn. [Beilinson-Bernstein-Drinfeld]
(1) a t-structure of triang. cat. $\mathcal{D}$ is to give a full subcat. $\mathcal{D}^{\le 0}\subset\mathcal{D}$ s.t.
① $\mathcal{D}^{\le 0}[1]\subset \mathcal{D}^{\le 0}$
② ${}^\forall E\in \mathrm{Ob}\,\mathcal{D}$ ${}^\exists$ dist. triang. $A\to E\to B\to A[1]$
with $A\in_{\mathrm{Ob}}\mathcal{D}^{\le 0}$, $B\in_{\mathrm{Ob}}\mathcal{D}^{\ge 1} := \{F\in \mathrm{Ob}\,\mathcal{D};\ \mathrm{Hom}(\mathcal{D}^{\le 0},F)=0\}$
(2) The core of t-str.
$:= \mathcal{D}^{\le 0}\cap(\mathcal{D}^{\ge 1}[1])$ //

Fact. The core of t-str. is abelian.

E.g. (std t-str. of $D(X)$)
$D^{\le 0}(X) := \{E\in \mathrm{Ob}\,D(X);\ H^i(E)=0\ \ {}^\forall i>0\}$
is a t-str. of $D(X)$
The core $= Coh(X)$ //

Dfn. A stab. cond. on triang. cat. $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ of
- t-str. on $\mathcal{D}$ ($\mathcal{A}$: the core)
- $Z : K(\mathcal{A}) \to \mathbb{C}$ grp. hom. (central charge)

  ↳ Grothendieck grp.

s.t. $0 \neq {}^\forall E \in \mathrm{Ob}\,\mathcal{A}$

① $Z(E) \in \mathbb{H}' := \{ re^{\sqrt{-1}\pi\phi} ; r, \phi \in \mathbb{R},\ r > 0,\ 1 \geq \phi > 0 \}$ ($\phi(E)$ (phase))

② $^\exists$ filtr. $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ in $\mathcal{A}$

s.t. · $F_i := E_i / E_{i-1}$ is semistable $^\forall i$

$\left[\overset{\text{dfn.}}{\Longleftrightarrow}\ 0 \neq {}^\forall F \subsetneq F_i \quad \phi(F) \leq \phi(F_i) \right]$ (##)

· $\phi(F_1) > \cdots > \phi(F_n)$ //

Dfn $E \in \mathrm{Ob}\,D(X)$ is σ-semistable $\Leftrightarrow$ E is a semistable obj. of $\mathcal{A}$ in the sense of (##) //

Construction of stab. cond.

Dfn. [Happel-Reiten-Smalø]

$\mathcal{A}$: abelian cat.

(1) A torsion pair is a pair $(\mathcal{T}, \mathcal{F})$ of full subcat. of $\mathcal{A}$

s.t. ① $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$

② $^\forall E \underset{\mathrm{Ob}}{\in} \mathcal{A}$ $\quad 0 \to T \to E \to F \to 0$ exact, $T \in \mathrm{Ob}\,\mathcal{T}$, $F \in \mathrm{Ob}\,\mathcal{F}$

(2) Tilting $\mathcal{A}^\#$ of $\mathcal{A}$ by a torsion pair $(\mathcal{T}, \mathcal{F})$ is a full subcat. of $D^b\mathcal{A}$

s.t. $\mathrm{Ob}\,\mathcal{A}^\# = \left\{ E \in \mathrm{Ob}\,D^b\mathcal{A} : \begin{array}{l} H^0(E) \in \mathrm{Ob}\,\mathcal{T} \\ H^{-1}(E) \in \mathrm{Ob}\,\mathcal{F} \\ H^i(E) = 0 \quad ^\forall i \neq 0, -1 \end{array} \right\}$ //

Rmk. $(\mathcal{T}, \mathcal{F})$: torsion pair of $\mathcal{A}$

$E \in \mathrm{Ob}\,\mathcal{A}$ $\quad 0 \to T \to E \to F \to 0 \rightsquigarrow T \to E \to F \to T[1]$ dist. tr. in $D^b\mathcal{A}$

$E' \in \mathrm{Ob}\,\mathcal{A}^\#$ $\quad F'[1] \to E' \to T' \to F'[2]$ dist. triang. in $D^b\mathcal{A}$, $F' \in \mathrm{Ob}\,\mathcal{F}$, $T' \in \mathrm{Ob}\,\mathcal{T}$ //

Fact [HRS] Tilting $\mathcal{A}^\#$ is the core of $^{\exists 1}$t-str. of $D^b\mathcal{A}$

($\Rightarrow \mathcal{A}^\#$ is abelian) //

Fact [Bridgeland]

X : abel. or K3. surf. H : ample div. on X

$\beta \in NS(X)_{\mathbb{R}} := NS(X) \otimes \mathbb{R}$ $\quad \omega \in Amp(X)_{\mathbb{R}}$

Then $\sigma(\beta.\omega) = (\mathcal{A}_{(\beta.\omega)}, Z_{(\beta.\omega)})$ is a stab. cond. on $D(X)$,

where $Z_{(\beta.\omega)}(E) = \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$

t-str. : tilting $\mathcal{A}_{(\beta.\omega)}$ of $Coh(X)$ by $(\mathcal{T}_{(\beta.\omega)}, \mathcal{F}_{(\beta.\omega)})$

$\mathcal{T}_{(\beta.\omega)} = \langle \beta$-tw. H-stable sheaves E, $Z_{(\beta.\omega)}(E) \in \mathbb{H}' \rangle$

$\mathcal{F}_{(\beta.\omega)} = \langle$ — , $-Z_{(\beta.\omega)}(E) \in \mathbb{H}' \rangle$ //

Rmk. $E \in Coh(X)$ is β-tw. H-[semi]stable

$\overset{def}{\iff}$ pure & $p_H(\mathcal{O}_X(-\beta) \otimes F) \underset{(=)}{<} p_H(\mathcal{O}_X(-\beta) \otimes E)$ $\quad 0 \neq {}^{\forall}F \subsetneq E$

β-tw. stability enjoys similar properties of the classical stab. //

Prop. 1. (1) (large volume limit)

$E \in Ob D(X)$ $\quad (\omega^2) \gg \langle v(E)^2 \rangle$

Then E : $\sigma_{(\beta.\omega)}$-semistable

$\iff E \in Coh(X)$. and β-tw. semistable

(2) (preservation of stability by FMT)

For Any FMT Φ on $D(X)$ ~~preserves~~

$E : \sigma_{(\beta.\omega)}$-s.s. $\Rightarrow \Phi(E) : \sigma_{(\beta'.\omega')}$-s.s.

$((\beta'.\omega')$: determined by Φ & $(\beta.\omega))$ //

§3 wall-chamber str.

X : abel. surf. $/\mathbb{C}$

$\{(\beta.\omega)\} = NS(X)_{\mathbb{R}} \times Amp(X)_{\mathbb{R}}$ space of stab. cond.

(with Euclid top.)

Dfn. Fix $v \in H^{ev}(X.\mathbb{Z})_{alg}$. Consider v_1 with $v_1 \notin \mathbb{Q}v$, $\langle v_1^2 \rangle \geq 0$, $\langle (v-v_1)^2 \rangle \geq 0$.

$\langle v_1, v-v_1 \rangle > 0$

(1) A wall for v of type v_1

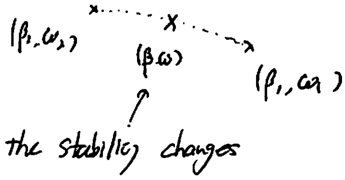
$W_{v_1,v} := \{(\beta.\omega) ; \mathbb{R}Z_{(\beta.\omega)}(v) = \mathbb{R}Z_{(\beta.\omega)}(v_1)\}$

(2) A chamber for v := a conn. comp. of $NS(X)_{\mathbb{R}} \times Amp(X)_{\mathbb{R}} \setminus \bigcup_{v_1} W_{v_1,v}$

//

Lem. The set of walls for v is locally finite //

Rmk. E : $\sigma_{(\beta_1,\omega_1)}$-semistable, not $\sigma_{(\beta_2,\omega_2)}$-semistable, $v = v(E)$



$\Rightarrow$ $0 \to E_1 \to E \to E_2 \to 0$ HNF in $\mathcal{A}(\beta,\omega)$

with E_1, E_2 : $\sigma_{(\beta,\omega)}$-semistable.

$\phi(E_1) = \phi(E_2) = \phi(E)$

$\leftarrow$ phase w.r.t. $Z_{(\beta,\omega)}$

$\Rightarrow$ Setting $v_1 := v(E_1)$, $v(E_2) = v - v_1$

$\langle v_1^2 \rangle \geq 0 \iff \exists E_1 \in \mathrm{Ob}\,\mathcal{A}(\beta,\omega)$

$\langle (v-v_1)^2 \rangle \geq 0 \iff \exists E_2$ 〃

$\langle v_1, v - v_1 \rangle > 0 \iff \mathrm{Ext}^1(E_2, E_1) > 0$

$\mathbb{R} Z_{(\beta,\omega)}(v_1) = \mathbb{R} Z_{(\beta,\omega)}(v) \iff \phi(E_1) = \phi(E)$ //

Lem. ~~For $\forall$~~ $\mathcal{C}$: chamber for v

Then $\sigma_{(\beta,\omega)}$-s.s. is indep. of the choice of $(\beta,\omega) \in \mathcal{C}$ //

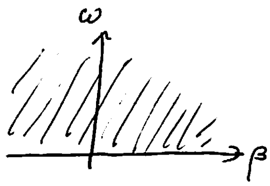
Dfn. $M_{(\beta,\omega)}(v) :=$ moduli space of S-equiv. class of $\sigma_{(\beta,\omega)}$-ss. objects E with $v(E) = v$

(proj. sch.)

$M_{\mathcal{C}}(v) := M_{(\beta,\omega)}(v)$ with $(\beta,\omega) \in \mathcal{C}$ //

● Now assume $NS(X) = \mathbb{Z}H$

Then $\{(\beta,\omega)\} = \{(sH, tH) ; s \in \mathbb{R}, t \in \mathbb{R}_{>0}\}$



Lem. (1) A wall $W_{v_1,v}$ is of the form

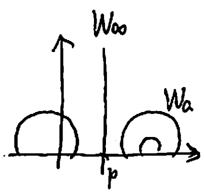
$W_a := \{(sH, tH) ; (s-a)^2 + t^2 = (p-a)^2 - q^2\}$

or $W_\infty := \{(sH, tH) : s = p\}$

with $p \in \mathbb{Q}$, $q \in \mathbb{Q}_{>0}$ determined by v

$a \in \mathbb{Q}$ 〃 v, v_1

(2) $W_a \cap W_{a'} = \phi$ for $a \neq a'$ //

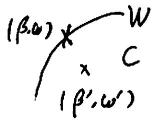


Dfn. For a wall W for v,

$$\operatorname{codim} W := \min_{v=\sum v_i} \Big\{ 1 + \sum_{i<j} \langle v_i, v_j \rangle - \sum_i \big(\dim \mathcal{M}_X^{H,\beta'}(v_i)^{ss} - \langle v_i^2 \rangle \big) \Big\}$$

with $v = \sum_{i=1}^{\ell} v_i$, $\ell \geqq 2$, $\phi_{(\beta,\omega)}(v) = \phi_{(\beta,\omega)}(v_i)$ $\forall i$

$\phi_{(\beta',\omega')}(v_i) > \phi_{(\beta',\omega')}(v_j)$ $\forall i<j$



$(\beta,\omega) \in W$ $\qquad$ $\phi_{(\beta,\omega)}$ = phase of $Z_{(\beta,\omega)}$

$(\beta',\omega') \in$ adj. chamb. of W

$\mathcal{M}_X^{H,\beta'}(v_i)^{ss}$: moduli ~~space~~ stack of β'-tw. H-s.s. sheaves //

Rmk. $\operatorname{codim} W = \operatorname{codim}_{M_C(v)} \{$destabilizing obj. when crossing $W\}$ //

Claim. $\{ W_{v_1,v} : \text{codim. } 0 \text{ wall for } v \} \overset{1:1}{\longleftrightarrow} \{$solution $(\ell_1, \ell_2, v_1, v_2)$ of numerical eqn. for $v\}$ //

$$\begin{bmatrix} v = \ell_1 v_1 - \ell_2 v_2 \\ (\ell_1 - 1)(\ell_2 - 1) = 0 \\ \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0,\ \langle v_1, v_2 \rangle = -1 \end{bmatrix}$$

Example. codim. 0 wall for $v = (1, 0, -\ell) = v(I_Z)$

$n := (H^2)/2$ $\qquad$ $\ell = \text{length}(Z)$

Then

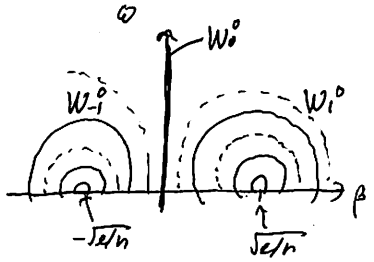
$\{$codim. 0 wall for $(1, 0, -\ell)\} = \{ W_m^0 : m \in \mathbb{Z} \}$

with $W_m^0 := \{ (sH, tH) ; (s - \frac{1}{\sqrt{n}} \frac{b_m}{a_m})(s - \frac{1}{\sqrt{n}} \frac{\ell a_m}{b_m}) + t^2 = 0 \}$

$W_0^0 := \{ \quad - \quad ; \quad s = 0 \}$

$$\begin{pmatrix} b_m & \ell a_m \\ a_m & b_m \end{pmatrix} = \begin{pmatrix} q & \ell p \\ p & q \end{pmatrix}^m$$

$p, q \in \mathbb{Z}_{>0}$, $\quad q^2 - \ell p^2 = \pm 1$ //

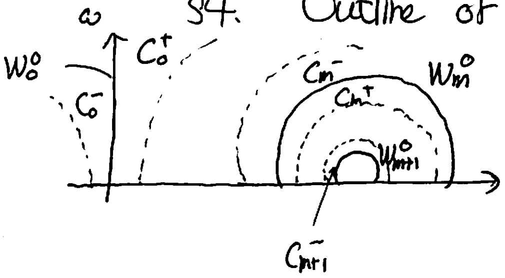


$\frac{b_m}{a_m} \underset{(m \to +\infty)}{\longrightarrow} \sqrt{\ell}$ $\qquad$ (center of W_m^0) $\underset{m \to +\infty}{\longrightarrow} \sqrt{\ell/n}$

— : codim 0-wall

-‐-- : codim >0-wall

§4. Outline of Prf. of Thm. B.



$C_m^\pm$: adj. chamber ~~of~~ W_m^0

$$\mathcal{M}_m := \mathcal{M}_{C_m^-}(1,0,-\ell) \cap \mathcal{M}_{C_{m-1}^+}(1,0,\ell)$$

<u>Prop. 2.</u> (1) $\mathcal{M}_m \neq \phi$ $\quad \mathcal{M}_m \xrightarrow{\text{birat.}} \mathcal{M}_{C_m^-}, \mathcal{M}_{C_{m-1}^+}$ $\quad (\forall m \in \mathbb{Z})$

(2) $\mathcal{M}_0 \cong \hat{X} \times \mathrm{Hilb}^\ell(X)$ $\quad (= \mathcal{M}_Y^H(1,0,-\ell)$: Prop.1.(1) large vol. lim.)

(3) $\exists$ isom. of schemes

$$\cdots \to \mathcal{M}_{m-1} \xrightarrow{\Phi_{m-1}} \mathcal{M}_m \xrightarrow{\Phi_m} \mathcal{M}_{m+1} \to \cdots$$

with Φ_m is of the form $\Phi_{Y\to X}^{\mathcal{E}} D_Y \Phi_{X\to Y}^{\mathcal{E}^\vee}[1]$

$D_Y = \mathbb{R}\mathcal{H}om(\cdot, \mathcal{O}_Y)$

$Y = \mathcal{M}_X^H(\cdot) \in FM(X)$

$\mathcal{E}$: univ. fam on Y //

<u>Sketch of Prf. of Thm B</u> $(\ell=2)$ $\quad v : \langle v^2 \rangle = 2\ell$

① $\# CL(\ell) = 1$. & Prop. 1. (2) (preserv. of stab. via FMT)

$\Rightarrow$ $\exists$FMT inducing $\mathcal{M}_X^H(v) \xrightarrow{\sim} \mathcal{M}_{(\beta,\omega)}(1,0,-\ell)$

with some $(\beta,\omega) \in \mathcal{C}$. chamber of $(1,0,-\ell)$

② $\nexists$ wall for $(1,0,-\ell)$ of codim $\geqq 1$ $\quad (\because \ell=2)$

$\Rightarrow$ $\mathcal{C} = C_{m-1}^+ = C_m^-$ $\quad (\exists m \in \mathbb{Z})$

Then $\mathcal{M}_X^H(v) \underset{①}{\xrightarrow{\sim}} \mathcal{M}_{(\beta,\omega)}(1,0,-\ell) \underset{②}{=} \mathcal{M}_m$

$$\underset{\text{Prop.2.(3)}}{\cong} \mathcal{M}_0 \underset{\text{Prop.2.(1)}}{\cong} \hat{X} \times \mathrm{Hilb}^\ell(X) \cong X \times \mathrm{Hilb}^\ell(X) \quad //$$

X : prin. pol.