量子アフィンKZ方程式とAskey-Wilson方程式との双スペクトル対応とその特殊化について

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§0. 背景

(ルート系に付随)

Knizhnik-Zamolodchikov eq. (KZ eq.) $\xleftrightarrow{\ \mathrm{dAH}\ }$ Heckman-Opdam type diff. eq (CS-model)

Cherednik-松尾 対応

$\Big\downarrow$ 　　　　　　　　　　　　　　　$\Big\downarrow$ q差分化

qKZ eq. $\xleftrightarrow[\ \text{Cherednik}\ ]{\ \mathrm{AH}\ }$ Macdonald's q diff. eq.

van Meer & Stokman

bispectral qKZ $\xleftrightarrow{\ \mathrm{DAHA}\ }$ bispectral Macdonald-Ruijsenaars eq.

§.1. bqKZ eq. of type $(C_1^\vee, C_1)$

㋑ $(C_1^\vee, C_1)$ 型の affine root system. [Macdonald, 2003]

$(V = \mathbb{R}\varepsilon, \langle , \rangle)$: 1次元 Euclid. sp. by $\langle \varepsilon, \varepsilon \rangle = 1$

$\hat{V} \simeq V \oplus \mathbb{R}c$: V 上の aff. lin. map の空間.

$$\hat{V} \ni (F : u \longmapsto \langle u, v \rangle + r) \longmapsto v + rc \in V \oplus \mathbb{R}c$$

$D : \hat{V} \longrightarrow V$, $v + rc \longmapsto v$

$f, g \in \hat{V}$ に対して

$\langle f, g \rangle := \langle D(f), D(g) \rangle$.

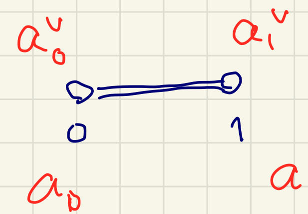
$$S(C_1^\vee, C_1) := \left\{ m\left(\pm\varepsilon + \tfrac{1}{2}nc\right) \;\middle|\; m \in \{1,2\},\ n \in \mathbb{Z} \right\} < \hat{V}$$

$$a_1 := \varepsilon \quad , \quad a_0 := \tfrac{1}{2}c - \varepsilon \qquad \text{とすれば}$$

$$\rightsquigarrow \quad \hat{V} = \mathbb{R}a_1 \oplus \mathbb{R}a_0$$

$$a_i^\vee := 2a_i / \langle a_i, a_i \rangle = 2a_i \in S(C_1^\vee, C_1) \qquad (i = 0, 1)$$

Dynkin diag.



$$s_i : V \longrightarrow V$$

by $\quad s_i(v) = v - a_i(v)\, D(a_i^\vee)$

Eg.

$$s_0(r\varepsilon) = (1-r)\varepsilon \quad , \quad s_1(r\varepsilon) = -r\varepsilon$$

$$W_0 := \langle s_1 \rangle \simeq \mathfrak{S}_2$$

$$\cap$$

$$W := W_0 \ltimes t(\Lambda) = \langle s_0, s_1 \rangle$$ 拡大アフィン Weyl 群.

$\Lambda = \mathbb{Z}\varepsilon < V$: weight lattice

$W_0 \curvearrowright t(\Lambda) := \left\{ t(\lambda) \;\middle|\; \begin{array}{l} t(\lambda)t(\mu) = t(\lambda+\mu) \\ \lambda, \mu \in \Lambda \end{array} \right\}$

by $s_1 t(\lambda) s_1 = t(s_1 \lambda)$

$$t(\varepsilon) = s_0 s_1 \quad , \quad t(-\varepsilon) = s_1 s_0 \quad \in W$$

Double affine Hecke alg. of type $(C_1^\vee, C_1)$.

$$\underline{k} = (k_1, k_0) \;,\; \underline{u} = (u_1, u_0) \in \mathbb{C}^\times$$

$H = H(\underline{k}) := \mathbb{C}\langle T_1, T_0 \rangle_{\mathbb{C}\text{-alg}}.$ 拡大アフィンHecke alg.

rel: $(T_i - k_i)(T_i + k_i^{-1}) = 0$

$Y := T_0 T_1$, $Y^{-1} := T_1^{-1} T_0^{-1} \in H$

$\uparrow$ 変形 $\uparrow$

$t(\varepsilon)$ $t(-\varepsilon)$

$\rightsquigarrow$ $\mathbb{C}[Y^{\pm 1}] = ZH \subset H$, $H_0 := \mathbb{C}T_1 + \mathbb{C}\cdot 1$

$\rightsquigarrow$ $H = \mathbb{C}[Y^{\pm 1}] \otimes_{\mathbb{C}} H_0$

Basic rep of H. $q^{1/2} \in \mathbb{C}^\times$ Fix

(Noumi)

$$W \curvearrowright \mathbb{C}[x^{\pm 1}] \quad \text{by} \quad \begin{cases} (s_1 f)(x) = f(x^{-1}) \\ (s_0 f)(x) = f(q x^{-1}) \\ (t(\varepsilon) f)(x) = f(qx) \end{cases}$$

$$\exists \rho : H \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x^{\pm 1}]) \quad \mathbb{C}\text{-alg. emb.}$$

$$\rho(T_i) := c(x_i; k_i; u_i) s_i + b(x_i; k_i; u_i) \qquad (i = 0, 1)$$

$$(x_1 = x^2, \quad x_0 = q x^{-2})$$

$$c(z; k; u) := k^{-1} \frac{(1 - k u z^{1/2})(1 + k u^{-1} z^{1/2})}{1 - z}$$

$$b(z; k; u) := k - c(z; k; u)$$

以後

$\rho(T_i) = T_i \qquad (i = 0, 1)$ とかく.

$$\mathbb{H} := \mathbb{H}(\underline{k}, \underline{u}, q) = \langle \mathbb{C}[X^{\pm 1}], T_0, T_1 \rangle \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$$

$$\left(X^{\pm} \cdot f(x) = x^{\pm 1} f(x) \right)$$

$$\mathbb{H} = \mathbb{C}\langle T_1, T_0, T_1^\vee, T_0^\vee \rangle_{\mathbb{C}\text{-alg}}. \qquad \begin{cases} T_1^\vee := X^{-1} T_1^{-1} \\ T_0^\vee := q^{-1/2} T_0^{-1} X \end{cases}$$

rel :

- $(T_i - k_i)(T_i + k_i^{-1}) = 0$
- $(T_i^\vee - u_i)(T_i^\vee + u_i^{-1}) = 0 \qquad (i = 0, 1)$
- $T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}$

$\exists !$ $\quad * : \mathbb{H}(\underline{k}, \underline{u}, q) \longrightarrow \mathbb{H}(\underline{k}^*, \underline{u}^*, q)$

the duality anti-inv.

$$T_1^* = T_1, \quad (Y^\lambda)^* = X^{-\lambda}, \quad (X^\lambda)^* = Y^{-\lambda}$$

$$(k_1^*, k_0^*, u_1^*, u_0^*) = (k_1, u_1, k_0, u_0)$$

bqKZ eq. $\left(\mathbb{T} := \mathrm{Hom}_{\text{Group}}(\Lambda, \mathbb{C}^\times)\right)$

$$\mathbb{L} := \mathbb{C}(\mathbb{T}\times\mathbb{T}) = \mathbb{C}[x^{\pm1}]\otimes\mathbb{C}[\xi^{\pm1}] \curvearrowright \mathbb{H}$$

$\cap$ by $(f\otimes g).h := f(X)\,h\,g(Y)$

$\mathbb{K} := \mathcal{M}(\mathbb{T}\times\mathbb{T})$: $\mathbb{T}\times\mathbb{T}$ 上の有理型函数全体.

$$H_0^{\mathbb{K}} := \mathbb{K}\otimes_{\mathbb{C}} H_0 \cong \mathbb{K}\otimes_{\mathbb{L}}\mathbb{H}$$

$W := \mathbb{Z}_2 \ltimes (W\times W) \curvearrowright \mathbb{K}$ by $\iota(x,\xi) = (\xi^{-1}, x^{-1})$

ι : non triv.

$$R_i^L(z) := C_i(z)^{-1}\left(\eta_L(T_i) - b_i(z)\right)$$

$$R_i^R(z) := C_i^*(z)^{-1}\left(\eta_R(T_i^*) - b_i^*(z)\right) \qquad i = 0, 1$$

Yang-Baxter eq. の解.

∘ $\eta_L : H \longrightarrow \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$

$$\eta_L(A)\left(\sum_{w\in W_0} f_w T_w\right) = \sum_{w\in W_0} f_w A T_w$$

∘ $\eta_R : H^* \longrightarrow \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$

$$\eta_R(A)\left(\sum_{w} f_w T_w\right) = \sum_{w\in W_0} f_w T_w A$$

$$C_{1,0} := R_0^L(qx^{-2})\, R_1^L(q^2 x^{-2}), \qquad C_{0,1} := R_0^R(q\xi^2)\, R_0^R(q^2\xi^2)$$

W の $GL_{\mathbb{K}}(H_0^{\mathbb{K}})$ に値をもつ 1-コサイクル.

$f(x,\xi) \in H_0^{\mathbb{K}}$ に対して.

$$\begin{cases} C_{1,0}\, f(q^{-1}x, \xi) = f(x,\xi) \\ C_{0,1}\, f(x, q\xi) = f(x,\xi) \end{cases} \quad \cdots (*)$$

bqKZ 方程式
$(C_1^\vee, C_1)$ 型.

$$SOL_{bqkz}(\underline{k}, \underline{u}, q) := \{ f \in H_0^{\mathbb{K}} \mid f \text{ satisfies } (*) \} \subset H_0^{\mathbb{K}}.$$

§.2. ノンスペクトル Askey-Wilson q 差分方程式.

$$D^x_{AW} := A(x)(T_{q,x}-1) + A(x^{-1})(T_{q,x}^{-1}-1)$$

$$D^\xi_{AW} := A^*(\xi^{-1})(T_{q,\xi}-1) + A^*(\xi)(T_{q,\xi}^{-1}-1)$$

$$T_{q,z}f(z) = f(qz) \quad : \quad q\text{差分作用素}$$

- $$A(z) := \frac{(1-k_1u_1z)(1+k_1u_1^{-1}z)(1-k_0u_0q^{1/2}z)(1+k_0u_0^{-1}q^{-1/2}z)}{(1-z^2)(1-q^{-1}z^2)}$$

- $$A^*(z) := A(z)\Big|_{k_0 \leftrightarrow u_1}$$

$$\mathcal{D}_q^W := W \ltimes \mathbb{C}(x,\xi) \qquad (t(\xi)f)(x) = f(qx))$$

$$\cup$$

$$\mathcal{D}_q := (t(\Lambda)\times t(\Lambda)) \ltimes \mathbb{C}(x,\xi)$$: $\mathbb{C}(x,\xi)$上のq差分作用素環.

$$\mathcal{D}_q^W \ni D = \sum_{w\in W} f_w\, w = \sum_{s\in W_0\times W_0} D_s \cdot s$$

$$f_w \in \mathbb{C}(\mathbb{T}\times\mathbb{T}) \cong \mathbb{C}(x,\xi)$$

$$D_s = \sum_{t\in t(\Lambda)\times t(\Lambda)} g_{ts}\, t \in \mathcal{D}_q$$

$$\mathrm{Res} : \mathcal{D}_q^W \longrightarrow \mathcal{D}_q, \qquad \mathrm{Res}(D) := \sum_{s\in W_0\times W_0} D_s$$

$$\rho^{x}_{1/\underline{k},\,1/\underline{u},\,q} : H(1/\underline{k}) \longrightarrow \mathbb{C}(x)[W \times \{e\}] \subset \mathcal{D}_q^{W}$$

$$\rho^{\xi}_{\underline{k}^*,\,\underline{u}^*,\,1/q} : H(\underline{k}^*) \longrightarrow \mathbb{C}(\xi)[\{e\} \times W] \subset \mathcal{D}_q^{W}$$

$$p_1 := p_1(Y) = Y + Y^{-1} \in \mathbb{C}[Y^{\pm 1}]^{W_0} = \mathbb{C}[Y + Y^{-1}]$$

$$\rightsquigarrow \quad D^{x}_{p_1} := \rho^{x}_{1/k,\,1/u,\,q}(Y + Y^{-1}) \in \mathcal{D}_q^{W}$$

$$D^{\xi}_{p_1} := \rho^{\xi}_{\underline{k}^*,\,\underline{u}^*,\,1/q}(Y + Y^{-1}) \in$$

$$\mathrm{Res}(D^{x}_{p_1}) =: L^{x}_{p_1}, \qquad \mathrm{Res}(D^{\xi}_{p_1}) =: L^{\xi}_{p_1}$$

$$\rightsquigarrow \quad L_{p_1}^{x} = k_1 k_0 + (k_1 k_0)^{-1} + (k_1 k_0)^{-2} D_{AW}^{x}$$

$$L_{p_1}^{\xi} = k_1 u_1 + (k_1 u_1)^{-1} + (k_1 u_1)^{-2} D_{AW}^{\xi}$$

$$\begin{cases} (L_{p_1}^{x} f)(x,\xi) = p_1(\xi^{-1}) f(x,\xi) \\ (L_{p_1}^{\xi} f)(x,\xi) = p_1(x) f(x,\xi) \end{cases} \quad \cdots\cdots (**)$$

双スペクトル Askey-Wilson 方程式

(bAW eq.)

$$SOL_{bAW}(\underline{k}, \underline{u}, q) := \{ f \in \mathbb{K} \mid f \text{ satisfies } (**) \} \subset \mathbb{K}$$

$\lrcorner$

§3. 双スペクトル対応

$$\begin{array}{ccc} \chi_+ : H_0 & \longrightarrow & \mathbb{C} \\ \cup & & \cup \\ T_1^{r} & \longmapsto & k_1^{r} \end{array} \quad \overset{\text{extended}}{\rightsquigarrow} \quad \begin{array}{ccc} \chi_+ : H_0^{\mathbb{K}} & \longrightarrow & \mathbb{K} \\ \cup & & \cup \\ \sum_{w \in W_0} f_w T_w & \longmapsto & \sum_{w \in W_0} f_w \, \chi_+(T_w) \end{array}$$

Thm (Stokman)

$0 < q < 1$

$$\chi_+ : \mathrm{SOL}_{\mathrm{bqKZ}}(\underline{k}, \underline{u}, q) \hookrightarrow \mathrm{SOL}_{\mathrm{bAW}}(\underline{k}, \underline{u}, q)$$

inj. $\mathbb{F}$-lin. $\mathbb{W}_0$-equiv.

$$\mathbb{F} := \left\{ f \in \mathbb{K} \;\middle|\; \begin{array}{c} (t(\lambda), t(\mu)) f = f \\ (\lambda, \mu) \in \Lambda \times \Lambda \end{array} \right\} \subset \mathbb{K}$$

$$\mathbb{W}_0 := \mathbb{Z}_2 \ltimes (W_0 \times W_0) < \mathbb{W}$$

$\lrcorner$

§. 4. 特殊化.

$$\begin{tikzcd}[column sep=small]
 & & (C_1^\vee, C_1)\ \mathrm{AW} \arrow[dl] \arrow[dr] & & \\
 & (C_1^\vee, BC) \arrow[ddl] \arrow[dd] \arrow[ddrr] & & (BC_1, C_1) \arrow[dd] \arrow[dr] & \\
 & & & & BC_1 \arrow[dllll] \arrow[d] \\
A_1 & A_1 & & A_1 & A_1
\end{tikzcd}$$

(k_1, k_0, u_1, u_0)

Cont. q-Jacobi

A_1: $(k_1^{1/2}, 1, k_1^{1/2}, 1)$ $\quad O_1$

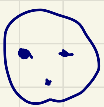
A_1: $(k_1, 1, 1, 1)$ $\quad O_2$ Rogers

A_1: $(1, k_1^{1/2}, 1, k_1^{1/2})$ $\quad O_3$

A_1: $(1, k_1, 1, 1)$ $\quad O_4$

Thm (Y–Yanagida) The specialization $(k_1, k_0, u_1, u_0) = (k, 1, 1, 1)$

$$
\begin{tikzcd}
\mathrm{SOL}^{(C_1^\vee, C_1)}_{\mathrm{bqKZ}}(k,1,1,1,q) \arrow[r, hook, "\chi_+^{(C_1^\vee, C_1)}"] \arrow[d, "\mathrm{sp}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{SOL}^{(C_1^\vee, C_1)}_{\mathrm{bAW}}(k,1,1,1,q) \curvearrowleft sW_7 \arrow[d, "\int"] \\
\mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k,q) \arrow[r, hook, "\chi_+^{A_1}"'] & \mathrm{SOL}^{A_1}_{\mathrm{bMR}}(k = t^{1/2}, q) \ni {}_2\varphi_1
\end{tikzcd}
$$



$$\mathbb{H}^{(C_1^\vee, C_1)}(k,1,1,1,q) \xrightarrow{\sim} \mathbb{H}^{A_1}(k,q)$$

$$\mathbb{H}^{A_1}(k,q) = \mathbb{C}\langle T, U, X \rangle_{\mathbb{C}\text{-alg.}}$$

$$T_1 \longmapsto T$$
$$T_0 \longmapsto U$$
$$T_0^\vee \longmapsto q^{-1/2} U X$$

$$\begin{cases} (T-k)(T-k^{-1}) = 0 \\ U^2 = 1, \quad TXT = X^{-1} \end{cases}$$

$(\quad U x u = q^{1/2} x^{-1}$