"Equivariant K-homology of the symplectic affine Grassmannian"

Kohei YAMAGUCHI (山口 航平)
(Nagoya Univ.)

2023 / 10 / 24

表現論の組合せ論的側面とその周辺 @ 早稲田大学

Joint work with Takeshi IKEDA and Shinsuke IWAO, Mark SHIMOZONO.

Main object : K-homology of the symplectic affine Grassmannian.

$$K_*^T(Gr_{Sp_{2n}(\mathbb{C})}) = \bigoplus_{w \in W_{af}^0} R(T)\,[\underbrace{\mathcal{O}_{X_w}}]$$

Structure sheaf of Schubert variety $X_w \subset Gr_{Sp_{2n}(\mathbb{C})}$

I want to calculate the structur constant c_{w_1,w_2}^v !

$$[\mathcal{O}_{X_{w_1}}]\cdot[\mathcal{O}_{X_{w_2}}] = \sum_{v \in W_{af}^0} c_{w_1,w_2}^v\,[\mathcal{O}_{X_v}], \quad c_{w_1,w_2}^v \in R(T).$$

K-theoretic Peterson isomorphism

$$\begin{tikzcd} QK_T(G/B) \arrow[r, "gr."] \arrow[d, phantom, "\cong"] & QH_T^*(G/B) \arrow[d, phantom, "\cong"] \left(\xrightarrow{q=0} H_T^*(G/B) \right) \\ \boxed{K_*^T(Gr_G)} \arrow[r, "gr."] & H_*^T(Gr_G) \end{tikzcd}$$

"Quantum = affine"

Main theorem [Ikeda - Iwao - Shimozono Y.]

There is an isomorphism of $R(T)$-algebras

$$K_*^T(Gr_G) \cong g\Gamma_{(n)}^{R(T)}, \quad \text{s.t.}$$

geometric $\longleftrightarrow$ algebraic, combinatorial

$$[\mathcal{O}_w] \longmapsto \widetilde{gP}_w^{(n)}(y|b), \quad [\mathcal{I}_{\partial X_w}] \longmapsto gP_w^{(n)}(y|b).$$

- $$\widetilde{gP}_w^{(n)}(y|b) = D_w(1)$$

Demazure operator

Notation

$G := Sp_{2n}(\mathbb{C}) = \{A \in GL_{2n}(\mathbb{C}) \mid {}^tAJA = J\}$, $J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} n \end{matrix}$

$\cup$ The symplectic group.

B : Borel subgroup.

$\cup$

$T := \left\{ \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & O & \\ & & \ddots & & & & \\ & & & d_n & & & \\ & & & & d_n^{-1} & & \\ & O & & & & \ddots & \\ & & & & & & d_1^{-1} \end{pmatrix} \middle| \; d_1, \dots, d_n \in \mathbb{C}^\times \right\}$: Maximal torus.

Root system $C_n^{(1)}$

5/22

A set of roots

- $\alpha_0 := -2a_1,\ \alpha_1 := a_1 - a_2,\ \dots,\ \alpha_{n-1} := a_{n-1} - a_n,\ \alpha_n := 2a_n \in \Phi$

  ($\delta = 0$) : Simple roots.

  $a_i \leftrightarrow \varepsilon_i$

- $Q^\vee := \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$: Coroot lattice of G.

- $W := \langle s_1, s_2, \dots, s_n \rangle_{grp.} \cong \{\pm 1\}^n \rtimes S_n$

$\cap$

- $W_{af} := t(Q^\vee) \rtimes W = \langle s_0, s_1, \dots, s_n \rangle_{grp}$: Affine Weyl grp.

$\cup$

- $W_{af}^\circ := \{ w \in W_{af} \mid \ell(ws_i) = \ell(w) + 1,\ \forall s_i \in W \} \cong W_{af}/W$

Example: $C_2^{(1)} \rightsquigarrow W_{af}^\circ = \{ s_0,\ s_1 s_0,\ s_2 s_1 s_0,\ s_0 s_1 s_0,\ \cdots \}$

Affine Grassmannian of $Sp_{2n}(\mathbb{C})$

- $Gr_G := G(\mathbb{K})/G(\mathbb{O})$, $\mathbb{O} := \mathbb{C}[[t]]$, $\mathbb{K} := \operatorname{Frac} \mathbb{O} = \mathbb{C}((t))$.

  (ind-scheme) : The symplectic affine Grassmannian.

- $I := ev_0^{-1}(B) \subset G(\mathbb{O})$: Iwahori subgroup.

  ($ev_0 : G(\mathbb{O}) \to G \quad (t \mapsto 0)$)

$$\rightsquigarrow \quad Gr_G = \bigsqcup_{w \in W_{af}^{\circ}} \underbrace{IwG(\mathbb{O})}_{\text{Schubert cell}}$$

- $X_w := \overline{IwG(\mathbb{O})} \subset Gr_G$: Schubert variety.

  ($\dim X_w = \ell(w)$)

- $\mathcal{O}_w := \mathcal{O}_{X_w}$: Structure sheaf of X_w

- $\mathcal{J}_{\partial X_w}$:= The ideal sheaf of the boundary

$$\partial X_w := \bigcup_{w \not\geq v} X_v$$

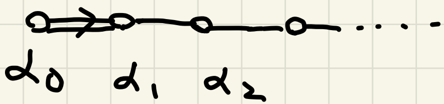
Main theorem [Ikeda - Iwao - Shimozono - Y.]

There is an isomorphism of $R(T)$-algebras

$$K_*^T(\mathrm{Gr}_G) \cong g\Gamma_{(n)}^{R(T)}, \quad \text{s.t.}$$

$$[\mathcal{O}_w] \longmapsto \widetilde{gP}_w^{(n)}(y|b), \quad [\mathcal{J}_{\partial X_w}] \longmapsto gP_w^{(n)}(y|b).$$

K-theoretic equivariant dual Schur's P-function.

C_∞ : 

Notation

- $x \oplus y := x + y + \beta xy$ • $x \ominus y := \dfrac{x-y}{1+\beta y}$ • $\overline{x} := 0 \ominus x = \dfrac{-x}{1+\beta x}$
- $b = (b_1, b_2, \cdots)$, $b_i := \dfrac{1-e^{\beta a_i}}{-\beta}$ $\left(\overline{b_i} = \dfrac{1-e^{-\beta a_i}}{-\beta} \right)$
  (equivariant parameters) $(\beta = 0 \rightsquigarrow b_i \to a_i)$
- $R := \mathbb{C}[b_i, \overline{b_i} \mid i \geqq 1]$
- $[\![u \mid b]\!]^k := (u \oplus u)(u \oplus b_1) \cdots (u \oplus b_{k-1})$ $(k \geqq 1)$
- $SP := \{ \lambda = (\lambda_1, \lambda_2, \cdots) \in \mathbb{Z}_{\geq 0} \mid \lambda_1 > \lambda_2 > \cdots, \lambda_k = 0 \Rightarrow \lambda_{k+1} = 0 \}$

- $g\Gamma := \left\{ f \in \mathbb{C}[\beta][y_1, y_2, \cdots] \;\middle|\; \begin{array}{l} f: \text{symmetric.} \\ f(t, -t-\beta, y_3, y_4, \cdots) \text{ does not depend on } t \end{array} \right\}$

$$= \bigoplus_{\lambda \in SP} \mathbb{C}[\beta]\, gP_\lambda(y)$$

[Nakagawa - Naruse]

- $$\Omega(x|y) := \prod_{i,j} \frac{1 - \bar{x}_i y_j}{1 - x_i y_j} = \sum_{\lambda \in SP} GQ_\lambda(x)\, gP_\lambda(y) = \sum_{\lambda \in SP} GQ_\lambda(x|b)\, gP_\lambda(y|b).$$

K - theoretic (factorial) Schur's Q - function
(if $\beta = -1$) [Ikeda - Naruse]

- $$\Omega(u|y) := \prod_{j \geq 1} \frac{1 - \bar{u} y_j}{1 - u y_j} = \sum_{k \geq 0} GQ_k(u|b)\, gP_k(y|b)$$

$$= 1 + \sum_{k=1}^{\infty} [\![u|b]\!]^k\, gP_k(y|b)$$

One variabele

Example

- $y=(y_1,y_2)$, $\beta=-1$, $b_1=0$, $b_2=0$.

$$gP_{(0)}(y)=1\ ,\quad gP_{(1)}(y)=y_1+y_2\ ,$$
$$gP_{(2)}(y)=(y_1+y_2-1)(y_1+y_2)\ ,$$
$$gP_{(3)}(y)=(y_1+y_2-1)(-y_1+y_1^2-y_2+y_1y_2+y_2^2)$$

- $$gP_{(1)}(y|b)=gP_{(1)}(y)-\bar{b}_1\,gP_{(2)}(y)+\bar{b}_1^{\,2}\,gP_{(3)}(y)-\cdots\cdot$$
$$\in g\Gamma^{R}:=R\,\hat{\otimes}\,g\Gamma=\hat{\bigoplus_{\lambda\in SP}}R\,gP_\lambda(y|b)$$

Demazure operators

- $W_\infty := \langle s_0, s_1, s_2, \dots \rangle_{grp.} \curvearrowright R = \mathbb{C}[b_i, \bar{b}_i \mid i \geq 1]$

  by $\quad s_i^b(b_j) := b_{s_i(j)}$.

- $W_\infty \curvearrowright g\Gamma^R$

  by — $s_0(c \otimes f) = \Omega(b_1|y)(s_{2a_1}^b(c) \otimes f)$

  — $s_i(c \otimes f) = s_i^b(c) \otimes f$

  $(\ c \in R,\ f \in g\Gamma\ )$

- $T_i := \dfrac{1 - s_i}{e(\alpha_i)}$ $(i = 0, 1, \cdots)$, $e(\alpha_0) := \overline{b_1} \oplus \overline{b_1}$, $e(\alpha_i) := b_i \oplus \overline{b_{i+1}}$

- $D_i := T_i - \beta = \dfrac{(1 + \beta e(\alpha_i)) - s_i}{e(\alpha_i)}$ (Demazure op.)

$\rightsquigarrow$ T_i, $D_i \curvearrowright g\Gamma^R$, $T_i(f) := \dfrac{f - s_i . f}{e(\alpha_i)}$

Theorem (c.f. [Nakagawa-Naruse])

$$T_i(gP_\lambda(y|b)) = \begin{cases} gP_{\lambda \cup \{i\}}(y|b) \\ -gP_\lambda(y|b) \\ 0 \end{cases}$$

Example

$\lambda = (6, 4, 3, 1) \longleftrightarrow$

λ

0	1	2	3	4	5	6	7	...		
	0	1	2	3	4	5	6	7	...	
		0	1	2	3	4	5	6	7	...
			0	1	2	3	4	5	6	...
				0	1	2	3	4	5	...

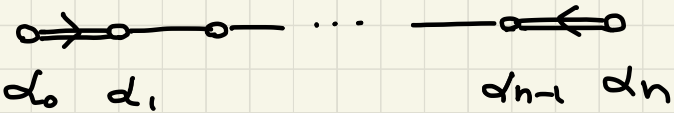
- $T_4(gP_\lambda(y|b)) = gP_{(6,5,3,1)}(y|b)$
- $T_2(gP_\lambda(y|b)) = -gP_\lambda(y|b)$
- $T_3(gP_\lambda(y|b)) = 0$

Proposition

- $T_{i-1}(gP_{i-1}(y|b)) = gP_i(y|b)$.
- $gP_i(y|b) = T_{i-1} \cdots T_1 T_0(1)$.

$\widetilde{gP}^{(n)}_w(y|b)$ & $gP^{(n)}_w(y|b)$

$C^{(1)}_n$:



- $\pi : R \longrightarrow R(T) := \mathbb{C}[b_i, \overline{b}_i \mid 1 \leq i \leq n]$,

$$b_j \longmapsto \begin{cases} b_i & (j \equiv i \mod 2n,\ 1 \leq i \leq n) \\ \overline{b}_{2n-i+1} & (j \equiv i \mod 2n,\ n+1 \leq i \leq 2n) \end{cases}$$

2n+1

- $b^{(n)} := (\pi(b_1), \pi(b_2), \dots\dots)$
$= (b_1, b_2, \dots, b_n, \overline{b}_n, \overline{b}_{n-1}, \dots, \overline{b}_1, b_1, b_2, \dots)$

- $R(T) \otimes_R g\Gamma^R = \widehat{\bigoplus_{\lambda \in SP}} R(T)\, gP_\lambda(y \mid b^{(n)})$

$$T_n := \frac{1 - s_n}{e(\alpha_n)} \curvearrowright R(T) \otimes_R g\Gamma^R$$

- $e(\alpha_n) := b_n \oplus b_n$
- $s_n.(c \otimes f) := s_n^b(c) \otimes f$
- $s_n^b(b_i) := \begin{cases} \bar{b}_n & i = n \\ b_i & i \neq n \end{cases}$

$\rightsquigarrow$

- $T_i^2 = -T_i \quad (i = 0, 1, \ldots, n)$
- $T_i T_j \cdots = T_j T_i \cdots$ braid rel. of $C_n^{(1)}$

For $w = s_{i_1} \cdots s_{i_\ell} \in \mathring{W}_{af}$, define

$$T_w := T_{i_1} \cdots T_{i_\ell}, \qquad D_w := D_{i_1} \cdots D_{i_\ell}.$$

- $\widetilde{gP}^{(n)}_w(y|b) := D_w(1)$.
- $gP^{(n)}_w(y|b) := T_w(1)$.
- $$g\Gamma^{R(T)}_{(n)} := \bigoplus_{w \in \mathring{W}_{af}} R(T)\, \widetilde{gP}^{(n)}_w(y|b) = \bigoplus_{w \in \mathring{W}_{af}} R(T)\, gP^{(n)}_w(y|b)$$

$$\big\downarrow$$

$$R(T) \otimes_R g\Gamma^R$$

$\underline{T_0, T_1, T_2, \ldots, T_n}$

$\overset{\circ}{W}_{af} \overset{1:1}{\longleftrightarrow} \mathcal{P}_c^n$

- $$\rho_i := \begin{cases} s_{i-1}\cdots s_1 s_0 & (1 \leqq i \leqq n) \\ s_{2n+1-i}\cdots s_{n-1}s_n s_{n-1}\cdots s_1 s_0 & (n+1 \leqq i \leqq 2n) \end{cases} \in \overset{\circ}{W}_{af}$$

- $$\mathcal{P}_c^n := \left\{ \underset{\text{partition}}{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)} \;\middle|\; \begin{array}{l} \lambda_1 \leqq 2n, \\ \lambda_i \leqq n \Rightarrow \lambda_{i+1} < \lambda_i \end{array} \right\} \cong \overset{\circ}{W}_{af}$$

$$\begin{array}{ccc} \mathcal{P}_c^n & & \overset{\circ}{W}_{af} \\ \Psi & & \Psi \\ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) & \longmapsto & w_\lambda := \rho_{\lambda_\ell}\cdots\rho_{\lambda_2}\rho_{\lambda_1}. \end{array}$$

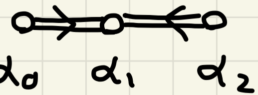
Prop.

$$\overset{\circ}{W}_{af} \ni w_\lambda \longleftrightarrow \lambda \in \mathcal{P}_c^{(n)}, \quad \ell(w_\lambda) \leqq 2n$$

$$\Longrightarrow \quad gP^{(n)}_{w_\lambda}(y|b) = gP_\lambda(y|b^{(n)}) \qquad (\beta = -1)$$

[Nakagawa–Naruse].

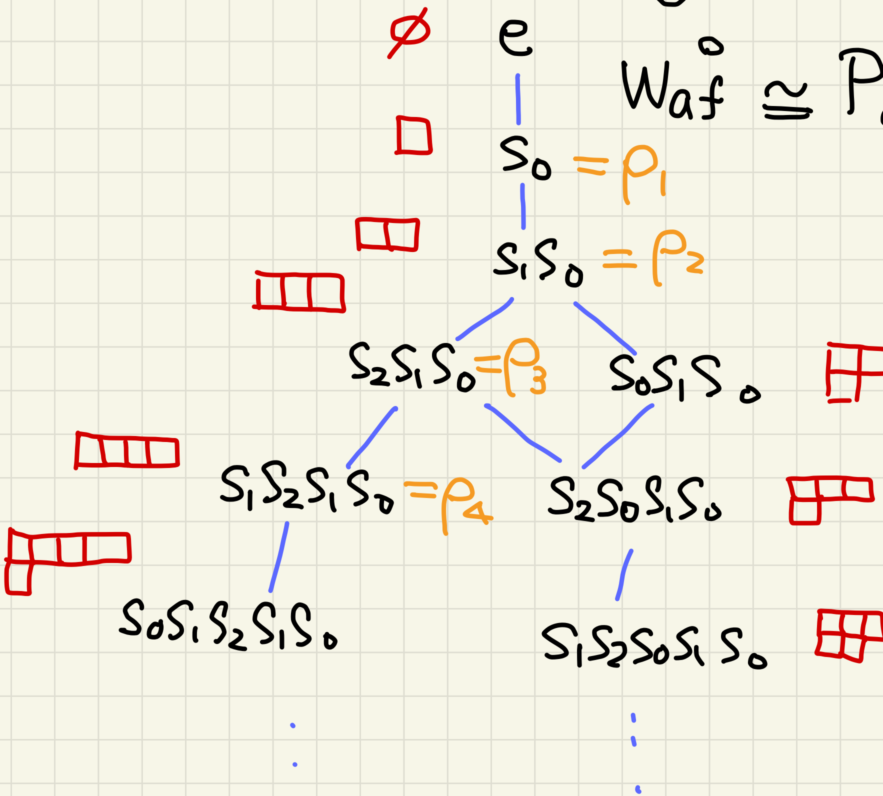
$C_2^{(1)}$



$W_{af} = \langle s_0, s_1, s_2 \rangle$

$\cup$

$W_{af}^{0} \cong P_c^2 := \{ \lambda : \text{part.} \mid \lambda_1 \leq 4 \ \& \ \lambda_i \leq 2 \Rightarrow \lambda_{i+1} < \lambda_i \}$



$\boxed{\beta = -1}$

<u>Main theorem</u> [Ikeda - Iwao - Shimozono - Y.]

There is an isomorphism of $R(T)$-algebras

$$\Phi : K_*^T(Gr_G) \cong g\Gamma_{(n)}^{R(T)}, \quad \text{s.t.}$$

$$[\mathcal{O}_w] \longmapsto \widetilde{gP}_w^{(n)}(y|b), \quad [\mathcal{I}_{\partial X_w}] \longmapsto gP_w^{(n)}(y|b).$$

$\boxed{\beta = 0}$

[Ikeda - Iwao - Shimozono]

$$H_*^T(Gr_G) \cong \hat{\Gamma}_{(n)}^{S} \qquad (S := H_T^*(pt) = \mathbb{C}[a_1, \dots, a_n])$$

$$[X_w] \longmapsto \hat{P}_w^{(n)}(y|a) \quad \left(= gP_w^{(n)}(y|b)\Big|_{\beta=0} \right)$$

Idea of Proof

- $K_*^T(\mathrm{Gr}_G)_{\mathrm{rat}} := \mathrm{Frac}(R(T)) \otimes K_*^T(\mathrm{Gr}_G)$
$$= \bigoplus_{\gamma \in Q^\vee} \mathrm{Frac}(R(T))\, t_\gamma$$

- $g\Gamma^{R(T)}_{(n),\mathrm{rat}} := \mathrm{Frac}(R(T)) \otimes g\Gamma^{R(T)}_{(n)}$
$$= \mathrm{Frac}(R(T))[\Omega(b_i|y)^{\pm 1} \mid 1 \leq i \leq 2n]$$

Key Lemma

$$\begin{array}{cccc} \Psi_{\mathrm{rat}} : & K_*^T(\mathrm{Gr}_G)_{\mathrm{rat}} & \xrightarrow{\simeq} & g\Gamma^{R(T)}_{(n),\mathrm{rat}} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & t_{\varepsilon_i} & \longmapsto & \Omega(b_i|y) \end{array}$$

as $\mathrm{Frac}(R(T))$-alg.

Lemma

$$t_{\varepsilon_i} = 1 + \sum_{k=1}^{i} (-1)^k \, [\![\bar{b}_i | b]\!]^k \, [I_{\partial x_{\rho_k}}] \qquad (1 \leq i \leq n).$$

Lemma

$$\Omega(y|b_i) = 1 + \sum_{k=1}^{i} (-1)^k [\![\bar{b}_i | b]\!]^k \, gp^{(n)}_{\rho_k}(y|b) \qquad (1 \leq i \leq n).$$

Example

$$\Omega(y|b_1) = 1 - (\bar{b}_1 \oplus \bar{b}_1)\, gp^{(n)}_{\rho_1}$$

$$\Omega(y|b_2) = 1 - (\bar{b}_2 \oplus \bar{b}_2)\, gp^{(n)}_{\rho_1}(y|b) + (\bar{b}_2 \oplus \bar{b}_2)(\bar{b}_2 \oplus b_1)\, gp^{(n)}_{\rho_2}(y|b)$$

$$\begin{tikzcd}
K_*^T(Gr_G)_{rat} \arrow[rr, "\sim", "\Phi_{rat}"'] & & g\Gamma_{(n),rat}^{R(T)} \\
K_*^T(Gr_G) \arrow[u, phantom, "\bigcup"] \arrow[rr, "\sim", "\Phi"'] & & g\Gamma_{(n)}^{R(T)} \arrow[u, phantom, "\bigcup"]
\end{tikzcd}$$

[Ikeda-Shimozono]

$$K_*^T(Gr_G) = R(T)\left[\frac{1-t_{\alpha^\vee}}{e(\alpha)} \,\middle|\, \alpha\in\Phi\right]$$

$$g\Gamma_{(n)}^{R(T)} = R(T)\left[gP_{\rho_i}^{(n)}(y|b) \,\middle|\, 1\leq i\leq 2n\right]$$

Conjecture.

Example

$$\Phi_{rat}\left(\frac{1-\overset{t_{\varepsilon_1}}{\overset{\|}{t_{\alpha_0^\vee}}}}{e(\alpha_0)}\right) = \frac{1-\Omega(b_1|y)}{e(\alpha_0)} = gP_{\underset{\overset{\|}{s_0}}{\rho_1}}^{(n)}(y|b) = \Phi([\mathcal{O}_{\rho_1}])$$