Plan.

Day 1. general theory of classical stability
§0. classification problem of vector bundles
§1. Mumford-Gieseker stability
§2. Some properties of moduli spaces of stable sheaves
§3. Two theorems on the structure of moduli spaces on abelian surfaces (Thm. A.B)

Day 2. Pf. of Thm. A.
Fourier-Mukai transforms.

Day 3. Pf. of Thm. B.
Bridgeland's stability conditions

Every variety/sch. is defined over $\mathbb{C}$.

§0.

Q. classify all the vector bundles with fixed rank and Chern classes on an alg. var.

rank=1 $\quad \xi \in NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$

- $Pic^{\xi}(X) := \{\text{line bdl. } L \text{ with } c_1(L) = \xi\} / \text{isom.}$
  $\cong H^1(X, \mathcal{O}_X)/H^1(X,\mathbb{Z})$: abelian var.
- $\exists\ \mathcal{E}$: line bdl.

$$\begin{array}{c}\mathcal{E}\\ \downarrow\\ Pic^{\xi}(X)\times X\end{array} \quad \text{s.t.} \quad \mathcal{E}|_{[L]\times X} \cong L$$

The moduli space $Pic^{\xi}(X)$ has the properties
① $\exists$ scheme str. with universal family $\mathcal{E}$
② (non-empty) projective
③ irreducible, smooth

higher rank. want to construct moduli spaces as schemes
obstruction: if a family $\{V\}$ of vect. bdl. is parametrized by a scheme, then $End(V) \supset \mathbb{C}$
$\Rightarrow$ cannot consider all the vector bundles at once!

§1.

X : proj. var. $/\mathbb{C}$

<u>Def.</u> For $E \in \mathrm{Coh}(X)$

(1) $\dim E := \dim \mathrm{Supp}(E)$ $\qquad \mathrm{Supp}(E) := \{x \in X : E_x \neq 0\}$

(2) E : pure $\overset{\text{def}}{\Longleftrightarrow}$ $\dim F = \dim E$ $\quad \forall F \subset E,\ F \neq 0$ //

Fix an ample div. ~~on~~ H on X.

<u>Fact.</u> $\forall E \in \mathrm{Coh}(X)$. $\exists \alpha_i(E) \in \mathbb{Z}$ $(i = 1, \dots, \dim E)$ $\binom{m}{i}$

s.t. $\chi(E \otimes \mathcal{O}_X(mH)) = \sum_{i=0}^{\dim(E)} \alpha_i(E) \boxed{\cancel{\frac{m^i}{i!}}}$ $\quad \forall m \in \mathbb{Z}$ //

$\chi(\cdot) = \sum (-1)^i \dim H^i(X, \cdot)$ (Hilbert polynomial)

<u>Def.</u> For $E \in \mathrm{Coh}(X)$, $m \in \mathbb{Z}$

$$p(E, m) = p(E)(m) = \frac{\chi(E \otimes \mathcal{O}_X(mH))}{\alpha_{\dim E}(E)}$$

(reduced Hilb. poly.) //

<u>Def.</u> (Mumford-Gieseker-Simpson stability)

$E \in \mathrm{Coh}(X)$ is (semi)stable

$\overset{\text{def}}{\Longleftrightarrow}$ pure & $p(E) \underset{(=)}{<} p(F)$ $\quad \forall F \subsetneq E,\ F \neq 0$ //

($f \leq g \overset{\text{def}}{\Longleftrightarrow} f(m) \leq g(m)$ for $m \gg 0$)

<u>Rmk. 1.</u>
- loc. free $\Rightarrow$ ~~on~~ torsion free $\Rightarrow$ pure.
- $\dim X = 1$ loc. free = torsion free = pure of dim 1.
- $\dim X = 2$ loc. free $\Rightarrow$ 〃 = 〃 2 /

<u>Rmk 2.</u> X : sm. proj. curve

$E \in \mathrm{Coh}(X)$ loc. free

$$\chi(E) = \int_X \mathrm{ch}(E)\,\mathrm{td}_X \qquad \text{(Grothendieck-Riemann-Roch)}$$

$$= \int (\mathrm{rk}(E) + c_1(E)) \cdot \left(1 - \frac{K_X}{2}\right)$$

$$= \deg(E) + \mathrm{rk}(E) \cdot \left(1 - \tfrac{1}{2}\deg(K_X)\right) = \deg(E) + \mathrm{rk}(E) \cdot (1 - g(X))$$

$$\therefore \chi(E \otimes \mathcal{O}_X(mH)) = \deg(E) + m \cdot \mathrm{rk}(E) \deg(H) + \mathrm{rk}(E) \cdot (1 - g(X))$$

$$\therefore p(E) = m \deg(H) + (1 - g(X)) + \frac{\deg(E)}{\mathrm{rk}(E)}$$

$\therefore$ E (semi)stable $\Leftrightarrow$ $\frac{\deg(F)}{rk(F)} \underset{(=)}{<} \frac{\deg(E)}{rk(E)}$ $\forall F \subsetneq E$ //

<u>Rmk 3.</u> X: sm. proj. surf.
E: torsion free

$$\chi(E) = \int ch(E) td_X = \int_X (rk(E) + c_1(E) + \tfrac{c_1^2(E)}{2} - c_2(E))(1 - \tfrac{1}{2}k_X + \chi(\mathcal{O}_X))$$
$$= rk(E)\cdot\chi(\mathcal{O}_X) + (c_1(E) \cdot -\tfrac{1}{2}k_X) + \tfrac{1}{2}c_1^2(E) - c_2(E)$$
$$\therefore \chi(E \otimes \mathcal{O}_X(mH)) = \chi(E) + m\cdot(H, c_1(E) - rk(E)\tfrac{1}{2}k_X) + \tfrac{m^2}{2}(H^2)\cdot rk(E)$$
$$\therefore p(E, m) = \frac{(H^2)}{2}m^2 + \left[(H, \tfrac{c_1(E)}{rk(E)}) + (H, -\tfrac{k_X}{2})\right] m + \frac{\chi(E)}{rk(E)}$$

$\therefore$ E (semi)stable
$\Leftrightarrow$ for any $0 \neq F \subsetneq E$.
$(c_1(F)/rkF . H) < (\frac{c_1(E)}{rk(E)} . H)$
or $\quad$ " $\quad$ = $\quad$ " $\quad$ and $\frac{\chi(F)}{rkF} \underset{(=)}{<} \frac{\chi(E)}{rkE}$ //

<u>Def.</u> ($\dim X$: arbitrary)
For $E \in Coh(X)$
$$\deg_H(E) := \alpha_{d-1}(E) - rk(E)\cdot \alpha_{d-1}(\mathcal{O}_X)$$
$$rk(E) := \alpha_d(E) - \alpha_d(\mathcal{O}_X) \quad (d := \dim E)$$
$$\mu_H(E) := \deg_H(E)/rk(E) \quad \text{(slope)}$$ //

<u>Def.</u> $E \in Coh(X)$ is μ-(semi)stable
$\overset{\text{def}}{\Leftrightarrow}$ pure & $\mu(F) \underset{(=)}{<} \mu(E)$ for $0 \neq \forall F \subsetneq E$ //

<u>Rmk.</u> $E \in Coh(X)$ pure
μ-stable $\Rightarrow$ stable $\Rightarrow$ semi-stable $\Rightarrow$ μ-semistable //

§ properties of stability.

<u>Lem.</u> For $E \in Coh(X)$ pure
E: (semi)stable $\Leftrightarrow$ $\forall E \twoheadrightarrow G$ with $\alpha_{\dim(G)}(G) > 0$.
$p(E) \underset{(=)}{<} p(G)$ //

Cor. $E, F \in Coh(X)$. semi-stable

1) $p(E) \gneq p(F) \Rightarrow Hom(E,F) = 0$

2) $p(E) = p(F)$ $E \xrightarrow{f} F$ non-trivial

then (a) f is injective if E is stable

(b) f is surj. if F is stable //

Cor. $E \in Coh(X)$ stable $\Rightarrow End(E) = \mathbb{C}$

Examples of stable sheaves

ex.1. $\forall$ line bdl is stable.

ex.2. X: sm proj. curve.

$0 \to L_0 \to E \to L_1 \to 0$ non-trivial ext. of line bdls. with $\deg L_0 = 0$ $\deg L_1 = 1$

$\Rightarrow E$: stable

∵ $\deg E = 1$. $rk E = 2$. $\mu(E) = \frac{1}{2}$.

$0 \neq F \subsetneq E$ $rk F = 1$ or 2

① $rk F = 2$:

E/F: $\dim(E/F) = 0$. $l := \deg(E/F) \gneq 0$

$\Rightarrow \mu(F) = \mu(E) - l/2 < \mu(E)$

② $rk F = 1$

$F \to E \to L_1$: trivial or injective

• trivial $\Rightarrow F \subset L_0 \Rightarrow \mu(F) \leq \mu(L_0) = 0 < \mu(E)$

◦ inj. $\Rightarrow \mu(F) \leq \mu(L_1) = 1$

$\mu(F) = 1 \Rightarrow F = L_1 \Rightarrow F \cong L_0 \oplus L_1$ absurd

$\therefore \mu(F) \leq 0 < \mu(E)$ //

Harder-Narasimhan filtration

X: proj. var. $/\mathbb{C}$. H: fix.

Thm. $\forall E \in Coh(X)$. pure. has a unique filtration

$0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_n(E) = E$

s.t. $F_i = HN_i(E)/HN_{i-1}(E)$ is semi-stable &

$p(F_1) > \cdots > p(F_n)$ //

Rmk/Ex. $X = \mathbb{P}^1$
$\forall$ vect. bdl E on X is of the form
$$E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \quad a_1 \geq a_2 \geq \cdots \geq a_n$$
$a_i \in \mathbb{Z}$

"Grothendieck's thm"
$$p(\mathcal{O}(a)) = m + 1 + a$$
//

● Jordan-Hölder filtration

Thm. $\forall E \in Coh(X)$ semistable, has a filtration
$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$
s.t. E_i/E_{i-1} stable, $p(E_i/E_{i-1}) = p(E)$ $\forall i$.
The assoc. graded $gr^{JH}(E) := \bigoplus_i E_i/E_{i-1}$
is indep. of the filtration //

Def. $E, E' \in Coh(X)$ semistable with $p(E) = p(E')$
are called S-equivalence if $gr^{JH}(E) = gr^{JH}(E')$ //

§2. moduli spaces of stable sheaves on surfaces

● X: sm. proj. surf. $/\mathbb{C}$. H: ample div. on X
$M_X^H := \{ E \in Coh(X) : \text{stable}, (rk(E), c_1, c_2) = c \}$

Thm. (Gieseker)
- $M_X^H(c)$ has a structure of quasi-proj. sch$/\mathbb{C}$.
- $M_X^H(c)$ is compactified to proj. sch. $\overline{M}_X^H(c)$
by attaching S-equiv classes of semistable sheaves //

Ex. $\mathrm{Hilb}^n X := \{ I_Z : \text{ideal sheaves of } \underset{\text{0-dim subsch.}}{Z} \subset X, \ \mathrm{length}(\mathcal{O}_X/I_Z) = n \}$
$$= M_X^H(1, 0, n)$$
2n-dim, smooth
/

Case of Calabi-Yau surface (X = abelian or K3)

Notation

- For $E \in \mathrm{Coh}(X)$

  $v(E) := ch(E)\sqrt{td_X}$ Mukai vector

  $= (rk(E), c_1(E), \frac{1}{2}c_1^2(E) - c_2(E) + \varepsilon \cdot rk(E))$ $\varepsilon = \begin{cases} 0 & X: \text{abel} \\ 1 & X: K3 \end{cases}$

  $\cap$

  $\underset{\mathbb{Z}}{\underset{\wr\wr}{H^0(X,\mathbb{Z})}} \oplus NS(X) \oplus \underset{\mathbb{Z}}{\underset{\wr\wr}{H^4(X,\mathbb{Z})}} =: H^{ev}(X,\mathbb{Z})_{alg}$

- For $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2) \in H^{ev}(X,\mathbb{Z})_{alg}$

  $\langle x, y \rangle := (x_1, y_1) - x_0 y_2 - x_2 y_0 \in \mathbb{Z}$

  $(H^{ev}(X,\mathbb{Z})_{alg}, \langle\ ,\ \rangle)$: Mukai lattice //

~~Prop.~~ **Rmk.** Grothendieck-Riemann-Roch

$\Leftrightarrow \chi(E,F)\ (:= \chi(E^\vee \otimes F)) = \langle v(E), v(F) \rangle$ //

Thm. (Mukai, 1984) (If $M_X^H(v) \neq \phi$, then)

(1) $M_X^H(v)$ is smooth, $\langle v^2 \rangle + 2 = \dim$

(2) $M_X^H(v)$ has a hol. symplectic str.

(3) v: primitive ($\overset{def}{\Leftrightarrow} \nexists w,\ v \in \mathbb{Z}w$)

H: general w.r.t. v

$\Rightarrow M_X^H(v) = \overline{M_X^H}(v)$

(so it is a cpt sympl. mfd.) //

Thm. (Yoshioka 2003)

$v = (r, \xi, a)$: positive ($\overset{def}{\Leftrightarrow}$ $r > 0$ or ($r = 0$, ξ: eff) or ($r = 0$, $\xi = 0$, $a > 0$))

then $M_X^H(v) \neq \phi \Leftrightarrow \langle v_0^2 \rangle \geq -2\varepsilon$

($v \in \mathbb{Z}_{>0} v_0$, v_0: primitive) //

References

Huybrechts-Lehn "The geometry of moduli spaces of sheaves" Cambridge.