2017 年度前期 数学演習 IX/X　4 月 14 日分演習/レポート問題[*1]

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1　導入：複素数、収束性、連続関数

テキストを本格的に読むのは来週からになります。今週は英語の文章に触れてみます。
まずは複素数に関する問題を英語で出しますので解いてみてください。

問題 1.1. 以下の問題 (テキスト I 章 Miscellaneous Examples 1,2) を解け。
(1) Show that the representative points of the complex numbers $1+4i$, $2+7i$, $3+10i$ on the complex plane are collinear.
(2) Show that a parabola can be drawn to pass through the representative points of the complex numbers

$$2+i,\ 4+4i,\ 6+9i,\ 8+16i,\ 10+25i.$$

次に一様収束性に関する文章を読んでみましょう。

問題 1.2. 以下の文章 (テキスト Ⅲ 章 §3.34) を和訳せよ。また書かれている証明を理解せよ。

3.34. *A condition, due to Weierstrass, for uniform convergence.*

A sufficient, though not *necessary*, condition for the uniform convergence of a series may be enunciated as follows: If, for all values of z within a domain, the moduli of the terms of a series

$$S = u_1(z) + u_2(z) + u_3(z) + \cdots$$

are respectively less than the corresponding terms in a convergent series of positive terms

$$T = M_1 + M_2 + M_3 + \cdots,$$

where M_n is INDEPENDENT of z, then the series S is uniformly convergent in this region. This follows from the fact that, the series T being convergent, it is always possible to choose n so that the remainder after the first n terms of T, and therefore the modulus of the remainder after the first n terms of S, is less than an assigned positive number ϵ; and since the value of n thus found is independent of z, it follows that the series S is uniformly convergent.

Example. The series

$$\cos z + \frac{1}{2^2}\cos^2 z + \frac{1}{3^2}\cos^3 z + \cdots$$

is uniformly convergent for all real values of z, because the moduli of its terms are not greater than the corresponding terms of the convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots,$$

whose terms are positive constants.

最後に Taylor 展開を複素解析を使って導く方法を復習しましょう。

問題 1.3. 以下の文章 (テキスト V 章 §5.4) を和訳せよ。また書かれている証明を理解せよ。

5.4. *Taylor's Theorem.*

Consider a function $f(z)$, which is analytic in the neighborhood of a point $z = a$. Let C be a circle with a as center in the z-plane, which does not have any singular point of the function $f(z)$ on or inside it; so

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that $f(z)$ is analytic at all points on and inside C. Let $z = a + h$ be any point inside the circle C. Then by Cauchy's theorem on the integral of a function round a contour, we have

$$\begin{aligned} f(a+h) &= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a-h} \\ &= \frac{1}{2\pi i}\int_C f(z)dz \left(\frac{1}{z-a} + \frac{h}{(z-a)^2} + \cdots + \frac{h^n}{(z-a)^n} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right) \\ &= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}. \end{aligned}$$

But when z is on C, the modulus of $\frac{f(z)}{z-a-h}$ is continuous, and will not exceed some finite real number M.

Therefore we have

$$\left|\frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}\right| \leq \frac{|h|^{n+1}}{2\pi}\frac{M \cdot 2\pi R}{R^{n+1}},$$

where R is the radius of the circle C, so that $2\pi R$ is the length of the path of integration in the last integral, and $R = |z-a|$ for points z on the circumference of C.

The right-hand side of the last inequality tends to zero as $n \to \infty$. We have therefore

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots,$$

which we can write

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots.$$

This result is known as Taylor's Theorem; and the proof given is due to Cauchy. It follows that *the radius of convergence of a power series is always at least so large as only just to exclude from the interior of the circle of convergence the nearest singularity of the function represented by the series.*

レポート問題

以下の問題のうち **1 題以上**を解いて提出して下さい。解答は日本語で構いません。
講義で分からなかった所、扱ってほしい話題などありましたらレポートに書いて下さい。

レポート問題 1.1 の文に誤りがありましたので訂正します。

レポート問題 1.1 (5 点, テキスト I 章 Miscellaneous Examples 3). Prove that if $\theta_1, \theta_2, \ldots$ are the arguments of the n-th primitive roots of 1, then $\sum_i \cos p\theta_i = 0$ when p is a positive integer less than $n/(ab\cdots c)$, where $a, b, \ldots, c$ are the different constituent primes of n; and that, when $p = n/(ab\cdots c)$, $\sum_i \cos p\theta_i = (-1)^\mu n/(ab\cdots c)$, where μ is the number of the constituent primes.

レポート問題 1.2 (5 点, テキスト II 章 Miscellaneous Examples 16). By converting the series

$$1 + \frac{8q}{1-q} + \frac{16q^2}{1+q^2} + \frac{24q^3}{1-q^3} + \cdots,$$

(in which $|q| < 1$), into a double series, show that it is equal to

$$1 + \frac{8q}{(1-q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1-q^3)^2} + \cdots. \qquad \text{(Jacobi)}$$

参考文献

Whittaker, Watson, *A course of modern analysis*, 4th edition, (Cambridge University Press, 1962) の Chap. I, II, III, V.

以上です。