Day 2.

Review of Day 1

- Stability of sheaves

  X : proj. var. $/\mathbb{C}$. H : ample div. on X

  $E \in Coh(X)$ is H-(semi)stable $\overset{def}{\iff}$ pure & $p(E) \underset{(=)}{>} p(F)$ $\forall F \subsetneq E$

  ← reduced Hilb. poly.

- moduli space of stable sheaves

  X : proj. sm. surf. $/\mathbb{C}$

  $v \in H^{ev}(X, \mathbb{Z})_{alg} = \underset{\mathbb{Z}}{\underset{\shortparallel}{H^0(X,\mathbb{Z})}} \oplus NS(X) \oplus \underset{\mathbb{Z}}{\underset{\shortparallel}{H^4(X,\mathbb{Z})}}$

  $M_X^H(v) := \{E \in Coh(X) ; H\text{-stable}, v(E) = v\}$

  : quasi-proj. sch. $/\mathbb{C}$ ← $ch(E)\sqrt{td_X}$

  X : K3 or abel. $\Rightarrow$ $M_X^H(v)$: sm, $\dim = \langle v^2 \rangle + 2$

Day 2.

§1. Theorem on birational types of moduli spaces of stable sheaves on abelian surfaces (Thm. A)

§2. Derived categories & Fourier-Mukai transforms (FMT)

§3. Semi-homogeneous (SH) sheaves

§4. Outline of Proof of Thm. A.

§1.

Dfn. (1) An integral quadratic form is

$$(x, y) \mapsto ax^2 + 2bxy + cy^2 \qquad a, b, c \in \mathbb{Z} \qquad (*)$$

(2) The discriminant $\Delta := b^2 - ac$

(3) $GL(2,\mathbb{Z})$-action on integral quadratic forms

$$ax^2 + 2bxy + cy^2 \underset{A \in GL(2,\mathbb{Z})}{\longmapsto} a'x^2 + 2b'xy + c'y^2$$

$$\begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} = {}^tA \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} A$$

(4) $CL(\ell) := \left\{ \begin{array}{l} \text{integral quad. form} \\ \text{with } \Delta = \ell \end{array} \right\} \Big/ \underset{GL(2,\mathbb{Z})}{\sim}$ //

<u>Thm A</u>. [Y.-Yoshioka (2009)]

Assume X : abel. surf./$\mathbb{C}$, principally polarized, $NS(X)=\mathbb{Z}H$

$v \in H^{ev}(X,\mathbb{Z})_{alg}$. positive, and

$\ell := \langle v^2\rangle/2 \ (\in \mathbb{Z})$ satisfies

$\ell > 0$ & $\# Cl(\ell) = 1$

Then $M_X^H(v) \underset{\text{birational}}{\dashrightarrow} X \times Hilb^{\ell}(X)$ //

<u>Rmk</u>. (1) can omit the assumption "principally polarized" (the claim is modified)

(2) For a very general abelian var. X $NS(X) \cong \mathbb{Z}$ //

§2. <u>Derived Categories</u>

X : alg. var.

$D(X) = D^b Coh(X)$

object : bounded cpx.

$$E = \cdots \to E^{n-1} \longrightarrow E^n \longrightarrow E^{n+1} \to \cdots$$

of coherent sheaves on X

morph. from E to F :

$$\begin{tikzcd} & G \arrow[dl, "qis"'] \arrow[dr] & \\ E & & F \end{tikzcd}$$

$G \xrightarrow{f} E$: quasi-isom. ($\overset{\text{def}}{\Leftrightarrow}$ $H^n(f): H^n(G) \xrightarrow{\sim} H^n(E)$)

Derived functors

(1) ~~$E, F \in Ob\,D(X)$ $i \in \mathbb{Z}$~~

~~$Hom^i(E,F) := Hom(E,$~~

For $f: X \to Y$. proj. morph. between sm. var.

$\mathbb{R}f_* : D(X) \to D(Y)$ derived push-forward

$\mathbb{L}f^* : D(Y) \to D(X)$ derived pull-back

<u>Rmk</u> f : flat $\Rightarrow \mathbb{L}f^* = f^*$

(2) For $E \in D(X)$

$\exists\, \mathbb{R}\mathcal{H}om_{\mathcal{O}_X}(E, \cdot) : D(X) \longrightarrow D(X)$ derived hom.

$\exists\, \cdot \overset{\mathbb{L}}{\otimes} E : D(X) \longrightarrow D(X)$ 〃 tensor product

FMT

Dfn. X, Y : sm. proj. var. $/\mathbb{C}$

(1) For $E \in \mathrm{Ob}\,D(X \times Y)$

$$\Phi_{X \xrightarrow{E} Y} : D(X) \longrightarrow D(Y)$$

$$\bullet \longmapsto \mathbb{R}p_{Y*}(p_X^*(\cdot) \overset{L}{\otimes} E)$$

(2) $\Phi_{X \xrightarrow{E} Y}$ is called an FMT if it gives an equivalence //

Fact. [Orlov] X, Y : sm. proj. var. $/\mathbb{C}$

$\Phi : D(X) \longrightarrow D(Y)$ equiv.

$\Rightarrow \exists E \in \mathrm{Ob}\,D(X \times Y)$ s.t. $\Phi \cong \Phi_{X \xrightarrow{E} Y}$ //

$D(X)$ as invariant

Dfn. X : sm. proj. var. $/\mathbb{C}$

$FM(X) := \{ Y : \text{sm. proj. var.} /\mathbb{C} \ ; \ D(X) \simeq D(Y) \} / \text{isom.}$

(The set of FM partners) //

Fact. (1) $Y \in FM(X) \Rightarrow \dim Y = \dim X$

(2) If K_X or $-K_X$ is ample, then $FM(X) = \{X\}$ //

Description of $FM(X)$

$\dim X \leqq 1 \Rightarrow FM(X) = \{X\}$

$\dim X \geqq 3$: generally unknown

$\dim X = 2$, X : minimal. ($\overset{\text{def}}{\Leftrightarrow} K_X \cdot C \geqq 0$, $\forall C \subset X$ curve)

$\Rightarrow$ · $FM(X) \neq \{X\}$ only if $X = K3$ or abelian

· If $X = K3$ [abel.], then $\forall Y \in FM(X)$ is $K3$ [abel.] and $Y \cong M^H(v)$

~~[illegible]~~

§3. X : abel. surf. $/\mathbb{C}$. (so $v(E) = ch(E)$)

Dfn. (1) E : loc. free sheaf on X, ($T_x : X \to X$ transl.; $y \mapsto y+x$)

is called SH if $\forall x \in X \ \exists L \in \mathrm{Pic}^0(X)$ $T_x^* E \cong E \otimes L$

(2) $E \in \mathrm{Coh}(X)$ is called SH if

① locally free & SH ((1))

or ② $\mathrm{Supp}\, E =: C$ is an elliptic curve, and $E|_C$ is loc. free on C.

or ③ $\mathrm{Supp}\, E = \{pt.\}$ //

<u>Fact</u> [Mukai (1978)]

(classification of 2-dim. moduli)

H : ample div. on X

$E \in Coh(X)$, $\langle v(E)^2 \rangle = 0$

(1) E : H-semistable $\Rightarrow$ E : SH

(2) E : H-stable $\Leftrightarrow$ $v(E)$: primitive //

<u>Fact</u> [Orlov (2002)]

(classification of FMTs)

X, Y : abel. surf. with $\exists \Phi : D(X) \to D(Y)$ equiv.

(1) $\exists v \in H^{ev}(X, \mathbb{Z})_{alg}$ s.t. positive, $\langle v^2 \rangle = 0$, $Y \cong M_X^H(v)$

($\Rightarrow M_X^H(v)$ consists of SH sheaves)

(2) $\exists \mathcal{E} \in Coh(X \times Y)$ (univ. fam. on Y) $\exists k \in \mathbb{Z}$

s.t. $\Phi \cong \Phi_{X \to Y}^{\mathcal{E}[k]}$ //

§4. Outline of Pf. of Thm. A.

X : abelian surf. $/\mathbb{C}$

<u>Dfn.</u> A <u>SH presentation</u> of $E \in Coh(X)$ is an exact seq.

$$0 \to E_1 \to E_2 \to E \to 0$$

or

$$0 \to E \to E_1 \to E_2 \to 0$$

with · E_1, E_2 : SH

· $(l_1 - 1)(l_2 - 1) = 0$, $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 0$, $\langle v_1, v_2 \rangle = -1$

($v(E_i) = l_i v_i$, $l_i \in \mathbb{Z}_{>0}$, v_i : primitive) //

<u>Rmk.</u> (1) "$l_1 = 1$ or $l_2 = 1$" and "$\langle v_i^2 \rangle = 0$"

$\underset{\text{Fact [Mukai]}}{\Rightarrow}$ E_1 or E_2 is stable

(2) $\langle v_1, v_2 \rangle = -1$ $\overset{\text{non-triv.}}{\Longrightarrow}$ $\exists$ FMT $\Phi_{X \to Y_i}^{\mathcal{E}_i} : D(X) \to D(Y_i)$

with $Y_i = M_X^H(v_i)$

$\mathcal{E}_i \in Coh(Y_i \times X)$ univ. fam.

$(i = 1, 2)$ //

<u>Claim</u>. ~~Fo~~ In the case $0 \to E_1 \to E_2 \to E \to 0$, $l_1 = 1$
we have
$$\Phi_{X\to Y_2}^{\mathcal{E}_2^\vee[1]} : D(X) \xrightarrow{\sim} D(Y_2)$$
$$\mathcal{E}_2^\vee := \mathbb{R}\mathcal{H}om(\mathcal{E}_2, \mathcal{O}_{Y_2\times X}) \qquad E \longmapsto I_Z \otimes L$$
$Z \subset X$: 0-dim. subsch.
length$(Z) = l$
$L \in \mathrm{Pic}^0(Y_2) = \hat{Y}_2$ //

In the ~~…~~ other 3 cases of SH presentation, similar arguments hold.

So if $\exists$SH presen. then
$\exists$FMT $\quad E \longmapsto I_Z \otimes L$
$\qquad\qquad\qquad\qquad \cap$
$\qquad\qquad\qquad\qquad \mathrm{Hilb}^l(Y) \times \hat{Y}$

Existence Criterion for SH presentation

<u>Dfn</u>. The numerical equation for v is
(#) $\quad v = l_1 v_1 - l_2 v_2$
with $\begin{cases} l_1, l_2 \in \mathbb{Z}_{>0} & (l_1 - 1)(l_2 - 1) = 0 \\ v_1, v_2 \in H^{ev}(X, \mathbb{Z})_{alg} & \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0,\ \langle v_1, v_2 \rangle = -1 \end{cases}$ //

<u>Prop</u>. Assume $NS(X) = \mathbb{Z}H$ and v : positive. $\langle v^2 \rangle > 0$
If (#) has a solution (l_1, l_2, v_1, v_2)
then a general member of $M_X^H(v)$ has an SH presen. //

Sketch of Pf of Thm. A

① X : abel. surf. $\quad NS(X) = \mathbb{Z}H$
By Prop. & Claim
if <u>(#) has a solution</u> (☆)
then $\exists$FMT, inducing $M_X^H(v) \overset{\text{birat.}}{\dashrightarrow} \hat{Y} \times \mathrm{Hilb}^l(Y)$
$Y = M_X^H(v_i) \in FM(X)$
$\langle v_i^2 \rangle = 0$

② Put $v = (r, dH, a)$, $n := (H^2)/2$
(☆) $\iff$ (☆☆) $\exists$ sol. $(x, y) \in \mathbb{Z}^2$ of
$$rx^2 - 2ndxy + ay^2 = \pm 1$$

③ Now, if X : princ. pol., then

(1) $n=1$. so

$$\#CL(\ell)=1 \Rightarrow rx^2-2dxy+ay^2 \underset{GL(2,\mathbb{Z})}{\sim} x^2-\ell y^2$$

$\left[\begin{array}{l}\Delta = d^2-ar \\ = \langle v^2\rangle/2\end{array}\right]$ $\quad$ $\uparrow$ ${}^\exists$sol. $(x,y)=(1,0)$ for $x^2-\ell y^2=\pm 1$

$\Rightarrow$ ${}^\exists$sol (x,y) for $rx^2-2dxy+ay^2=\pm 1$

$\Rightarrow$ (✱✱) satisfied

(2) $FM(X)=\{X\}$ $\quad$ (c.f. $\#FM(X)=2^{n-1}$ for X princ. pol.)

$Y, \hat{Y} \in FM(X)$ $\quad$ $\therefore Y \cong \hat{Y} \cong X$

$\therefore {}^\exists M_X^H(v) \dashrightarrow X\times Hilb^\ell(X)$ //

Similarly one has

<u>Thm. A'</u> X : abel. surf. /$\mathbb{C}$ princ. pol. $NS(X)=\mathbb{Z}H$
Then

$$\#CL(\ell) \geq \#\left\{\begin{array}{l}\text{birat. cls of} \\ M_X^H(v) \text{ with } \langle v^2\rangle/2=\ell\end{array}\right\}$$ //

<u>Eg.</u> $\#CL(\ell)=\begin{cases} 2 & \ell=1 & \{x^2-y^2,\ 2xy\} \\ 1 & \ell=2,3,4,6,7,8. & \{x^2-\ell y^2\} \\ 2 & \ell=5,9 & \{x^2-\ell y^2,\ 2x^2+2xy-\text{(?)}y^2\}\end{cases}$

$\#\{\text{birat. cls.}\}=\begin{cases} 1 & \ell=1,2,3,4,6,7,8. & \{X\times Hilb^\ell X\} \\ 2 & \ell=5,9. & \{X\times Hilb^\ell X,\ M_X^H(2,H,\cdot)\}\end{cases}$ //

<u>Conj.</u> (Mukai) $\quad \#CL(\ell) \overset{?}{=} \#\{\text{birat. cls}\}$ for $\ell>1$ //

<u>Rmk.</u> (Gauss's Conj.) $\quad \#\{\ell\in\mathbb{Z}_{>0} ; \#CL(\ell)=1\} \overset{?}{=} \infty$ //

<u>Day 3</u> : ∘ Thm. B. (isom. cls. of moduli)
∘ Bridgeland's stability cond. //

<u>References</u> Huybrechts "Fourier-Mukai transforms in algebraic geometry" Oxford
Y.-Yoshioka "Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces" to appear in Crelle.