Macdonald-Koornwinder polynomials and affine Hecke algebras (Macdonald-Koornwinder 多項式とアフィン Hecke 代数)

Kohei YAMAGUCHI[1]

2023/01/13

[1]Graduate School of Mathematics, Nagoya University. Furocho, Chikusaku, Nagoya, Japan, 464-8602.
e-mail address: d20003j@math.nagoya-u.ac.jp

Contents

Preface

This thesis studies Macdonald-Koornwinder polynomials through the representation theory of affine Hecke algebras, mainly focusing on the structure of Koornwinder polynomials and their behavior under parameter specialization.

Macdonald-Koornwinder polynomials are multivariable q-orthogonal polynomials associated to each affine root system, which integrate Macdonald polynomials introduced by Macdonald [M87] in the late 1980s and Koornwinder polynomials introduced by Koornwinder [Ko92] in the early 1990s. Here, the word "affine root system" means that in the sense of Macdonald [M71, M03]. Macdonald-Koornwinder polynomials appear in various branches of mathematics such as integrable systems, representation theory and mathematical physics, and form an important research subject in recent years.

Today, Macdonald-Koornwinder polynomials are formulated by the *Macdonald-Cherednik theory*, which is based on the representation theory of affine Hecke algebras. This theory was first developed for untwisted affine root systems by Cherednik [C92a, C95a, C95b, C95c]. By the works of Noumi [N95], Sahi [Sa99, Sa00], Stokman [St00] and others, the Macdonald-Cherednik theory is extended to non-reduced affine root systems, particularly to the type $(C_n^\vee, C_n)$ in the sense of Macdonald [M71, M03], and that one can recover Koornwinder polynomials as Macdonald polynomials of type $(C_n^\vee, C_n)$.

Koornwinder polynomials, the main object of this thesis, are q-orthogonal polynomials associated to the affine root systems of type $(C_n^\vee, C_n)$. Let us only introduce the symbol for them, and refer to § 1.3.1 for the precise explanation. Koornwinder polynomials (precisely speaking, the symmetric Koornwinder polynomials) are Laurent polynomials of n-variable, attached to partitions λ (dominant weights of type C_n), and have six complex or formal parameters q, t, t_n, t_0, u_n, u_0 (if $n = 1$, then we omit t and have five parameters). In this thesis, the monic symmetric Koornwinder polynomial of variable $x = (x_1, \ldots, x_n)$ attached to λ is denoted as

$$P_\lambda(x) = P_\lambda(x; q, t, t_n, t_0, u_n, u_0).$$

The five parameters t, t_n, t_0, u_n, u_0 will be called the Hecke parameter of Koornwinder polynomials.

Abstract of Chapter 1

In Chapter 1, we give an introduction to Macdonald-Koornwinder polynomials. Since these polynomials are multivariate analogue of one-variable q-hypergeometric orthogonal polynomials, we start with § 1.1 a brief recollection on hypergeometric orthogonal polynomials and their q-analogues. The total picture of these orthogonal families are depicted in Askey scheme of hypergeometric orthogonal polynomials (Figure 1.1.1) and its q-analogue (Figure 1.1.2). Among the q-hypergeometric orthogonal polynomials in Figure 1.1.2, we focus on Askey-Wilson polynomials, whose properties will be explained in detail. Askey-Wilson polynomials form one of the "mother" classes of q-hypergeometric orthogonal polynomials, and, as will be explained in § 1.3.1, Koornwinder polynomials are multivariate analogue of Askey-Wilson polynomials.

Next, we turn to the multivariate orthogonal polynomials. In § 1.2, we give a brief recollection the well-known three families of orthogonal symmetric polynomials, namely Schur polynomials $s_\lambda(x)$ (§ 1.2.1), Jack polynomials $P_\lambda(x;\beta)$ (§ 1.2.2), and Macdonald symmetric polynomials $P_\lambda(x;q,t)$ (§ 1.2.3). The last ones are 2-parameter generalization of the others, and these orthogonal families sit in the following degeneration scheme.

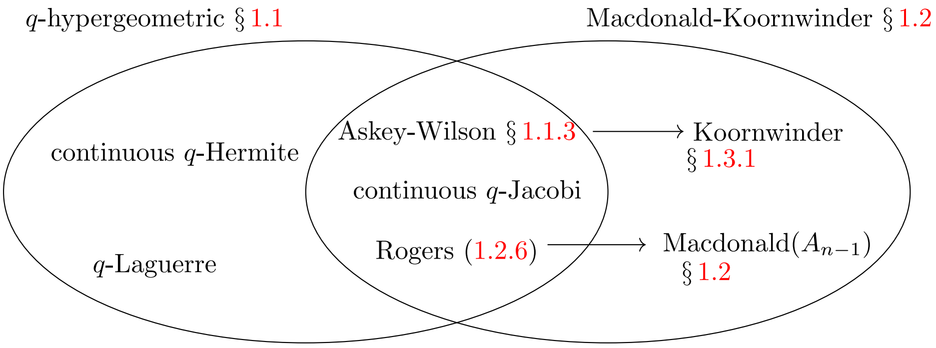
$$\text{Macdonald } P_\lambda(x;q,t) \xrightarrow[t=q^\beta,\ q\to 1]{} \text{Jack } P_\lambda(x;\beta) \xrightarrow[\beta=1]{} \text{Schur } s_\lambda(x)$$

Macdonald symmetric functions $P_\lambda(x;q,t)$ can be regarded as the Macdonald polynomial associated to the affine root system of type A. As mentioned in the beginning, there are analogous orthogonal families associated to other affine root systems, and they are now called Macdonald-Koornwinder polynomials. A unified formulation is established after the development of representation theoretic approach using the (double) affine Hecke algebras, and it is called the Macdonald-Cherednik theory.

We give an overview of this theory in § 1.3.1, and refer to [C05, H06, M03, St20] for the concise explanation.

In this thesis, we do not explain the Macdonald-Cherednik theory for arbitrary affine root systems, but treat the theory only for type $(C_n^\vee, C_n)$, i.e., the theory for Koornwinder polynomials. In § 1.3.2 and § 1.3.3, we will explain in detail how to define Koornwinder polynomials though the representation theory of affine Hecke algebra associated to the non-reduced affine root system $(C_n^\vee, C_n)$.

Below is the picture of relations among various orthogonal systems treated in Chapter 1 (which will also appear in the later chapters). The two arrows mean that the target is the multivariate analogue of the source.



Abstract of Chapter 2

Chapter 2 is based on the author's paper [Ya22]. We consider *Littlewood-Richardson coefficients* $c_{\lambda,\mu}^{\nu}$ of Koornwinder polynomials P_λ, that is the structure constants of the product in the invariant ring $\mathbb{K}[x^{\pm 1}]^{W_0}$:

$$P_\lambda P_\mu = \sum_{\nu} c_{\lambda,\mu}^{\nu} P_\nu.$$

Hereafter we call $c_{\lambda,\mu}^{\nu}$ *LR coefficients* for simplicity.

Let us recall what is known in the case of type A. The classical LR coefficients are the structure constants of the product $s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^{\nu} s_\nu$ of Schur polynomials s_λ (1.2.1) in the ring of symmetric polynomials. Regarding Schur polynomials s_λ as the characters of the irreducible representation V_λ of the general linear group, we can interpret the coefficient $c_{\lambda,\mu}^{\nu}$ as the multiplicity of the irreducible decomposition of the tensor product representation $V_\lambda \otimes V_\mu$. For Hall-Littlewood polynomials, which are t-deformations of Schur polynomials, we can also consider the LR coefficients $c_{\lambda,\mu}^{\nu}$, and some explicit formulas are known. See [Ma95, Chap. II, (4.11)] for example.

Although Macdonald polynomial of type A is a q-deformation of Hall-Littlewood polynomial, no explicit formula for the corresponding LR coefficient $c_{\lambda,\mu}^{\nu}$ had been unknown for a long time. In [Ma95, Chap. VI, §6], Macdonald derived some combinatorial formulas for *Pieri coefficients* using arms and legs of Young diagrams. Here Pieri coefficients mean the LR coefficients $c_{\lambda,\mu}^{\nu}$ with λ the one-row type (k) or the one-column type (1^l), where the weights are identified with Young diagrams or partitions.

On the LR coefficients of Macdonald polynomials, Yip [Yi12] made a great progress. Using *alcove walks*, an explicit formula of $c_{\lambda,\mu}^{\nu}$ is given in [Yi12, Theorem 4.4] for the Macdonald polynomials of untwisted affine root systems. Moreover, a simplified formula [Yi12, Corollary 4.7] is derived in the case λ is equal to a minuscule weight. In particular, this simplified formula recovers Macdonald's formula for Pieri coefficients of type A [Yi12, Theorem 4.9]. In Yip's study, the key ingredient is the notion of alcove walks, originally introduced by Ram [Ra06]. We will explain the relevant notations and terminology in § 2.1.

The main result of Chapter 2 is the following Theorem A, which is a natural $(C_n^\vee, C_n)$-type analogue of Yip's alcove walk formulas for LR coefficients in [Yi12, Theorem 4.4]. Let us prepare the necessary notations and terminology for the explanation.

Let A be the fundamental alcove of the extended affine Weyl group W (see (2.1.1)). Given an element $w \in W$, we take a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Given a bit sequence $b = (b_1, \ldots, b_r) \in \{0,1\}^r$ and an element $z \in W$, we call a sequence of alcoves of the form

$$p = \big(p_0 := zA,\ p_1 := zs_{i_1}^{b_1}A,\ p_2 := zs_{i_1}^{b_1}s_{i_2}^{b_2}A,\ \ldots,\ p_r := zs_{i_1}^{b_1}\cdots s_{i_r}^{b_r}A\big)$$

an *alcove walk of type* $\overrightarrow{w} := (i_1, \ldots, i_r)$ *beginning at* zA. We denote by $\Gamma(\overrightarrow{w}, z)$ the set of such alcove walks. See Example 2.1.0.1 of alcove walks.

For an alcove walk p, we call the transition $p_{k-1} \to p_k$ *the k-th step of* p. The k-th step of p is called a *folding* if $b_k = 0$ where the bit sequence b corresponds to the alcove walk p (see Table 2.1.1).

In our main result, we use *a colored alcove walk* introduced by Yip [Yi12]. It is an alcove walk equipped with the coloring of folding steps by either black or gray. We denote by $\Gamma_2^C(\overrightarrow{w}, z)$ the set of colored alcove walks whose steps belong to the dominant chamber $C \subset V := \mathbb{R}^n$.

Theorem A (Theorem 2.2.4.2)**.** Let $\lambda, \mu \in \Lambda_+$ be dominant weights, W_μ be the stabilizer of μ in the finite Weyl group W_0 (see (1.3.38)), and W^μ be the complete system of representatives of W_0/W_μ such that the shortest length element in each the quotient class (see (2.2.11)). Let also $W_\lambda(t)$ be the Poincaré polynomial of the stabilizer W_λ (see (1.3.41)). We take a reduced expression of the element $w(\lambda) \in W$ in (1.3.36). Then we have

$$P_\lambda P_\mu = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})} A_p B_p C_p P_{-w_0.\mathrm{wt}(p)}.$$

Here $w_0 \in W_0$ is the longest element, and the weight $\mathrm{wt}(p) \in \Lambda$ is determined from the element $e(p) \in W$ corresponding to the end of the colored alcove walks p as in (2.2.14). The coefficients A_p, B_p and C_p are factorized, and we have

$$A_p := \prod_{a \in w(\mu)^{-1}\mathcal{L}(v^{-1}, v_\mu^{-1})} \rho(a), \quad B_p := \prod_{a \in \mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0, e(p))} \rho(-a).$$

Here the term $\rho(\alpha)$ is given by

$$\rho(a) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \notin W.a_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \in W.a_n) \end{cases},$$

$$q^{\mathrm{sh}(a)} := q^{-k}, \quad t^{\mathrm{ht}(a)} := \prod_{\gamma \in R_+^s} t^{\frac{1}{2}\langle \gamma^\vee, \alpha \rangle} \prod_{\gamma \in R_+^\ell} (t_0 t_n)^{\frac{1}{2}\langle \gamma^\vee, \alpha \rangle} \quad (a = \alpha + kc \in S),$$

where we used $R_+^s := \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$ and $R_+^\ell := \{2\epsilon_i \mid 1 \le i \le n\}$. For the notation $\mathcal{L}$, see (2.1.3) in § 2.1. Finally the term C_p is given by $C_p = \prod_{k=1}^r C_{p,k}$ with the factor $C_{p,k}$ determined from the k-th step of the alcove walk p in Proposition 2.2.3.2. Here we display the relevant formulas for $C_{p,k}$:

$$\psi_i^\pm(z) := \mp \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - z^{\pm 1}} \qquad (i = 1, \ldots, n-1),$$

$$\psi_0^\pm(z) := \mp \frac{(u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}, \quad \psi_n^\pm(z) := \mp \frac{(t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})}{1 - z^{\pm 1}},$$

$$n_i(z) := \frac{1 - tz}{1 - z} \frac{1 - t^{-1}z}{1 - z} \qquad (a \in W.a_i,\ i = 1, \ldots, n-1),$$

$$n_0(z) := \frac{(1 - u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \frac{(1 + u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 - u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \quad (a \in W.a_0),$$

$$n_n(z) := \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \frac{(1 + t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 - t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \quad (a \in W.a_n).$$

Note that the term A_p actually depends only on $v \in W^\mu$, which corresponds to the beginning of the colored alcove walk p.

Abstract of Chapter 3

Chapter 3 is based on the collaboration paper [YY22] with S. Yanagida. The contents of [YY22] can be divided into two parts:

- Classification of parameter specializations of Koornwinder polynomials
- Re-derivation of Ram-Yip type formulas of Macdonald polynomials.

The author contributed mainly to the latter part.

The motivation of Chapter 3 is the comment by Macdonald given at [M03, p.12, (5.17)]: The Macdonald polynomials associated to all the subsystems of type $(C_n^\vee, C_n)$ can be obtained by specializing the five Hecke parameters of the Koornwinder polynomial in the way respecting the orbits of the extended affine Weyl group acting on the affine root systems. Seemingly, the detailed explanation of such parameter specialization is not given in literature. The aim of Chapter 3 is to clarify this point. The result is as follows.

Theorem B (Propositions 3.1.3.1, 3.1.4.1–3.1.4.9)**.** For each type X listed in Table 0.0.1 and for each (not necessarily) dominant weight μ of type C_n, the specialization of the Noumi parameters in the (non-symmetric) Koornwinder polynomial with weight μ yields the (non-symmetric) Macdonald polynomial with μ of type X in the sense of Definition 1.3.1.1.

reduced		t	t_0	t_n	u_0	u_n	non-reduced		t	t_0	t_n	u_0	u_n
B_n	§ 3.1.4	t_l	1	t_s	1	t_s	(BC_n, C_n)	§ 3.1.4	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	§ 3.1.4	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	§ 3.1.4	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	§ 3.1.3	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	§ 3.1.4	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	§ 3.1.4	t_l	t_s	t_s	t_s	t_s							
BC_n	§ 3.1.4	t_m	t_l^2	t_s	1	t_s							
D_n	§ 3.1.4	t	1	1	1	1							

Table 0.0.1: Specialization table

Hereafter we refer Table 0.0.1 as the *specialization table*.

In § 3.2, as a verification of the specializing Table 0.0.1, we check the obtained specializations by using explicit formulas of Macdonald-Koornwinder polynomials. We focus on *Ram-Yip type formulas* [RY11, OS18] which were mentioned before. These formulas give explicit description of the coefficients in the monomial expansion of non-symmetric Macdonald-Koornwinder polynomials $E_\mu(x) = E_\mu(x; q, t, t_0, t_n, u_0, u_n)$ as a summation of terms over the so-called *alcove walks*, the notion introduced by Ram [Ra06]. We do this check for Ram-Yip formulas of type B, C and D in the sense of [RY11]. The check is done just in case-by-case calculation, but since the situation is rather complicated due to the notational problem of affine root systems and parameters, we believe that it has some importance. The result is as follows.

Theorem C (Propositions 3.2.1.5, 3.2.2.4 and 3.2.3.5)**.** For each $\mu \in P_{C_n}$, we have

$$E_\mu(x; q, t_m^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}) = E_\mu^{B,\mathrm{RY}}(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}}),$$
$$E_\mu(x; q, t_m^{\mathrm{RY}}, 1, t_s^{\mathrm{RY}}, 1, 1) = E_\mu^{C,\mathrm{RY}}(x; q, t_s^{\mathrm{RY}}, t_m^{\mathrm{RY}}),$$
$$E_\mu(x; q, t, 1, 1, 1, 1) = E_\mu^{D,\mathrm{RY}}(x; q, t).$$

Here the left hand sides denote specializations of the non-symmetric Koornwinder polynomials $E_\mu(x)$, and the right hand side denotes the non-symmetric Macdonald polynomials of type B, C and D in the sense of [RY11]. For the detail, see the beginning of § 3.2 for the explanation. Comparing these identities with the specialization Table 0.0.1, we find that $E_\mu^{B,\mathrm{RY}}(x)$ is equivalent to the polynomial of type B_n, $E_\mu^{C,\mathrm{RY}}(x)$ is to that of type $C_n^\vee$, and $E_\mu^{D,\mathrm{RY}}(x)$ is to that of type D_n in the sense of Definition 1.3.1.1.

Abstract of Chapter 4

Chapter 4 is based on the proceeding draft [YY] of the author's talk in the conference "Recent developments in Combinatorial Representation Theory" at RIMS, Kyoto University held in November 7th–11th, 2022, written with S. Yanagida.

The purpose of Chapter 4 is to give a review of the bispectral correspondence [vMS09, vM11, St14] between quantum affine Knizhnik-Zamolodchikov equations and the eigenvalue problems of Macdonald type, delivered in § 4.1 and § 4.2. We also study the relation of the bispectral correspondence and the parameter specialization explained in Chapter 3, and it is fulfilled in § 4.3.

Rank one review of bispectral correspondence

The first part (§ 4.1, § 4.2) is devoted to the review of the bispectral correspondence between quantum affine Knizhnik-Zamolodchikov (QAKZ for short) equations and Macdonald-type eigenvalue problems, established by the works [vM11, vM11, St14].

Let us begin with the recollection on the original Cherednik's correspondence. We refer to [C05, §1.3] for an exposition of this correspondence. In [C92b], Cherednik introduced his QAKZ equations for arbitrary reduced root systems (in the sense of Bourbaki [B68b]) and for the type GL_n. Let $H = H(k, q)$ be the affine Hecke algebra of the concerning root systems, and let $T := \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$ be the algebraic torus associated to the weight lattice Λ. Then the QAKZ equations are q-difference equations for functions of torus variable $t \in T$ valued in a (left) H-module M satisfying certain conditions. In [C92a], Cherednik constructed a correspondence between solutions of the QAKZ equations for the principal series representation M_γ with central character $\gamma \in T$, and eigenfunctions of the q-difference operators of Macdonald type.

Cherednik's correspondence for the type GL_n is now described as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{qKZ}}(k, q)_\gamma \longrightarrow \mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma. \qquad (*)$$

A bispectral analogue of Cherednik's correspondence is investigated by van Meer and Stokman [vMS09] for type GL, who introduced the bispectral QAKZ equations using Cherednik's duality anti-involution $* \colon \mathbb{H} \to \mathbb{H}$ of the double affine Hecke algebra (DAHA) $\mathbb{H}$ (see (1.3.9)). The bispectral QAKZ equations are consistent systems of q-difference equations for functions on the product torus $T \times T$, and splits up into two subsystems. Denoting by $(t, \gamma) \in T \times T$ the variable, we have:

- The first subsystem only acts on t, and for a fixed γ, the equations in t are Cherednik's QAKZ equations for the principal series representation M_γ of the affine Hecke algebra $H \subset \mathbb{H}$.
- For a fixed $t \in T$, the equations in γ are essentially the QAKZ equations for $M_{t^{-1}}$ of the image $H^* \subset \mathbb{H}$.

This argument can be extended to arbitrary reduced and non-reduced root systems, as done by van Meer [vM11] for reduced types and by Takeyama [T10] for the non-reduced type $(C_n^\vee, C_n)$.

After the build-up of bispectral QAKZ equations, it is rather straightforward, except for one issue, to make an analogue of Cherednik's construction of correspondence to the bispectral eigenvalue problems of Macdonald-type. Mimicking $(*)$, the resulting bispectral correspondence is depicted as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(k, q) \longrightarrow \mathrm{SOL}_{\mathrm{bMac}}(k, q).$$

The issue here is the existence of (some nice) asymptotic free solutions of the bispectral QAKZ equations, i.e., non-emptiness of the source, which was carefully done for type GL_n in [vM11, §5, Appendix]. The same argument works with minor modification for reduced and non-reduced root types (see [St14, §3]).

In § 4.1 and § 4.2, we give a review of the bispectral correspondence explained so far, focusing on type A_1 and type $(C_1^\vee, C_1)$, respectively.

Specializing parameters in the rank one bispectral problems

The second part (§ 4.3) is a complement of the first part, and is also a continuation of Chapter 3 (the paper [YY22]) on the parameter specialization of Macdonald-Koornwinder polynomials. There we classify all the specializations based on the affine root systems appearing as subsystems of the type $(C_n^\vee, C_n)$ system. The obtained parameter specializations are compatible with degenerations of the Macdonald-Koornwinder inner product to the subsystem inner products.

In the rank one case § 3.1.6 ([YY22, §2.6]), where the concerned polynomials are Askey-Wilson polynomials, we discovered four ways of specialization of the type $(C_1^\vee, C_1)$ parameters to recover the type A_1. Table 0.0.2 is the excerpt from Table 3.1.1.

type	Dynkin	orbits	Hecke parameters			
$(C_1^\vee, C_1)$ Askey-Wilson	$\overset{*}{\underset{0}{\circ}} = \overset{*}{\underset{1}{\circ}}$	$O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$	k_0	k_1	l_0	l_1
A_1	$\overset{0}{\circ} = \overset{1}{\circ}$	O_1	1	t	1	t
		O_3	t	1	t	1
Rogers		O_2	1	t^2	1	1
		O_4	t^2	1	1	1

Table 0.0.2: Type A_1 subsystems in $(C_1^\vee, C_1)$ and parameter specializations

In § 4.3, we study the relation between our parameter specializations and the bispectral correspondence. To begin with, let us recall that the bispectral correspondence is built using the duality anti-involution $*$ of the DAHA $\mathbb{H}$. As reviewed in § 4.2.1 (4.2.16), the duality anti-involution $*$ of $\mathbb{H}$ affects on the Hecke parameters in the way

$$(k_1^*, k_0^*, l_1^*, l_0^*) = (k_1, l_1, k_0, l_0).$$

Then, we see from Table 4.0.1 that the specialization corresponding to the orbit O_2 is the only one which is compatible with the bispectral correspondence reviewed in the first part. Under this specialization, we establish the following commutative diagram (Theorem 4.3.1.2).

$$\begin{tikzcd}
\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)} \arrow[r, hook, "\chi_+^{(C_1^\vee, C_1)}"] \arrow[d, hook, "\mathrm{sp}"'] & \mathrm{SOL}_{\mathrm{bAW}} \arrow[d, hook, "\mathrm{sp}"] \\
\mathrm{SOL}_{\mathrm{bqKZ}}^{A_1} \arrow[r, hook, "\chi_+^{A_1}"'] & \mathrm{SOL}_{\mathrm{bMR}}
\end{tikzcd}$$

Acknowledgements. The author would like to thank my advisor Shintarou Yanagida for reviewing the manuscript several times, giving useful advice and for your dedication and guidance. He would also like to thank Masatoshi Noumi for the explanation on the Macdonald-Cherednik theory and Koornwinder polynomials given in the master course in Kobe University and also thank Satoshi Naito for his important comments. He would like to thank Takeshi Ikeda for listening to my research and encouraging me.

Chapter 1

Macdonald-Koornwinder polynomials

1.0 Global notation

Let us explain the notation used throughout in this thesis.

- We denote by $\mathbb{Z}$ the ring of integers, by $\mathbb{N} = \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$ the set of non-negative integers, by $\mathbb{Q}$ the rational number field, by $\mathbb{R}$ the real number field, and by $\mathbb{C}$ the complex number field.
- We denote by $\delta_{i,j}$ the Kronecker delta on a set $I \ni i, j$.
- We denote by e or 1 the unit of a group.
- We denote $\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$, regarded as the multiplicative group.
- We denote an action of a group G on a set S by $g.s$ for $g \in G$ and $s \in S$, and denote the G-orbit of s by $G.s$ or by Gs.
- A ring or an algebra means a unital associative one unless otherwise stated.
- For a commutative ring k and a family of commutative variants $x = (x_1, x_2, \ldots)$, we denote by $k[x^{\pm 1}]$ the Laurent polynomial ring $k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots]$.
- Linear spaces will be those over the complex number field $\mathbb{C}$ unless otherwise stated, and we denote by $\mathrm{Hom}(V, W)$ and $\mathrm{End}(V)$ the linear spaces of $\mathbb{C}$-linear homomorphisms $V \to W$ and of endomorphisms $V \to V$. We also denote by $\otimes$ the standard tensor product $\otimes_{\mathbb{C}}$ over $\mathbb{C}$.

1.1 Hypergeometric orthogonal polynomials and the q-analogue

In this section, we give a brief review of one-variable hypergeometric orthogonal polynomials and their q-analogue. The main references are [KLS10], [GR04], [青 13] and [三 04].

1.1.1 Hypergeometric notation

We begin with the Gauss hypergeometric series [三 04, p.363]. It is defined using the factorial symbol $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ as

$$ {}_2F_1\left[\begin{matrix}\alpha,\ \beta \\ \gamma\end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n. $$

We have the following generalization, called the generalized hypergeometric series [三 04, p.363]:

$$ {}_{s+1}F_s\left[\begin{matrix}\alpha_1, & \cdots, & \alpha_{s+1} \\ \beta_1, & \cdots, & \beta_s\end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_{s+1})_n}{n!(\beta_1)_n \cdots (\beta_s)_n} z^n. $$

We use Gasper and Rahman's notation [GR04] for q-shifted factorials

$$ (x;q)_\infty := \prod_{n=0}^{\infty}(1 - xq^n), \quad (x_1, \ldots, x_r; q)_\infty := \prod_{i=1}^{r}(x_i;q)_\infty, \tag{1.1.1} $$

which are understood as complex numbers if they converge (e.g., if $x, x_i, q \in \mathbb{C}$ and $|q| < 1$), and as formal series of q otherwise. For $n \in \mathbb{N}$, we set

$$(x;q)_n := \frac{(x;q)_\infty}{(xq^{n+1};q)_\infty}, \quad (x_1,\dots,x_r;q)_n := \prod_{i=1}^{r}(x_i;q)_n. \tag{1.1.2}$$

We also use the symbol in [GR04] of the basic hypergeometric series

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix}; q,\ z\right] := \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n} z^n.$$

1.1.2 One-variable orthogonal polynomials

Let us recall here the definition of a one-variable orthogonal polynomial. Take a function $w(z)$ such that

$$\int_a^b w(z)dz > 0.$$

For one-variable polynomial functions $f(z)$ and $g(z)$ defined on the closed interval $[a,b]$, the inner product is determined as follows:

$$\langle f, g\rangle := \int_a^b f(z)g(z)w(z)dz.$$

When a family $\{p_n(z)\}_{n\in\mathbb{N}}$ of polynomials satisfies the following property, it is called an orthogonal polynomial system and $w(x)$ is called a weight function.
(1) $p_n(z)$ is a polynomial of degree n.
(2) $\langle p_m(z), p_n(z)\rangle = 0 \quad (m \neq n)$.

Wilson polynomials and Racah polynomials have the most parameters among all one-variable hypergeometric orthogonal polynomials. It is known that by specializing them appropriately, we can obtain various one-variable hypergeometric orthogonal polynomials. These polynomials and the specialization behavior are summarized in the *Askey scheme* Figure 1.1.1, cited from.

There is also known a q-analogue of the Wilson polynomial, called *the Askey-Wilson polynomial* [KLS10, p.415–419], [青 13, p.126–141], which is the q-hypergeometric orthogonal polynomial with the most parameters. By appropriately specializing the Askey-Wilson polynomial, various q-analogue of Jacobi polynomials can be recovered. The degenerate scheme is given in Figure 1.1.2, called *the q-Askey scheme*.

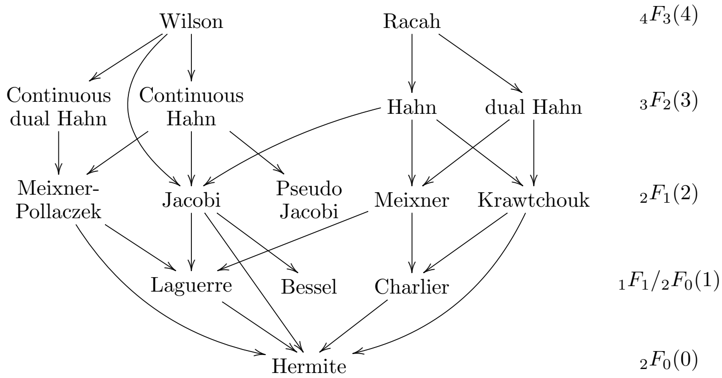


Figure 1.1.1: Askey scheme of hypergeometric orthogonal polynomials [KLS10, p.182]

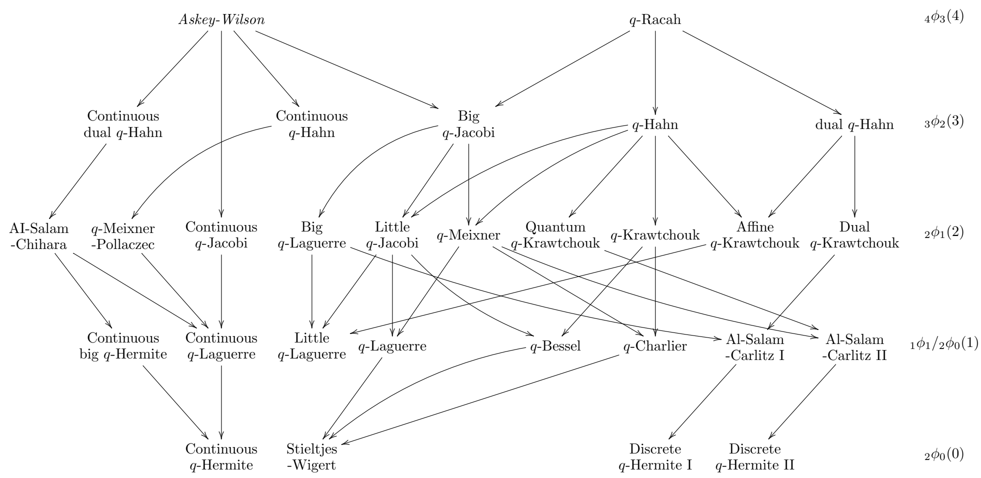


Figure 1.1.2: Askey scheme of q-hypergeometric orthogonal polynomials [KLS10, p.412]

1.1.3 Askey-Wilson polynomials and its specialization

We will now introduce Askey-Wilson polynomials, continuous q-Jacobi polynomials [KLS10, p.463] and Jacobi polynomials [青 13, 三 04], as an example of a one-variable hypergeometric orthogonal polynomials and its q-analogues.

Askey-Wilson polynomials

Askey-Wilson polynomials [AW85] are q-hypergeometric orthogonal polynomials of one-variable equipped with extra parameters (a, b, c, d), which recover various q-analogue of Jacobi polynomials by specialization of the parameters.

(1) Explicit formula:

$$P_l(z; a, b, c, d; q) = P_l(z) := \frac{(ab, ac, ad; q)_l}{a^l} {}_4\phi_3 \left[\begin{matrix} q^{-l},\ abcdq^{l-1},\ ax,\ a/x \\ ab,\ ac,\ ad \end{matrix} ; q,\ q \right] \quad (l \in \mathbb{N}).$$

with $z := (x + x^{-1})/2$.

(2) Orthogonality: For generic parameters $a, b, c, d \in \mathbb{C}$,

$$\int_{-1}^{1} P_m(z) P_n(z) \frac{w(z)}{2\pi\sqrt{1-z^2}} dz = 0, \quad m \neq n,$$

where the weight function $w(z)$ is given by

$$w(z) := \frac{\prod_{k=0}^{\infty}(1 - (2z^2 - 1)q^k + q^{2k})}{h(z, a)h(z, b)h(z, c)h(z, d)}, \quad h(z, \alpha) := \prod_{k=0}^{\infty}(1 - 2\alpha z q^k + \alpha^2 q^{2k}).$$

(3) 3-term recursive relation:

$$2x\tilde{P}_l(z) = A_l \tilde{P}_{l+1}(z) + \left(a + a^{-1} - (A_l + C_l)\right) \tilde{P}_l(z) + C_l \tilde{P}_{l-1}(z),$$

$$\begin{aligned} \tilde{P}_l(z) &:= \frac{a^l (abcd; q)_l}{(ab, ac, ad; q)_l} P_l(z; q, a, b, c, d), \\ A_l &:= \frac{(1 - abq^l)(1 - acq^l)(1 - adq^l)(1 - abcdq^{l-1})}{a(1 - abcdq^{2l-1})(1 - abcdq^{2l})}, \\ C_l &:= \frac{a(1 - q^{l-1})(1 - bcq^{l-1})(1 - bdq^l)(1 - cdq^{l-1})}{(1 - abcdq^{2l-2})(1 - abcdq^{2l-1})}. \end{aligned}$$

Remark 1.1.3.1. Jacobi polynomials can be recovered by specializing the Askey-Wilson parameters as follows [KLS10]:

$$(a,b,c,d) \to (q^{(2\alpha+1)/4}, q^{(2\alpha+3)/4}, -q^{(2\beta+1)/4}, -q^{(2\beta+3)/4}), \quad q \to 1.$$

Continuous q-Jacobi polynomials

Continuous q-Jacobi polynomials can be recovered by specializing the Askey-Wilson parameters as follows:

$$(a,b,c,d) \to (q^{(2\alpha+1)/4}, q^{(2\alpha+3)/4}, -q^{(2\beta+1)/4}, -q^{(2\beta+3)/4}).$$

(1) Explicit formula:

$$P_l^{(\alpha,\beta)}(z;q) = P_l(z;q) := \frac{(q^{\alpha+1};q)_l}{(q;q)_l} {}_4\phi_3\left[\begin{matrix} q^{-l},\ q^{l+\alpha+\beta+1},\ q^{\frac{1}{2}\alpha+\frac{1}{4}}x,\ q^{\frac{1}{2}\alpha+\frac{1}{4}}x^{-1} \\ q^{\alpha+1},\ q^{\frac{1}{2}(\alpha+\beta+1)},\ -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix}; q,\ q\right] \quad (l \in \mathbb{N}).$$

with $z := (x+x^{-1})/2$.

(2) Orthogonality: For parameters $\alpha \geq -\frac{1}{2},\ \beta \geq -\frac{1}{2}$,

$$\int_{-1}^{1} P_m(z;q)P_n(z;q)\frac{w(z)}{2\pi\sqrt{1-z^2}}dz = 0, \quad m \neq n,$$

where the weight function $w(z)$ is given by

$$w(z) := \frac{h(z,1)h(z,-1)h(z,q^{\frac{1}{2}})h(z,-q^{\frac{1}{2}})}{h(z,q^{\frac{1}{2}\alpha+\frac{1}{4}})h(z,q^{\frac{1}{2}\alpha+\frac{3}{4}})h(z,-q^{\frac{1}{2}\beta+\frac{1}{4}})h(z,-q^{\frac{1}{2}\beta+\frac{3}{4}})}, \quad h(z,\alpha) := \prod_{k=0}^{\infty}(1-2\alpha z q^k + \alpha^2 q^{2k}).$$

(3) 3-term recursive relation:

$$2x\tilde{P}_l(z;q) = A_l\tilde{P}_{l+1}(z;q) + \left(q^{\frac{1}{2}\alpha+\frac{1}{4}} + q^{-\frac{1}{2}\alpha-\frac{1}{4}} - (A_l + C_l)\right)\tilde{P}_l(z) + C_l\tilde{P}_{l-1}(z;q),$$

$$\begin{aligned}
\tilde{P}_l(z) &:= \frac{(q^{\alpha+1};q)_l}{(q;q)_l}P_l(z;q), \\
A_l &:= \frac{(1-q^{l+\alpha+1})(1-q^{l+\alpha+\beta+1})(1+q^{l+\frac{1}{2}(\alpha+\beta+1)})(1+q^{l+\frac{1}{2}(\alpha+\beta+2)})}{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^{2l+\alpha+\beta+1})(1-q^{2l+\alpha+\beta+2})}, \\
C_l &:= \frac{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^l)(1-q^{l+\beta})(1-q^{l+\frac{1}{2}(\alpha+\beta)})(1+q^{l+\frac{1}{2}(\alpha+\beta+1)})}{(1-q^{2l+\alpha+\beta})(1-q^{2l+\alpha+\beta+1})}.
\end{aligned}$$

Remark 1.1.3.2. Jacobi polynomials can be recovered by specializing $q \to 1$[KLS10]:

Jacobi polynomials

(1) Explicit formula:

$$P_l^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_l}{l!} {}_2F_1\left[\begin{matrix} -l,\ \alpha+\beta+l+1 \\ \alpha+1 \end{matrix}; \frac{1-z}{2}\right] \quad (l \in \mathbb{N}). \tag{1.1.3}$$

(2) Orthogonality: The weight function $w(z)$ given by

$$w(z) := (1-z)^\alpha(1-z)^\beta.$$

In other words, the following holds:

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(z)P_n^{(\alpha,\beta)}(z)(1-z)^\alpha(1-z)^\beta dz = 0 \quad (m \neq n).$$

(3) 3-term recursive relation:

$$\begin{aligned}
A_l P_l^{(\alpha,\beta)}(z) &= B_l P_{l-1}^{(\alpha,\beta)}(z) + C_l P_{l-2}^{(\alpha,\beta)}(z), \quad (P_{-1}^{(\alpha,\beta)}(z) := 0) \\
A_l &:= 2l(l+\alpha+\beta)(2l+\alpha+\beta-2), \\
B_l &:= (2l+\alpha+\beta-1)\big((2l+\alpha+\beta)(2l+\alpha+\beta-2)z + \alpha^2 - \beta^2\big), \\
C_l &:= -2l(l+\alpha-1)(n+\beta-1)(2l+\alpha+\beta).
\end{aligned}$$

1.2 Macdonald symmetric polynomials

So far, we have discussed one-variable hypergeometric orthogonal polynomials and their q-analogues. In particular, we explained the "mother" family of Askey-Wilson polynomials. As we will explain in §1.3.1, the multivariate version of Askey-Wilson polynomial is formulated as the Macdonald polynomial associated to the non-reduced affine root system of type $(C_n^\vee, C_n)$, and is called the Koornwinder polynomial. Before introducing Macdonald polynomials associated to arbitrary affine root systems, we explain in this section the proto-typical theory of Macdonald symmetric polynomials, which can be regarded as the Macdonald polynomials of type GL. The main references for this section are [M87, Ma95].

1.2.1 Schur polynomials

Let $x = (x_1, \dots x_n)$ be a family of independent indeterminates. The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{C}[x] := \mathbb{C}[x_1, \dots, x_n]$ by permuting the x_i's, and we denote

$$\mathbb{C}[x]^{\mathfrak{S}_n} := \{f(x) \in \mathbb{C}[x] \mid w(f(x)) = f(x) \text{ for all } w \in \mathfrak{S}_n\}$$

for the subring of symmetric polynomials in $\mathbb{C}[x]$. We denote by

$$\Lambda_+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}$$

the set of dominant weights of type A_n (the reduced irreducible root system in the sense of Bourbaki [B68b]). The dominance order $\geq$ on Λ_+ is defined as follows:

$$\lambda \geq \mu \iff \textstyle\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i \text{ and } \sum_{i=1}^r \lambda_i \geq \sum_{i=1}^r \mu_i \text{ for all } r = 1, \dots, n-1. \tag{1.2.1}$$

For a dominant weight $\lambda \in \Lambda$, the monomial symmetric polynomial $m_\lambda(x)$ is defined by

$$m_\lambda(x) := \sum_{\mu \in \mathfrak{S}_n . \lambda} x_1^{\mu_1} \cdots x_n^{\mu_n} \tag{1.2.2}$$

where S_n acts on Λ_+ by permuting its components. Then, $\{m_\lambda(x)\}_{\lambda \in \Lambda_+}$ is a $\mathbb{C}$-basis of $\mathbb{C}[x]^{\mathfrak{S}_n}$.

For a dominant weight $\lambda \in \Lambda_+$, Schur polynomial $s_\lambda(x)$ is the symmetric polynomial defined by

$$s_\lambda(x) := \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{\det(x_i^{n-j})_{1 \leq i,j \leq n}} \in \mathbb{C}[x]^{\mathfrak{S}_n}. \tag{1.2.3}$$

The following is a list of remarkable properties of Schur polynomials.

- Combinatorial explicit formula: n-semi-standard tableaux of Young diagram $D(\lambda)$ corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of length at most n are those in which $1, 2, \dots, n$ are written in each box of $D(\lambda)$ with the following rules.
  - The entries in each row are weakly increasing.
  - The entries in each column are strictly increasing.

  For example, if $n = 7$ and $\lambda = (4, 2, 1)$, then the following are 7-semi-standard tableaux.

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}\ , \qquad \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 3 & 5 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

  We denote by $\mathrm{SST}_n(\lambda)$ the set of the all n-semi-standard tableaux of $D(\lambda)$. Then, Schur polynomials yield the following formula

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in \mathrm{SST}_n(\lambda)} x^T, \quad x^T = \prod_{i=1}^n x_i^{\#\{\text{i's in } T\}}.$$

- $\{s_\lambda(x)\}_{\lambda \in \Lambda_+}$ is an orthogonal basis of $\mathbb{C}[x]^{S_n}$: For $\lambda, \mu \in \Lambda_+$,

$$\frac{1}{n!} \int_T \overline{s_\lambda(x)} s_\mu(x) \Delta(x) dx = 0 \quad (\lambda \neq \mu)$$

with

$$T := \{x \in \mathbb{C}^n \mid |x_1| = \cdots = |x_n| = 1\}, \quad dx := \prod_{i=1}^{n} \frac{dx_i}{2\pi\sqrt{-1}x_i}$$

and the weight function Δ given by

$$\Delta(x) := \prod_{i \neq j}(1 - x_i/x_j).$$

- Triangular expansion: For $\lambda \in \Lambda_+$, Then

$$s_\lambda(x) = m_\lambda(x) + \sum_{\lambda > \mu} K_{\lambda,\mu} m_\mu(x) \quad (K_{\lambda,\mu} \in \mathbb{Z})$$

  where $>$ is the dominance order (1.2.1).
- Small example:

$$\begin{gathered} s_{(1)}(x) = s_1(x), \quad s_{(2)}(x) = m_{(2)}(x) + m_{(1,1)}(x), \\ s_{(3)}(x) = m_{(3)}(x) + m_{(2,1)}(x) + m_{(1,1,1)}(x), \\ s_{(2,1)}(x) = m_{(2,1)}(x) + m_{(1^3)}(x). \end{gathered}$$

1.2.2 Jack polynomials

Jack polynomials are symmetric polynomials $P_\lambda(x;\beta)$, indexed by dominant weights $\lambda \in \Lambda_+$ and depending on parameter $\beta \in \mathbb{C}$, which form a β-deformed family of Schur polynomials. Jack polynomials do not have a simple explicit formula like Schur polynomials (§1.2.1), but they are uniquely characterized by the following two conditions:

- Triangular expansion: For $\lambda \in \Lambda_+$, we have

$$P_\lambda(x;\beta) = m_\lambda(x) + \sum_{\lambda > \mu} c_{\lambda,\mu}(\beta) m_\mu(x) \quad (c_{\lambda,\mu}(\beta) \in \mathbb{C})$$

  where the order $\leq$ is dominance order.
- Differential eigen-equation: The (gauged) Calogero-Sutherland differential operator

$$D^{\mathrm{Jack}} := \frac{\beta^{-1}}{2} \sum_{i=1}^{n} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{1 \leq i \neq j \leq n} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i} \quad (\beta > 0). \tag{1.2.4}$$

  has Jack polynomials as eigenfunctions. More precisely, for each $\lambda \in \Lambda_+$, the polynomial $P_\lambda(x;\beta)$ satisfies the differential eigen-equation

$$D^{\mathrm{Jack}} P_\lambda(x;\beta) = P_\lambda(x;\beta) c_\lambda$$

  with eigenvalue c_λ given by

$$c_\lambda := \frac{\beta^{-1}}{2} \sum_{i=1}^{n} \lambda_i(\lambda_i - 1) - \sum_{i=1}^{n} (i-1)\lambda_i + (n-1) \sum_{i=1}^{n} \lambda_i.$$

We list some properties of Jack polynomial.

- $\{P_\lambda(x;\beta)\}_{\lambda \in \Lambda_+}$ is an orthogonal basis of $\mathbb{C}[x]^{S_n}$: For $\lambda, \mu \in \Lambda_+$,

$$\frac{1}{n!} \int_T \overline{P_\lambda(x;\beta)} P_\mu(x;\beta) \Delta(x;\beta) dx = 0 \quad (\lambda \neq \mu)$$

  with

$$T := \{x \in \mathbb{C}^n \mid |x_1| = \cdots = |x_n| = 1\}, \quad dx := \prod_{i=1}^{n} \frac{dx_i}{2\pi\sqrt{-1}x_i}$$

  and the weight function Δ given by

$$\Delta(x;\beta) := \prod_{i \neq j} (1 - x_i/x_j)^\beta, \quad -\pi < \theta_i \leq \pi, \quad \theta_1 < \cdots < \theta_n < \theta_1 + 2\pi, \quad \arg(1) = 0,$$

  where $\theta_i := \arg(x_i)$ and $(1 - x_i/x_j)^\beta := e^{\beta \log(1 - x_i/x_j)} = e^{\log|1 - x_i/x_j| + i \arg(1 - x_i/x_j)}$.

- Relationship with other polynomials:
  - $P_\lambda(x;\beta=0) = m_\lambda(x)$ (monomial symmetric polynomials (1.2.2))
  - $P_\lambda(x;\beta=1) = s_\lambda(x)$ (Schur polynomials § 1.2.1)
  - $P_\lambda(x;\beta=\frac{1}{2})$ = Zonal polynomials associated to $\mathrm{GL}(n)/\mathrm{SO}(n)$ [Ma95, §VII].
  - $P_\lambda(x;\beta=2)$ = Zonal polynomials associated to $\mathrm{GL}(2n)/\mathrm{Sp}(n)$ [Ma95, §VII].
  - $P_l(x;\beta) = P_l^{(\alpha=\beta,\beta)}(x)$ (Jacobi polynomials (1.1.3)) $(n=1,\ l\in\mathbb{N})$.
- Small example:

$$
\begin{aligned}
P_{(1)}(x;\beta) &= m_1(x), \quad P_{(2)}(x;\beta) = m_{(2)}(x) + \frac{2\beta}{1+\beta}m_{(1,1)}(x),\\
P_{(3)}(x;\beta) &= m_{(3)}(x) + \frac{3\beta}{2+\beta}m_{(2,1)}(x) + \frac{6\beta^2}{(1+\beta)(2+\beta)}m_{(1,1,1)}(x),\\
P_{(2,1)}(x;\beta) &= m_{(2,1)}(x) + \frac{6\beta}{1+2\beta}m_{(1^3)}(x).
\end{aligned}
$$

1.2.3 Macdonald symmetric polynomials

In this part we briefly discuss Macdonald symmetric polynomials. Macdonald symmetric polynomials $P_\lambda(x;q,t)$ are a family of multivariate q-orthogonal symmetric polynomials introduced by I. G. Macdonald in late 1980s [M87], and the family is a generalization of Schur and Jack polynomials. Macdonald polynomials, like the Jack polynomials, are uniquely characterized by of the following two conditions.

- Triangular expansion: For $\lambda \in \Lambda_+$, Then

$$P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\lambda>\mu} d_{\lambda,\mu}(q,t)m_\mu(x) \quad (d_{\lambda,\mu}(q,t)\in\mathbb{C})$$

  where the order $\leq$ is dominance order.
- q-difference eigen-equation: The Macdonald-Ruijsenaars q-difference operator [R87, Ma95] is given by

$$D^{\mathrm{Mac}} := \sum_{i=1}^{n}\prod_{j\neq i}\frac{tx_i - x_j}{x_i - x_j}T_{q,x_i}, \tag{1.2.5}$$

  where T_{q,x_i} denotes the q-shift operator:

$$(T_{q,x_i}f)(x_1,\dots,x_i,\dots,x_n) = f(x_1,\dots,qx_i,\dots,x_n), \quad f(x)\in\mathbb{C}[x].$$

  Then, for each $\lambda\in\Lambda_+$, the polynomial $P_\lambda(x;q,t)$ satisfies the eigen-equation

$$D^{\mathrm{Mac}}P_\lambda(x;q,t) = P_\lambda(x;q,t)c_\lambda$$

  with eigenvalue c_λ given by

$$c_\lambda := \sum_{i=1}^{n} q^{\lambda_i}t^{n-1}.$$

We list some properties of Macdonald symmetric polynomial.

- $\{P_\lambda(x;q,t)\}_{\lambda\in\Lambda_+}$ is an orthogonal basis of $\mathbb{C}[x]^{S_n}$: For $\lambda,\mu\in\Lambda_+$,

$$\frac{1}{n!}\int_T \overline{P_\lambda(x;q,t)}P_\mu(x;q,t)\Delta(x;q,t)dx = 0 \quad (\lambda\neq\mu)$$

  with

$$T := \{x\in\mathbb{C}^n \mid |x_1| = \cdots = |x_n| = 1\}, \quad dx := \prod_{i=1}^{n}\frac{dx_i}{2\pi\sqrt{-1}x_i}$$

  and the weight function Δ given by

$$\Delta(x;q,t) := \prod_{1\leq i<j\leq n}\frac{(x_i/x_j;q)_\infty(x_j/x_i;q)_\infty}{(tx_i/x_j;q)_\infty(tx_j/x_i;q)_\infty}.$$

- Relationship with other polynomials:
  - $P_\lambda(x; q=t) = s_\lambda(x)$ (Schur polynomials § 1.2.1)
  - $\lim_{q\to 1} P_\lambda(x; t=q^\beta) = P_\lambda(x;\beta)$ (Jack polynomials § 1.2.2)
- A_1 cases: For $l \in \Lambda_+ = \mathbb{N}$, we have

$$P_l(x;q,t) = x^l{}_2\phi_1\left[\begin{matrix} t,\ q^{-l} \\ q^{1-l}/t \end{matrix}; q,\ \frac{q}{tx^2}\right], \tag{1.2.6}$$

  which is also called Rogers polynomial.
- Small example:

$$P_{(1)}(x;q,t) = m_1(x), \quad P_{(2)}(x;q,t) = m_{(2)}(x) + \frac{(1+q)(1-t)}{1-qt}m_{(1,1)}(x),$$
$$P_{(3)}(x;q,t) = m_{(3)}(x) + \frac{(1-q^3)(1-t)}{(1-q)(1-q^2t)}m_{(2,1)}(x) + \frac{(1-q^2)(1-q^3)(1-t)^2}{(1-q)^2(1-qt)(1-q^2t)}m_{(1,1,1)}(x),$$
$$P_{(2,1)}(x;q,t) = m_{(2,1)}(x) + \frac{(1-t)(2+q+t+2qt)}{1-qt^2}m_{(1^3)}(x).$$

1.3 Macdonald-Koornwinder polynomials

Macdonald symmetric functions $P_\lambda(x;q,t)$ reviewed in the previous § 1.2 can be regarded as the Macdonald polynomial of type GL_n (or type A affine root system). As mentioned in the beginning, there are analogous orthogonal families associated to other affine root systems, and they are now called Macdonald-Koornwinder polynomials. A unified formulation is established after the development of representation theoretic approach using the (double) affine Hecke algebras [C92a, C95a, C95b, C95c, C97a, N95, Sa99, Sa00, St00, vD96], and it is called the Macdonald-Cherednik theory. There are now several versions of such formulation, and we give an overview in § 1.3.1. We refer to [C05, H06, M03, St20] for the concise explanation.

In this thesis, we only use the Macdonald-Cherednik theory for the non-reduced affine root system of type $(C_n^\vee, C_n)$, i.e., the theory for Koornwinder polynomials. In § 1.3.2 and § 1.3.3, we will explain in detail how to define Koornwinder polynomials though the representation theory of affine Hecke algebra associated to the non-reduced affine root system $(C_n^\vee, C_n)$. Let us explain the organization of this part. In § 1.3.2, we explain the root system R of type C_n and the affine root system of type $(C_n^\vee, C_n)$. In § 1.3.3, we introduce the affine Hecke algebra H of type $(C_n^\vee, C_n)$, and review the basic representation constructed by Noumi [N95]. Then we introduce the double affine Hecke algebra $\mathbb{H}$ of type $(C_n^\vee, C_n)$, and explain the non-symmetric Koornwinder polynomials E_λ (Fact 1.3.3.2). Finally we introduce Koornwinder polynomials P_λ in § 1.3.3 (Fact 1.3.3.4).

1.3.1 Overview of the Macdonald-Cherednik theory

In [M87], Macdonald introduced families of multivariate q-orthogonal polynomials associated to various root systems, which are today called the Macdonald polynomials. Each family has additional t-parameters corresponding to the Weyl group orbits in the root system. The family of Macdonald symmetric polynomials, explained in § 1.2, is the GL_n-version of these families. Following this work, in [Ko92], Koornwinder introduced a multivariate analogue of Askey-Wilson polynomial, having additional five parameters aside from q, which is today called the Koornwinder polynomial. It was also shown in [Ko92] that by specializing these five parameters, we can obtain the Macdonald polynomials of type (BC_n, B_n) and (BC_n, C_n) in the sense of [M87]. Today, these families of multivariate q-orthogonal polynomials are called the Macdonald-Koornwinder polynomials [C05, H06, M03, St20].

After the development of the representation theoretic approach [C92a, C95a, C95b, C95c, C97a, N95, Sa99, Sa00, St00, vD96] using the (double) affine Hecke algebras, there appeared several versions of unified formulation of the Macdonald-Koornwinder polynomials [C05, H06, M03, St20]. These studies are now called the Macdonald-Cherednik theory.

The specialization argument given by Koornwinder in [Ko92] is now understood in a more general form. First, after the studies in [N95, Sa99, Sa00, St00, vD96], the Koornwinder polynomial can be formulated as the Macdonald polynomial associated to the affine root system of type $(C_n^\vee, C_n)$ in the sense of [M03]. See also [St04, St20] for the relevant explanation. Then, as mentioned in [M03, p.12, (5.17)], the Macdonald polynomials associated to all the subsystems of type $(C_n^\vee, C_n)$ can be obtained by specializing the five parameters of the Koornwinder polynomial in the way respecting the orbits of the extended affine Weyl group acting on the affine root systems. See also the comment in [H06, 6.19].

However, it seems that the detailed explanation of the specialization argument is not given in literature. The aim of this paper is to clarify this point.

What troubled the authors in the early stages of the study is that there are tremendously many notations for the affine root systems and the parameters of Macdonald-Koornwinder polynomials, and that even for the work [M87] and the book [M03] both by Macdonald, there seems no explicit comparison in literature. To the authors' best knowledge, in the present writing this paper, the most general framework of the theory of Macdonald-Koornwinder polynomials is given by Stokman [St20], which is based on the approach of Haiman [H06]. It treats uniformly the four classes of Macdonald-Koornwinder polynomials: GL_n, the untwisted case, the twisted case, and the Koornwinder case. The formulation by Macdonald in [M03] treats the latter three cases along this classification.

Although it would be the best to work in the framework of [St20], we gave up to do so due to the following reasons. First, since we are interested in the specialization of Koornwinder polynomials, we may ignore GL_n case, and the formulation of [M03] will be enough. Second, we are also motivated by Ram-Yip type formulas of non-symmetric Macdonald-Koornwinder polynomials [RY11, OS18], and will check our specialization argument in the level of those formulas. The calculations in the check are based on the recent paper [Ya22] by the second named author, which mainly follows the notation in [M03]. Let us mention that some specialization arguments are given in [St20, Example 9.3.28, Remark 9.3.29].

After these considerations, we decided to use the notation in the following literature:

(1) [M03] for affine root systems.
(2) [N95] for the parameters of Koornwinder polynomials.

Let us explain (1) in detail. We use the word "affine root system" in the sense of [M03, §1.2], which originates in [M71]. The word "irreducible finite root system" means an irreducible root system in [M03]. We also denote by $\vee$ the dualizing of finite and affine root systems. Then, as explained in [M03, §1.3], similarity classes of irreducible affine root systems are divided into three cases:

- reduced and of the form $S(R)$ with R an irreducible finite root system.
- reduced and of the form $S(R)^\vee$ with R an irreducible finite root system.
- non-reduced and of the form $S_1 \cup S_2$ with S_1 and S_2 reduced affine root systems.

The appearing R is one of the types A_n, B_n, C_n, D_n, BC_n, E_6, E_7, E_8, F_4 and G_2. According to the type of R, we say

- $S(R)$ is of type X if R is of type X,
- $S(R)^\vee$ is of type $X^\vee$ if R is of type X,
- a non-reduced system $S_1 \cup S_2$ is of type (X, Y) if S_1 and S_2 are of type X and Y, respectively.

We refer [M03, (1.3.1)–(1.3.18))] for explicit descriptions of these irreducible affine root systems, although some of them will be displayed in the main text.

As explained in [M03, §1.4], Macdonald developed a unified formulation to associate a family of q-orthogonal polynomials to each of the following pairs (S, S') of irreducible affine root systems.

(a) $(S, S') = (S(R), S(R^\vee))$ with R an irreducible finite root system.
(b) $S = S' = S(R)^\vee$ with R an irreducible finite root system.
(c) $S = S'$ is non-reduced of type (X, Y).

For each pair (S, S'), we have the associated non-symmetric [M03, §5.2] and symmetric [M03, §5.3] Macdonald polynomials. For the reference in the main text, let us introduce:

Definition 1.3.1.1. We call the non-symmetric and symmetric Macdonald polynomials associated to (S, S') in the class (a), (b) and (c) *the non-symmetric and symmetric Macdonald polynomials of type* X*,* $X^\vee$ *and* (X, Y), respectively.

1.3.2 Affine root system of type $(C_n^\vee, C_n)$

In this part, we describe the affine root system $(C_n^\vee, C_n)$ by which we can define Koornwinder polynomials from the viewpoint of affine Hecke algebras.

Root systems for type C_n

We consider the n-dimensional real Euclidean space $(V, \langle\cdot,\cdot\rangle)$ with

$$V = \bigoplus_{i=1}^{n} \mathbb{R}\epsilon_i, \quad \langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j}.$$

The set R of roots is given by

$$R := \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid i = 1, \ldots, n\} \subset V. \tag{1.3.1}$$

For each root $\alpha \in R$, we denote the associated coroot by $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle \in V$. The set $R^\vee$ of coroots is given by

$$R^\vee := \{\alpha^\vee \mid \alpha \in R\} = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm\epsilon_i \mid i = 1, \ldots, n\} \subset V. \tag{1.3.2}$$

We use the following choice of the subset $R_+ \subset R$ of positive roots and the subset $R_+^\vee \subset R^\vee$ of positive coroots.

$$R_+ := \{\epsilon_i \pm \epsilon_j \mid i < j\} \cup \{2\epsilon_i \mid i = 1, \ldots, n\}, \quad R_+^\vee := \{\epsilon_i \pm \epsilon_j \mid i < j\} \cup \{\epsilon_i \mid i = 1, \ldots, n\},$$

We have $R = R_+ \sqcup -R_+$ and $R^\vee = R_+^\vee \sqcup -R_+^\vee$. The simple roots $a_i \in R$ $(i = 1, \ldots, n)$ are given by

$$a_1 := \epsilon_1 - \epsilon_2, \ \ldots, \ a_{n-1} := \epsilon_{n-1} - \epsilon_n, \ a_n := 2\epsilon_n.$$

The coroots for simple roots are

$$a_1^\vee = \epsilon_1 - \epsilon_2, \ \ldots, \ a_{n-1}^\vee = \epsilon_{n-1} - \epsilon_n, \ a_n^\vee = \epsilon_n.$$

We call $a_i^\vee$ simple coroots.

For $\alpha \in R$, we write s_α the reflection by the hyperplane $H_\alpha := \{x \in V \mid \langle \alpha^\vee, x \rangle = 0\}$ in V. That is,

$$s_\alpha.x := x - \langle \alpha^\vee, x \rangle \alpha, \quad x \in V.$$

We write $s_i := s_{\alpha_i}$ for $i = 1, \ldots, n$. The finite Weyl group W_0 is defined to be the subgroup of $\mathrm{GL}(V)$ generated by $s_1, \ldots, s_n$. As an abstract group, W_0 is a Coxeter group with generators $s_1, \ldots, s_n$ and relations

$$\begin{aligned} s_i^2 &= 1 \quad (i = 1, \ldots, n), \\ s_i s_j &= s_j s_i \quad (|i - j| > 1), \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad (i = 1, \ldots, n-2), \\ s_{n-1} s_n s_{n-1} s_n &= s_n s_{n-1} s_n s_{n-1}. \end{aligned}$$

Next we introduce notation for weights of the root system of type C_n. For $i = 1, \ldots, n$, we define $\omega_i := \epsilon_1 + \cdots + \epsilon_i \in V$, and call them the fundamental weights. Then we have $\langle a_i^\vee, \omega_j \rangle = \delta_{i,j}$ for $i, j = 1, \ldots, n$. We define the root lattice Q and the weight lattice Λ by

$$Q := \bigoplus_{i=1}^{n} \mathbb{Z} a_i \subset \Lambda := \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i \subset V. \tag{1.3.3}$$

The action of $W_0 \subset \mathrm{GL}(V)$ on V preserves the weight lattice Λ. We denote this action by $\lambda \mapsto w.\lambda$ for $w \in W_0$ and $\lambda \in \Lambda$.

Affine root system of type $(C_n^\vee, C_n)$

Here we introduce the notation for the affine root system of type $(C_n^\vee, C_n)$ in the sense of Macdonald [M03], following Chapter 1 of loc. cit.

Let F be the space of affine real functions on V, which is identified with real vector space $V \oplus \mathbb{R}c$ by the map $(u \mapsto \langle v, u \rangle + r) \mapsto v + rc$ for $u, v \in V$ and $r \in \mathbb{R}$. Using the gradient map

$$\overline{\cdot} : \ F \longrightarrow V, \quad \overline{v + rc} := v, \tag{1.3.4}$$

we extend the inner product $\langle\cdot,\cdot\rangle$ on V to a positive semi-definite bilinear form on F by $\langle f,g\rangle := \langle\overline{f},\overline{g}\rangle$ for $f,g\in F$. We define the affine root system $S = S(C_n^\vee, C_n)$ of type $(C_n^\vee, C_n)$ in the sense of [M03, (1.3.18)] and [Sa00] by

$$S(C_n^\vee, C_n) := \{\pm\epsilon_i + \frac{k}{2}c, \pm 2\epsilon_i + kc \mid k\in\mathbb{Z},\ i=1,\dots,n\} \cup \{\pm\epsilon_i \pm \epsilon_j + kc \mid k\in\mathbb{Z}, 1\le i<j\le n\} \subset F. \tag{1.3.5}$$

We also define the subset $S_+\subset S$ of positive roots by

$$\begin{aligned} S_+ :=& \{\alpha + kc, \alpha^\vee + \frac{k}{2}c \mid \alpha\in R_+, \alpha^\vee\in R_+^\vee, k\in\mathbb{N}\} \\ &\cup \{\alpha + kc, \alpha^\vee + \frac{k}{2}c \mid \alpha\in R_-, \alpha^\vee\in R_-^\vee, k\in\mathbb{N}\setminus\{0\}\}. \end{aligned} \tag{1.3.6}$$

We then have $S = S_+ \sqcup S_-$ with $S_- := -S_+$. We also denote the set $\overline{S} := R\cup R^\vee$, and denote the set

$$\overline{S}_+ := R_+\cup R_+^\vee, \quad \overline{S}_- := -\overline{S}_+. \tag{1.3.7}$$

We denote by $\mathrm{t}(\Lambda) = \{\mathrm{t}(\lambda)\mid\lambda\in\Lambda\}$ the abelian group with relations $\mathrm{t}(\lambda)\,\mathrm{t}(\mu) = \mathrm{t}(\lambda+\mu)$ $(\lambda,\mu\in\Lambda)$. We define the action of $\mathrm{t}(\Lambda)$ on F by

$$\mathrm{t}(\lambda).(\mu+mc) := \mu + (m - \langle\mu,\lambda\rangle)c, \quad \mu+mc\in F.$$

The relation of $w\in W_0$ and $\mathrm{t}(\lambda)\in\mathrm{t}(\Lambda)$ in the group $\mathrm{GL}(F)$ is then given by $w\,\mathrm{t}(\lambda)\,w^{-1} = \mathrm{t}(w.\lambda)$. The subgroup $W\subset\mathrm{GL}_{\mathbb{R}}(F)$ generated by $\mathrm{t}(\Lambda)$ and W_0 is called *the extended affine Weyl group*. That is,

$$W := \mathrm{t}(\Lambda)\rtimes W_0 \subset \mathrm{GL}_{\mathbb{R}}(F). \tag{1.3.8}$$

The action of the element $s := \mathrm{t}(\epsilon_1)s_{2\epsilon_1}\in W$ on Λ is given by $s.\epsilon_1 = c-\epsilon_1$ and $s.\epsilon_i = \epsilon_i$ $(i=2,\dots,n)$, which is the same as the reflection $s_0 := s_{a_0}$ with respect to the hyperplane $H_{a_0} := \{x\in V\mid\langle a_0^\vee, x\rangle = 0\}$ for the affine root $a_0 := c-2\epsilon_1\in S$. For each $a\in S$, we denote the associated affine coroot by $a^\vee := 2a/\langle a,a\rangle\in S$. As an abstract group, W is a Coxeter group with generators $s_0, s_1,\dots,s_n$ and relations

$$\begin{aligned} s_i^2 &= 1 && (i=0,\dots,n), \\ s_is_j &= s_js_i && (|i-j|>1), \\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} && (i=1,\dots,n-2), \\ s_is_{i+1}s_is_{i+1} &= s_{i+1}s_is_{i+1}s_i && (i=0,n-1). \end{aligned}$$

We define the length $\ell(w)$ of an element $w\in W$ to be the length of the reduced expression of w by the generators $s_0,\dots,s_n$. We also denote by $\preccurlyeq_B$ the corresponding *Bruhat order*. The reduced expressions of $\mathrm{t}(\epsilon_i)$ $(i=1,\dots,n)$ are given by

$$\begin{aligned} \mathrm{t}(\epsilon_1) &= s_0s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_2s_1, \\ \mathrm{t}(\epsilon_2) &= s_1s_0s_1\cdots s_ns_{n-1}\cdots s_2, \\ \mathrm{t}(\epsilon_i) &= s_{i-1}\cdots s_0s_1\cdots s_ns_{n-1}\cdots s_i, \\ \mathrm{t}(\epsilon_n) &= s_{n-1}\cdots s_1s_0s_1\cdots s_n. \end{aligned} \tag{1.3.9}$$

1.3.3 Koornwinder polynomials

Affine Hecke algebras of type $(C_n^\vee, C_n)$ and polynomial representations

Recall the affine root system S of type $(C_n^\vee, C_n)$ and the extended affine Weyl group W explained in § 1.3.2. Let $\{t_a\mid a\in S\}$ be parameters satisfying the condition $t_a = t_{a'}$ for $a'\in W.a$. Since the W-orbits in S are given by

$$W.a_i = W.a_i^\vee\ (i=1,\dots,n-1),\ W.a_n,\ W.a_0,\ W.a_n^\vee,\ W.a_0^\vee,$$

we can replace the family $\{t_a\}$ by

$$(t_{a_i}=t_{a_i^\vee},t_{a_n},t_{a_0},t_{a_n^\vee},t_{a_0^\vee})=(t,t_n,t_0,u_n,u_0). \tag{1.3.10}$$

We will also denote $t_1,\dots,t_{n-1}:=t$. Now we set the base field $\mathbb{K}$ as

$$\mathbb{K}:=\mathbb{Q}(q^{\frac12},t^{\frac12},t_0^{\frac12},t_n^{\frac12},u_0^{\frac12},u_n^{\frac12}), \tag{1.3.11}$$

and all the linear spaces, their tensor products, and the algebras will be those over $\mathbb{K}$ unless otherwise stated.

The extended affine Hecke algebra $H=H(q,t_i^{\frac12})$ is the associative algebra generated by $T_0,T_1,\dots,T_n$ subject to the following relations.

$$\begin{aligned}
(T_i-t_i^{\frac12})(T_i+t_i^{-\frac12})&=0 && (i=0,\dots,n),\\
T_iT_j&=T_jT_i && (|i-j|>1,(i,j)\notin\{(n,0),(0,n)\}), && (1.3.12)\\
T_iT_{i+1}T_i&=T_{i+1}T_iT_{i+1} && (i=1,\dots,n-2), && (1.3.13)\\
T_iT_{i+1}T_iT_{i+1}&=T_{i+1}T_iT_{i+1}T_i && (i=0,n-1). && (1.3.14)
\end{aligned}$$

The relations (1.3.12)–(1.3.14) are called *the braid relations.*

Given an element $w\in W$ together with a reduced expression $w=s_{i_1}\cdots s_{i_r}$, we define $Y^w\in H$ by

$$Y^w:=T_{i_1}^{\varepsilon(b_1)}\cdots T_{i_r}^{\varepsilon(b_r)},\quad b_k:=s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(a_{i_k})\quad(k=1,\dots,r), \tag{1.3.15}$$

where the map $\varepsilon:S\to\{\pm1\}$ is defined by

$$\varepsilon(a):=\begin{cases}+1 & (\overline{a}\in\overline{S}_-)\\ -1 & (\overline{a}\in\overline{S}_+)\end{cases},\quad a\in S.$$

The decomposition of Y^w by T_i's is independent of the choice of a reduced expression of w. By the relations of H, we find that the family $\{Y^w\mid w\in W\}$ is mutually commutative [N95, §2].

As explained in [M03, §3], we can calculate $Y^{\mathrm{t}(\epsilon_i)}$ using the reduced expression of $\mathrm{t}(\epsilon_i)$ in (1.3.9). The result is

$$\begin{aligned}
Y^{\mathrm{t}(\epsilon_1)}&=T_0\cdots T_nT_{n-1}\cdots T_1,\\
Y^{\mathrm{t}(\epsilon_2)}&=T_1^{-1}T_0\cdots T_{n-1}T_nT_{n-1}\cdots T_2,\\
Y^{\mathrm{t}(\epsilon_i)}&=T_{i-1}^{-1}\cdots T_1^{-1}T_0\cdots T_{n-1}T_nT_{n-1}\cdots T_i,\\
Y^{\mathrm{t}(\epsilon_n)}&=T_{n-1}^{-1}\cdots T_1^{-1}T_0T_1\cdots T_n.
\end{aligned} \tag{1.3.16}$$

Now we denote by

$$\mathbb{K}[Y^{\pm1}]=\mathbb{K}[Y_1^{\pm1},\dots,Y_n^{\pm1}]\subset H,\quad Y_i:=Y^{\mathrm{t}(\epsilon_i)}\quad(i=1,\dots,n)$$

the ring of Laurent polynomials in $Y_1,\dots,Y_n$. Then we have an isomorphism $H\simeq H_0\otimes\mathbb{K}[Y^{\pm1}]$, where H_0 is the Hecke algebra of the finite Weyl group W_0. The latter is the subalgebra of H generated by $T_1,\dots,T_n$.

Next we review the basic representation of the affine Hecke algebra H introduced by Noumi [N95]. Let $\mathbb{K}(x)=\mathbb{K}(x_1,\dots,x_n)$ be the field of rational functions with n variables. Then the mapping

$$\begin{aligned}
T_i&\longmapsto t_i^{\frac12}+t_i^{-\frac12}\frac{1-t_ix_i/x_{i+1}}{1-x_i/x_{i+1}}(s_i-1)\quad(i=1,\dots,n-1),\\
T_0&\longmapsto t_0^{\frac12}+t_0^{-\frac12}\frac{(1-u_0^{\frac12}t_0^{\frac12}q^{\frac12}x_1^{-1})(1+u_0^{-\frac12}t_0^{\frac12}q^{\frac12}x_1^{-1})}{1-qx_1^{-2}}(s_0-1),\\
T_n&\longmapsto t_n^{\frac12}+t_n^{-\frac12}\frac{(1-u_n^{\frac12}t_n^{\frac12}x_n)(1+u_n^{-\frac12}t_n^{\frac12}x_n)}{1-x_n^2}(s_n-1)
\end{aligned} \tag{1.3.17}$$

defines a ring homomorphism $\rho : H \to \mathrm{End}(\mathbb{K}(x))$. Moreover its image is contained in the endomorphism algebra $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x^{\pm 1}]) \subset \mathrm{End}_{\mathbb{K}}(\mathbb{K}(x))$ of the Laurent polynomials. We call ρ *the basic representation of* H. Hereafter we identify H and its image under ρ, and regard H as a subalgebra of $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x^{\pm 1}])$. The right hand sides of (1.3.17) are q-difference operators called *Dunkl operators of type* $(C_n^\vee, C_n)$.

Let us give a simplified description of (1.3.17). Using

$$u_i := \begin{cases} 1 & (i = 1, \ldots, n-1) \\ u_0 & (i = 0) \\ u_n & (i = n) \end{cases}, \quad x^{a_i} := \begin{cases} x_i/x_{i+1} & (i = 1, \ldots, n-1) \\ q x_1^{-2} & (i = 0) \\ x_n^2 & (i = n) \end{cases},$$

we can rewrite T_i's as

$$T_i = t_i^{\frac{1}{2}} + t_i^{-\frac{1}{2}} \frac{(1 - u_i^{\frac{1}{2}} t_i^{\frac{1}{2}} x^{\frac{a_i}{2}})(1 + u_i^{-\frac{1}{2}} t_i^{\frac{1}{2}} x^{\frac{a_i}{2}})}{1 - x^{\alpha_i}} (s_i - 1), \tag{1.3.18}$$

where we identified the left and right hand sides in (1.3.17) as claimed before. Let us further define the rational functions $c_i(z), d_i(z) \in \mathbb{K}(z)$ by

$$c_i(z) := t_i^{-\frac{1}{2}} \frac{(1 - u_i^{\frac{1}{2}} t_i^{\frac{1}{2}} z^{\frac{1}{2}})(1 + u_i^{-\frac{1}{2}} t_i^{\frac{1}{2}} z^{\frac{1}{2}})}{1 - z}, \quad d_i(z) := t_i^{\frac{1}{2}} - c_i(z) = \frac{(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) + (u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}}) z^{\frac{1}{2}}}{1 - z}. \tag{1.3.19}$$

Then we can rewrite (1.3.17) or (1.3.18) as

$$T_i = t_i^{\frac{1}{2}} + c_i(x^{a_i})(s_i - 1) = t_i^{\frac{1}{2}} s_i + d_i(x^{a_i})(1 - s_i) = c_i(x^{a_i}) s_i + d_i(x^{a_i}). \tag{1.3.20}$$

For later use, we calculate the action of the element Y^a on 1 in the basic representation for an affine root $a = \alpha + kc \in S$ ($\alpha \in \overline{S}$, $k \in \mathbb{Z}$). Let us define

$$q^{\mathrm{sh}(\alpha + kc)} := q^{-k}, \quad t^{\mathrm{ht}(\alpha + kc)} := \prod\nolimits_{\gamma \in R_+^s} t^{\frac{1}{2}\langle \gamma^\vee, \alpha \rangle} \prod\nolimits_{\gamma \in R_+^\ell} (t_0 t_n)^{\frac{1}{2}\langle \gamma^\vee, \alpha \rangle}. \tag{1.3.21}$$

Here $R_+^s := \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$ denotes the set of positive short roots, and $R_+^\ell := \{2\epsilon_i \mid 1 \le i \le n\}$ denotes the set of of positive long roots. Then we can check

$$Y^a 1 = q^{\mathrm{sh}(a)} t^{\mathrm{ht}(a)}. \tag{1.3.22}$$

See also [St00, Proposition 4.5] for a more general formula.

Finally we recall *the Lusztig relations* in the basic representations of affine Hecke algebra. For each weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda$, we define $x^\lambda \in \mathbb{K}[x^{\pm 1}]$ by

$$x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathbb{K}[x^{\pm 1}]. \tag{1.3.23}$$

Fact 1.3.3.1 (Lusztig relations, [L89, Proposition 3.6])**.** For $i = 0, \ldots, n$ and $\lambda \in \Lambda$, we have

$$T_i x^\lambda - x^{s_i . \lambda} T_i = d_i(x^{a_i})(x^\lambda - x^{s_i . \lambda}),$$

where the rational function $d_i(z)$ is defined by (1.3.19).

Double affine Hecke algebras and non-symmetric Koornwinder polynomials

Next we review *the double affine Hecke algebra* $\mathbb{H}$ *of type* $(C_n^\vee, C_n)$ and *the non-symmetric Koornwinder polynomials* $E_\lambda(x)$, following [M03], [Sa99] and [Sa00].

As in the previous part, we regard H as a $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x^{\pm 1}])$ by the basic representation (1.3.17). We define *the double affine Hecke algebra* $\mathbb{H} \subset \mathrm{End}_{\mathbb{K}}(\mathbb{K}[x^{\pm 1}])$ as the $\mathbb{K}$-subalgebra generated by $\mathbb{K}[x^{\pm 1}]$, $H(W_0)$ and $\mathbb{K}[Y^{\pm 1}]$. Thus

$$\mathbb{H} := \langle \mathbb{K}[x^{\pm 1}],\, H_0,\, \mathbb{K}[Y^{\pm 1}] \rangle \subset \mathrm{End}_{\mathbb{K}}(\mathbb{K}[x^{\pm 1}]).$$

As in the case of untwisted affine root systems, the algebra $\mathbb{H}$ has *the Cherednik anti-involution* $* : \mathbb{H} \to \mathbb{H}$ [Sa99, §4]:

$$
\begin{aligned}
&x_i^* = Y_i^{-1}, \quad Y_i^* = x_i^{-1}, \quad T_i^* = T_i \quad (i = 1, \ldots, n), \\
&(t^*, t_n^*, t_0^*, u_n^*, u_0^*) = (t, t_n, u_n, t_0, u_0).
\end{aligned} \tag{1.3.24}
$$

On the element T_0 the anti-involution acts as $T_0^* = T_{s_{2\epsilon_1}}^{-1} x_1^{-1}$. In fact, we have $T_0 = Y_1 T_{s_{2\epsilon_1}}^{-1}$ and $T_{s_{2\epsilon_1}} = T_1 \cdots T_n T_{n-1} \cdots T_1$ by (1.3.16).

Next we introduce *the x- and Y-intertwiners* for $\mathbb{H}$ following [M03, §5.6]. Let $\widetilde{\mathbb{H}}$ be the coefficient extension of $\mathbb{H}$ by rational functions of x's and Y's. In other words, we set

$$
\widetilde{\mathbb{H}} := \big\langle \mathbb{K}(x),\, H_0,\, \mathbb{K}(Y) \big\rangle \subset \operatorname{End}_{\mathbb{K}}(\mathbb{K}(x)). \tag{1.3.25}
$$

Here $\mathbb{K}(x)$ and $\mathbb{K}(Y)$ are the fields of rational functions of x_i and Y_i $(i = 1, \ldots, n)$ respectively. For $i = 0, \ldots, n$, we define $S_i^x \in \widetilde{\mathbb{H}}$ by

$$
S_i^x := T_i + \varphi_i^+(x^{a_i} = T_i^{-1} + \varphi_i^-(x^{a_i}), \tag{1.3.26}
$$

where

$$
\varphi_i^{\pm}(z) := \mp \frac{(t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(u_i^{\frac{1}{2}} - u_i^{-\frac{1}{2}})}{1 - z^{\pm 1}} \in \mathbb{K}(z). \tag{1.3.27}
$$

We call S_i^x *the x-intertwiners.*

Let us explain some basic properties of x-intertwiners. Recalling the rational function $d_i(z)$ in (1.3.19) and the expression of T_i in (1.3.20), we have

$$
\varphi_i^+(z) = d_i(z), \quad S_i^x = T_i - d_i(x^{a_i}) = c_i(x^{a_i}) s_i. \tag{1.3.28}
$$

For each weight $\lambda \in \Lambda$, we have

$$
S_i^x x^{\lambda} = x^{s_i(\lambda)} S_i^x \tag{1.3.29}
$$

by the Lusztig relations (Fact 1.3.3.1). Moreover, by [M03, (5.5.2)], the x-intertwiners S_i^x $(i = 0, \ldots, n)$ satisfy the same braid relations as (1.3.12)–(1.3.14):

$$
\begin{aligned}
S_i^x S_j^x &= S_j^x S_i^x && (|i - j| > 1), \\
S_i^x S_{i+1}^x S_i^x &= S_{i+1}^x S_i^x S_{i+1}^x && (i = 1, \ldots, n-2), \\
S_i^x S_{i+1}^x S_i^x S_{i+1}^x &= S_{i+1}^x S_i^x S_{i+1}^x S_i^x && (i = 0, n-1).
\end{aligned} \tag{1.3.30}
$$

Given an element $w \in W$, choose a reduced expression $w = s_{i_1} \cdots s_{i_p}$, and set

$$
S_w^x := S_{i_1}^x \cdots S_{i_p}^x \in \widetilde{\mathbb{H}}. \tag{1.3.31}
$$

By the braid relations, S_w^x is independent of the choice of a reduced expression of w.

Next we introduce *Y-intertwiners*. First, note that the anti-involution $*$ can be extended to $\widetilde{\mathbb{H}}$. In fact, $\widetilde{\mathbb{H}}$ is the Ore localization of the non-commutative algebra $\mathbb{H}$ by the commutative subalgebras $\mathbb{K}[x^{\pm 1}]$ and $\mathbb{K}[Y^{\pm 1}]$, and $*$ is an isomorphism on these commutative subalgebras. We denote the extension of $*$ to $\widetilde{\mathbb{H}}$ by same symbol $*$. Now we define the Y-intertwiners $S_i^Y \in \widetilde{\mathbb{H}}$ by

$$
\begin{aligned}
S_i^Y &:= (S_i^x)^* = T_i + \psi_i^+(Y^{-a_i}) = T_i^{-1} + \psi_i^-(Y^{-a_i}) \quad (i = 1, \ldots, n), \\
S_0^Y &:= (S_0^x)^* = T_0^* + \psi_0^+(qY_1^2) = (T_0^*)^{-1} + \psi_0^-(qY_1^2),
\end{aligned} \tag{1.3.32}
$$

where the symbols $\psi_i^{\pm}(z)$ denote the following functions:

$$
\begin{aligned}
\psi_i^{\pm}(z) &:= \varphi_i^{\pm 1}(z) = \mp \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 1, \ldots, n-1), \\
\psi_0^{\pm}(z) &:= \mp \frac{(u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}, \\
\psi_n^{\pm}(z) &:= \mp \frac{(t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}.
\end{aligned} \tag{1.3.33}
$$

Note that we have $\psi_i^{\pm}(Y^{-a_i}) = \left(\varphi_i^{\pm}(x^{a_i})\right)^*$, where $*$ is the extended anti-involution.

We can deduce properties of S_i^Y's from those of S_i^x's. For example, applying the anti-involution $*$ to the relation (1.3.29), we have

$$S_i^Y Y^{\lambda} = Y^{s_i\lambda} S_i^Y \tag{1.3.34}$$

for each $i = 0, \dots, n$ and $\lambda \in \Lambda$. We can also see that S_i^Y's satisfy the same braid relations as (1.3.30). For an element $w \in W$, we can define $S_w^Y \in \widetilde{\mathbb{H}}$ by choosing a reduced expression $w = s_{i_1} \cdots s_{i_p}$ and

$$S_w^Y := S_{i_1}^Y \cdots S_{i_p}^Y \in \widetilde{\mathbb{H}}. \tag{1.3.35}$$

It is well-defined by the braid relations of S_i^Y's.

Finally we explain the non-symmetric Koornwinder polynomials. For each weight $\mu \in \Lambda$, we regard $\mathrm{t}(\mu)W_0 \subset W$ by the decomposition $W = \mathrm{t}(P) \rtimes W_0$ in (1.3.8). Then we define $w(\mu) \in W$ by the following description:

$$w(\mu) \text{ is the shortest element among } \mathrm{t}(\mu)W_0 \subset W. \tag{1.3.36}$$

In the case $\mu = \epsilon_i$, $i = 1, \dots, n$, the element $w(\epsilon_i)$ is given by

$$w(\epsilon_i) = s_{i-1} \cdots s_0. \tag{1.3.37}$$

Now we have:

Fact 1.3.3.2 ([Sa99, §6], [St00, Theorem 4.8])**.** For $\mu \in \Lambda$, the element

$$E_\mu(x) := S_{w(\mu)}^Y 1 \in \mathbb{K}(x)$$

belongs to $\mathbb{K}[x^{\pm 1}]$. We call it *the non-symmetric Koornwinder polynomial associated to* μ.

By (1.3.34), $E_\mu(x)$ is a simultaneous eigenfunction of the family $\left\{Y^\lambda \mid \lambda \in \Lambda\right\}$ of Dunkl operators. Note that our normalization of $E_\mu(x)$ is different from that in [Sa99, St00]. In loc. cit., the coefficient of x^μ is normalized to be 1.

Koornwinder polynomials

Now we introduce *Koornwinder polynomials* by symmetrizing non-symmetric Koornwinder polynomials.

First, we define the set $\Lambda_+ \subset \Lambda$ of dominant weights by

$$\Lambda_+ := \{\mu \in \Lambda \mid \langle a_i^\vee, \mu\rangle \geq 0,\ i = 1, \dots, n\}.$$

For a weight $\mu \in \Lambda$, we denote the stabilizer of μ in the finite Weyl group W_0 by

$$W_\mu := \{w \in W_0 \mid w.\mu = \mu\} \subset W_0, \tag{1.3.38}$$

and denote the longest element among W_μ by

$$w_\mu \in W_\mu. \tag{1.3.39}$$

Next, using the notations in § 1.3.2 and § 1.3.3, we define $t_w \in \mathbb{K}$ for each $w \in W$ by

$$t_w := \prod_{a \in \mathcal{L}(w)} t_a \in \mathbb{K}. \tag{1.3.40}$$

Here $\{t_a \mid a \in S\}$ is the W-invariant family of parameters (1.3.10), $\mathbb{K}$ is the base field (1.3.11), and $\mathcal{L}(w) \subset S$ is given by (2.1.2). If $w = s_{i_1} \cdots s_{i_r} \in W$ is a reduced expression, then we have $t_w = t_{i_1} \cdots t_{i_r}$. For a dominant weight $\mu \in \Lambda_+$, we define *the Poincaré polynomial* $W_\mu(t) \in \mathbb{K}$ of the stabilizer W_μ by

$$W_\mu(t) := \sum_{u \in W_\mu} t_u. \tag{1.3.41}$$

Lemma 1.3.3.3. For each element $\mu \in \Lambda_+$, we have

$$\sum_{u \in W_\mu} \left(\prod_{a \in \mathcal{L}(e,u)} t_a^{\frac{1}{2}} \frac{1 - t^{\mathrm{ht}(-a)} t_a^{-1}}{1 - t^{\mathrm{ht}(-a)}} \right) \left(\prod_{a \in \mathcal{L}(u, w_\mu)} t_a^{-\frac{1}{2}} \frac{1 - t^{\mathrm{ht}(-a)} t_a}{1 - t^{\mathrm{ht}(-a)}} \right) = t_{w_\mu}^{-\frac{1}{2}} W_\mu(t).$$

For a proof, see [Yi12, Lemma 3.4].

Next we define the *symmetrizer* U by

$$U := \sum_{w \in W_0} t_{w_0 w}^{-\frac{1}{2}} T_w. \tag{1.3.42}$$

By [M03, (5.5.9)], we then have

$$UT_i = U t_i^{\frac{1}{2}}, \quad T_i U = t_i^{\frac{1}{2}} U \quad (i = 1, \ldots, n). \tag{1.3.43}$$

Hereafter we denote the ring of W_0-invariant Laurent polynomials by

$$\mathbb{K}[x^{\pm 1}]^{W_0} := \left\{ f \in \mathbb{K}[x^{\pm 1}] \mid w.f = f,\, w \in W_0 \right\}.$$

Here W_0 acts on x^λ (1.3.23) by the action on the weight λ. Also recall that for each $\mu \in \Lambda_+ \subset \Lambda$ we defined $w(\mu) \in \mathrm{t}(\mu) W_0 \subset W$ by (1.3.36).

Fact 1.3.3.4 ([St00, Theorem 6.6])**.** For each dominant weight $\lambda \in \Lambda_+$, the element

$$P_\lambda(x) := \frac{1}{t_{w_\mu}^{-\frac{1}{2}} W_\lambda(t)} U S_{w(\lambda)}^Y 1 = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\mu(t)} U E_\lambda(x) \in \mathbb{K}(x)$$

belongs to $\mathbb{K}[x^{\pm 1}]^{W_0}$. We call $P_\lambda(x)$ *the (monic) Koornwinder polynomial associated to* λ.

Note that the coefficient of x^λ in $P_\lambda(x)$ is 1 since the coefficient of the top term x^λ in $U S_{w(\lambda)}^Y 1$ is $t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)$. To emphasize the root system $(C_n^\vee, C_n)$, we call $P_\lambda(x)$ *the Koornwinder polynomial of rank* n or *of type* $(C_n^\vee, C_n)$.

Chapter 2

Littlewood-Richardson coefficients

Chapter 2 is based on the author's publication [Ya22].

2.0 Introduction

As explained in Preface, Abstract of Chapter 2, the theme of this chapter is the Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$ of Koornwinder polynomials P_λ, that is the structure constants of the product in the invariant Laurent polynomial ring $\mathbb{K}[x^{\pm 1}]^{W_0}$ of the finite Weyl group W_0 of type C_n:

$$P_\lambda P_\mu = \sum_{\nu} c_{\lambda,\mu}^{\nu} P_\nu.$$

Hereafter we call $c_{\lambda,\mu}^{\nu}$ *LR coefficients* for simplicity.

The main result of this chapter is Theorem 2.2.4.2, which is a natural $(C_n^\vee, C_n)$-type analogue of Yip's alcove walk formulas for LR coefficients in [Yi12, Theorem 4.4]. Let us prepare the necessary notations and terminology for the explanation. See also the explanation of Theorem A in Preface.

Let us explain the outline of proof of Theorem A. We denote by $E_\mu(x) \in \mathbb{K}[X^{\pm 1}]$ *the non-symmetric Koornwinder polynomials* [Sa99, St00], which was introduced in § 1.3.2. We need the following two properties.

- $\{E_\mu(x) \mid \mu \in \Lambda\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x^{\pm 1}]$.
- $P_\mu(x)$ is obtained by symmetrizing $E_\mu(x)$ (Fact 1.3.3.2). More precisely, using the symmetrizer U in (1.3.42), we have
$$P_\mu(x) = \frac{1}{t_{w_\mu}^{-\frac{1}{2}} W_\mu(t)} U E_\mu(x).$$

The outline of proof is a straight $(C_n^\vee, C_n)$-type analogue of Yip's derivation in [Yi12]. The argument can be divided into four steps, and below we explain them abbreviating some coefficients and ranges of summations.

(i) For dominant weights $\lambda, \mu \in \Lambda_+$, we derive an expansion formula
$$x^\mu E_\lambda(x) = \sum_{p \in \Gamma^C} c_p E_{\varpi(p)}(x)$$
of the product of the non-symmetric Koornwinder polynomial $E_\lambda(x)$ and the monomial x^μ (Corollary 2.2.1.5). Here the index set Γ^C consists of alcove walks belonging to the dominant chamber C. The symbol $\varpi(p) \in \Lambda_+$ will be given in (2.2.7).

(ii) We use *Ram-Yip type formula* (Fact 2.2.3.1), an expansion formula for the non-symmetric Koornwinder polynomials in terms of monomials:
$$E_\mu(x) = \sum_{p \in \Gamma} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)}.$$
This formula was derived by Orr and Shimozono [OS18], based on the work of Ram and Yip [RY11] on the same type formula for the untwisted affine root systems.

(iii) Using (i) and (ii), we can calculate the product of the non-symmetric Koornwinder polynomial $E_\mu(x)$ and the Koornwinder polynomial $P_\lambda(x)$ in an extension $\widetilde{\mathbb{H}}$ of the double affine Hecke algebra $\mathbb{H}$, and express it as a sum over alcove walks (2.2.17). Then we can rewrite it as a sum over colored alcove walks and have (Proposition 2.2.3.2):

$$E_\mu(x)P_\lambda(x) = \sum_{v\in W^\lambda}\sum_{p\in\Gamma_2^C} A_p C_p E_{\varpi(p)}(x).$$

(iv) Theorem A is obtained by symmetrizing $E_\mu(x)$ in (iii) and switching $\lambda \leftrightarrow \mu$.

Organization

In § 2.1, we introduce alcove walks for type $(C_n^\vee, C_n)$, slightly modifying the original alcove walks introduced by Ram and Yip [RY11]. In § 2.2, we derive our main Theorem 2.2.4.2. previously explained, and the organization of § 2.2 follows that. In § 2.3, we derive several corollaries of the main Theorem 2.2.4.2. In § 2.3.1, we discuss the case of rank $n = 1$, that is the case of Askey-Wilson polynomials. In particular, we give a simplified formula for the Pieri coefficient (Proposition 2.3.1.3), and recover the recurrence formula of Askey-Wilson polynomials in [AW85] from our Pieri formula (Remark 2.3.1.4). In § 2.3.2, we discuss the Hall-Littlewood limit $q \to 0$, and show that LR coefficients are somewhat simplified (Proposition 2.3.2.1). In § 2.3.3 we display examples of LR coefficients in the case of rank $n = 2$.

Notation and terminology

In Table 2.0.1 below, we collect several symbols concerning Weyl groups which might be confusing.

R	(1.3.1)	the set of roots in the finite root system of type C_n.
$\Lambda := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$	(1.3.3)	the weight lattice of type C_n.
$W = \mathrm{t}(\Lambda) \rtimes W_0$	(1.3.8)	the extended affine Weyl group of type C_n.
$W_\mu \subset W_0$	(1.3.38)	the stabilizer of weight $\mu \in \Lambda$ in the finite Weyl group W_0.
W^μ	(2.2.11)	the distinguished complete system of representatives of W_0/W_μ.
$W_\mu(t)$	(1.3.41)	the Poincare polynomial of W_μ.
w_μ	(1.3.39)	the longest element of the stabilizer W_μ.
v_μ	(2.2.12)	the longest element of W^μ.
$w(\mu)$	(1.3.36)	the shortest element among $\mathrm{t}(\mu)W_0 \subset W$.

Table 2.0.1: Symbols concerning Weyl groups

2.1 Alcove walks

Alcove walks are introduced by Ram [Ra06] as analogue of Littelmann paths for affine Hecke algebras. They are valuable combinatorial objects, and used in Ram-Yip type formula [RY11, OS18] for non-symmetric Macdonald-Koornwinder polynomials, and in Yip's formula [Yi12] for Littlewood-Richardson rules of Macdonald polynomials in the untwisted affine root systems. In this part we introduce the notation of alcove walks which will be used throughout in the text. Basically we follow the notations in [Yi12, §2.2], but make slight modifications.

Let us regard an affine root $a = \alpha + kc \in S$ ($\alpha \in \overline{S}$, $k \in \frac{1}{2}\mathbb{Z}$) as an affine linear function on V by

$$a(v) = \langle \alpha, v\rangle + k \quad (v \in V).$$

An alcove is defined to be a connected component of the complement $V \setminus \bigcup_{a\in S} H_a$ of the hyperplanes $H_a := \{x \in V \mid a(x) = 0\}$. The fundamental alcove A is the alcove given by

$$A := \{x \in V \mid a_i(x) > 0 \ (i = 0, \dots, n)\}. \tag{2.1.1}$$

Its boundary consists of the hyperplanes $H_{a_0}, H_{a_1}, \dots, H_{a_n}$. Note that the mapping

$$W \ni w \longmapsto wA \in \pi_0(V \setminus \textstyle\bigcup_{a\in S} H_a)$$

is a bijection. An alcove wA is surrounded by $n+1$ hyperplanes, say H_{γ_i} ($\gamma_i \in S_+$; $i = 0, \dots, n$). We call the intersection $H_{\gamma_i} \cap \overline{wA}$ an *edge* of the alcove wA, where $\overline{wA}$ denotes the closure with respect to the Euclidean topology. Note that each hyperplane H_{γ_i} separates wA and another alcove vA, which can be written as $v = ws_j$ for some $j = 0, \dots, n$. Then the edge $H_{\gamma_i} \cap \overline{wA}$ is just the intersection $\overline{wA} \cap \overline{ws_j A}$, and has two sides, which we call *the wA-side* and *the ws_jA-side*.

Given an alcove wA, we give a sign $\pm$ to each of the two sides on an edge of wA. Let H_{γ_i} ($i = 0, \dots, n$) be the hyperplanes surrounding wA. By renaming the indices i if necessary, we can assume that the hyperplane H_{γ_i} separates wA and ws_iA. Then using the projection $\gamma_i \mapsto \overline{\gamma_i}$ in (1.3.4) and the symbols $\overline{S}_\pm$ in (1.3.7), we set the signs by the following rule.

- If $\overline{\gamma_i} \in \overline{S}_+$, then the wA-side of $H_{\gamma_i} \cap \overline{wA}$ is assigned by $+$ and the ws_iA-side is by $-$.
- If $\overline{\gamma_i} \in \overline{S}_-$, then wA-side is assigned by $-$ and the ws_iA-side is by $+$.

See Figure 2.1.1 for the assignment in the rank 2 case.

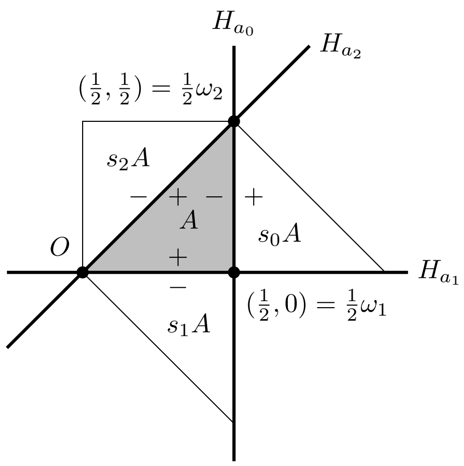


Figure 2.1.1: Signs for the edges of the fundamental alcove A in the rank 2 case

Given an element $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we define a subset $\mathcal{L}(w) \subset S$ by

$$\mathcal{L}(w) := \left\{a_{i_1},\, s_{i_1}a_{i_2},\, \dots,\, s_{i_1} \cdots s_{i_{r-1}}a_{i_r}\right\}. \tag{2.1.2}$$

The set $\{H_a \mid a \in \mathcal{L}(w)\}$ consists of the hyperplanes separating A and wA. Given elements $v, w \in W$ and their reduced expressions, we also set

$$\mathcal{L}(v, w) := (\mathcal{L}(v) \cup \mathcal{L}(w)) \setminus (\mathcal{L}(v) \cap \mathcal{L}(w)). \tag{2.1.3}$$

The set $\{H_a \mid a \in \mathcal{L}(v, w)\}$ consists of the hyperplanes separating vA and wA. If $v \preccurlyeq_B w$, where $\preccurlyeq_B$ is the Bruhat order explained at the line before (1.3.9), then we have

$$\mathcal{L}(v, w) = v.\mathcal{L}(e, v^{-1}w) = v.\mathcal{L}(v^{-1}w). \tag{2.1.4}$$

Let us again given $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Then the mapping

$$\{0,1\}^r \ni (b_1, \dots, b_r) \longmapsto s_{i_1}^{b_1} \cdots s_{i_r}^{b_r} \in \{v \in W \mid v \preccurlyeq_B w\}$$

is a surjection. Let us given extra $v, w \in W$ such that $v \preccurlyeq_B w$. We can write $v = s_{i_1}^{b_1} \cdots s_{i_r}^{b_r}$ with $b = (b_1, \dots, b_r) \in \{0,1\}^r$. We then consider the following sequence p of alcoves.

$$p = \left(p_0 := zA,\ p_1 := zs_{i_1}^{b_1}A,\ p_2 := zs_{i_1}^{b_1}s_{i_2}^{b_2}A,\ \dots,\ p_r := zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A\right).$$

The sequence p is called an *alcove walk of type* $\overrightarrow{w} = (i_1, \dots, i_r)$ *beginning at* zA, and we denote by $\Gamma(\overrightarrow{w}, z)$ the set of alcove walks of this kind. The symbol $\overrightarrow{w}$ emphasizes that we choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$.

Example 2.1.0.1 (Alcove walks in the rank 2 case)**.** For $w = s_1s_2s_1s_0$ and $z = e \in W$, the two alcove walks

$$p_1 := (A, A, s_2A, s_2s_1A, s_2s_1s_0A),\ p_2 := (A, s_1A, s_1s_2A, s_1s_2s_1A, s_1s_2s_1s_0A) \in \Gamma(\overrightarrow{w}, z)$$

are shown in Figure 2.1.2, where the gray region is the fundamental alcove A, and the number $i = 0, 1, 2$ on a hyperplane means that it belongs to the W-orbit of H_{a_i}.

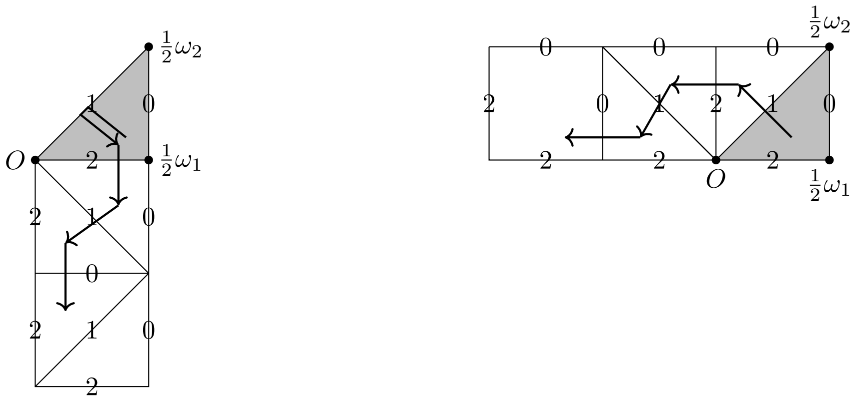


Figure 2.1.2: Alcove walks p_1 and p_2

For an alcove walk $p \in \Gamma(\overrightarrow{w}, z)$ and $k = 1, \dots, r$, the transition $p_{k-1} \to p_k$ is called *the k-th step of p*. The k-th step is called a *crossing* if $b_k = 1$, and called a *folding* if $b_k = 0$. The correspondence between the bit b_k and the k-th step is shown in Table 2.1.1, where we denote by $v_{k-1} \in W$ the element such that $p_{k-1} = v_{k-1}A$.

b_k	1	0
	crossing	folding
	p_{k-1} $\longrightarrow$ p_k	$p_{k-1} = p_k$ $\quad$ $v_{k-1}s_{i_k}A$

Table 2.1.1: Correspondence between bits and steps

Let us again given $z, w \in W$ with a reduced expression $w = s_{i_1} \cdots s_{i_r}$. For an alcove walk $p = (zA, \dots, zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A) \in \Gamma(\overrightarrow{w}, z)$, we define $e(p) \in W$ by

$$e(p) := zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}. \tag{2.1.5}$$

Thus $e(p)$ corresponds to the end of p. We also define $h_k(p) \in S$ for $k = 1, \dots, r$ so that the chosen root $h_k(p)$ is positive. Denote $v := s_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}}$ for simplicity, so that we have $p_{k-1} = vA$. Then we define

$$h_k(p) := \text{the affine root such that the corresponding hyperplane } H_{h_k(p)} \text{ separates } vA \text{ and } vs_{i_k}A. \tag{2.1.6}$$

Furthermore, we call the k-th step of $p \in \Gamma(\overrightarrow{w}, z)$ an *ascent* if $zs_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} \precneqq_B zs_{i_1}^{b_1} \cdots s_{i_k}^{b_k}$, and call it a *descent* if $zs_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} \succneqq_B zs_{i_1}^{b_1} \cdots s_{i_k}^{b_k}$. We denote the set of descent steps of p by

$$\mathrm{des}(p) := \{k = 1, \dots, r \mid \text{the } k\text{-th step is a descent}\}. \tag{2.1.7}$$

Recalling the sign on an edge of an alcove (see Figure 2.1.1 for an example), we can classify each step of an alcove walk p into four types as in Table 2.1.2, where we used the symbol $v_{k-1} \in W$ such that $p_{k-1} = v_{k-1}A$.

Using this classification, we define $\varphi_\pm(p) \subset \{1, \dots, r\}$ by

$$\begin{aligned} \varphi_+(p) &:= \{k \mid \text{the } k\text{-th step of } p \text{ is a positive folding}\}, \\ \varphi_-(p) &:= \{k \mid \text{the } k\text{-th step of } p \text{ is a negative folding}\}, \end{aligned} \tag{2.1.8}$$

and define $\xi_{\mathrm{des}}(p) \subset \{1, \dots, r\}$ by

$$\xi_{\mathrm{des}}(p) := \{k \mid \text{the } k\text{-th step of } p \text{ is a crossing and } k \in \mathrm{des}(p)\}. \tag{2.1.9}$$

Note that we fix a reduced expression $w = s_{i_1} \cdots s_{i_r}$ in the definitions of $\varphi_\pm(p)$ and $\xi_{\mathrm{des}}(p)$.

positive crossing	negative crossing	positive folding	negative folding
$- \mid +$	$+ \mid -$	$+ \mid -$	$- \mid +$
$p_{k-1} \longrightarrow p_k$	$p_{k-1} \longrightarrow p_k$	$p_{k-1} = p_k \Longleftarrow \mid \ v_{k-1}s_{i_k}A$	$p_{k-1} = p_k \Longleftarrow \mid \ v_{k-1}s_{i_k}A$

Table 2.1.2: Classification of steps in alcove walks

2.2 Littlewood-Richardson coefficients

Yip [Yi12, Theorem 4.4] derived a combinatorial explicit formula of LR coefficients for Macdonald polynomials $P_\lambda(x)$ in the case of untwisted affine root systems. In this section, we derive a $(C_n^\vee, C_n)$-analogue of Yip's formula. The outline of the derivation is quite similar to Yip's proof [Yi12, §§3.1–4.1], but we need non-trivial adjustments in each step.

2.2.1 Products of non-symmetric Koornwinder polynomials and monomials

In [Yi12, Theorem 3.3], Yip derived an expansion formula for the product of the monomial x^μ and the non-symmetric Macdonald polynomial $E_\lambda(x)$ in the case of untwisted affine root systems. In this subsection, we give its $(C_n^\vee, C_n)$-type analogue (Corollary 2.2.1.5).

We will use the notations in § 1.3.3. In particular, $\widetilde{\mathbb{H}}$ is the extension (1.3.25) of the double affine Hecke algebra $\mathbb{H}$ of type $(C_n^\vee, C_n)$, $S_i^Y \in \widetilde{\mathbb{H}}$ is the Y-intertwiner (1.3.32), and S_w^Y for $w \in W$ is the product of S_i^Y's (1.3.35). We also denote the Bruhat order in W by $\preccurlyeq_B$.

As a preparation of Proposition 2.2.1.3, we derive a product formula of the Y-intertwiners.

Proposition 2.2.1.1. For $w \in W$ and $i = 0, \dots, n$, we have the following relations in $\widetilde{\mathbb{H}}$.
(i) If $w \preccurlyeq_B s_i w$, then $S_i^Y S_w^Y = S_{s_i w}^Y$.
(ii) If $w \succcurlyeq_B s_i w$, then $S_i^Y S_w^Y = n_i(Y^{-a_i}) S_{s_i w}^Y$, where

$$\begin{aligned} n_0(Y^a) &:= \frac{(1 - u_n^{\frac12} u_0^{\frac12} Y^{\frac a2})(1 + u_n^{\frac12} u_0^{-\frac12} Y^{\frac a2})}{1 - Y^a} \frac{(1 + u_n^{-\frac12} u_0^{\frac12} Y^{\frac a2})(1 - u_n^{-\frac12} u_0^{-\frac12} Y^{\frac a2})}{1 - Y^a} && (a \in W.a_0), \\ n_i(Y^a) &:= \frac{1 - tY^a}{1 - Y^a} \frac{1 - t^{-1} Y^a}{1 - Y^a} && (a \in W.a_i,\ 0 < i < n), \\ n_n(Y^a) &:= \frac{(1 - t_n^{\frac12} t_0^{\frac12} Y^{\frac a2})(1 + t_n^{\frac12} t_0^{-\frac12} Y^{\frac a2})}{1 - Y^a} \frac{(1 + t_n^{-\frac12} t_0^{\frac12} Y^{\frac a2})(1 - t_n^{-\frac12} t_0^{-\frac12} Y^{\frac a2})}{1 - Y^a} && (a \in W.a_n). \end{aligned}$$

Proof. Fix $w \in W$ and choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$. By the definitions (1.3.35), (1.3.32) and the equation (1.3.28), we have

$$\begin{aligned} S_w^Y &= S_{i_1}^Y \cdots S_{i_r}^Y = (T_{i_1}^* + \psi_{i_1}^+(Y^{-a_{i_1}})) \cdots (T_{i_r}^* + \psi_{i_r}^+(Y^{-a_{i_r}})) \\ &= c_{i_1}^*(Y^{-a_{i_1}}) s_{i_1} \cdots c_{i_r}^*(Y^{-a_{i_r}}) s_{i_r} = c_{i_1}^*(Y^{-b_1}) \cdots c_{i_r}^*(Y^{-b_r}) w. \end{aligned}$$

Here we set $b_k := s_{i_1} \cdots s_{i_{k-1}}(a_{i_k})$ $(k = 1, \ldots, r)$ and

$$c_i^*(z) := \begin{cases} u_n^{-\frac{1}{2}} \dfrac{(1 - u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} & (i = 0), \\ t^{-\frac{1}{2}} \dfrac{1 - tz}{1 - z} & (0 < i < n), \\ t_n^{-\frac{1}{2}} \dfrac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} & (i = n). \end{cases}$$

Since $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression, we have $b_k \in S_+$ for $k = 1, \ldots, r$, where $S_+ \subset S$ denotes the set of positive affine roots (1.3.6). The product S_i^Y $(i = 0, \ldots, n)$ and S_w^Y is now calculated as

$$S_i^Y S_w^Y = c_i^*(Y^{-a_i}) s_i c_{i_1}^*(Y^{-b_1}) \cdots c_{i_r}^*(Y^{-b_r}) w. \tag{2.2.1}$$

If $\ell(s_i w) = \ell(w) + 1$, then the equation (2.2.1) becomes $S_i^Y S_w^Y = S_{s_i w}^Y$. If $\ell(s_i w) = \ell(w) - 1$, then there exists $k \in \{1, \ldots, r\}$ such that $s_i(b_{k-1}) \in S_+$ and $s_i(b_k) \in S_-$. Since we have $b_k = a_i$, the equation (2.2.1) becomes

$$\begin{aligned} S_i^Y S_w^Y &= c_i^*(Y^{-a_i}) s_i c_{i_1}^*(Y^{-b_1}) \cdots c_{i_r}^*(Y^{-b_r}) w \\ &= c_i^*(Y^{-a_i}) c_{i_1}^*(Y^{-s_i(b_1)}) \cdots c_{i_{k-1}}^*(Y^{-s_i(b_{k-1})}) s_i c_{i_k}^*(Y^{-a_i}) \cdots c_{i_r}^*(Y^{-b_r}) w \\ &= c_i^*(Y^{-a_i}) c_i^*(Y^{a_i}) c_{i_1}(Y^{-s_i(b_1)}) \cdots \widehat{c_{i_k}^*(Y^{a_i})} \cdots c_{i_r}^*(Y^{-s_i(b_r)}) s_i w = c_i^*(Y^{-a_i}) c_i^*(Y^{a_i}) S_{s_i w}^Y. \end{aligned}$$

Here the symbol $\widehat{\ }$ denotes skipping the term. Then the consequence follows from the equality $c_i^*(Y^{-a_i}) c_i^*(Y^{a_i}) = n_i(Y^{-a_i})$, which can be checked by a direct calculation. □

The same discussion shows the following statement.

Corollary 2.2.1.2. For $w \in W$ and $i = 0, \ldots, n$. we have the following relations in $\widetilde{\mathbb{H}}$
(i) If $w \preccurlyeq_B ws_i$, then $S_w^Y S_i^Y = S_{ws_i}^Y$.
(ii) If $w \succcurlyeq_B ws_i$, then $S_w^Y S_i^Y = S_{ws_i}^Y n_i(Y^{-a_i})$, where $n_i(Y^{-a_i})$ is given in Proposition 2.2.1.1.

Next we recall the notations on alcove walks in § 2.1. Given $z, w \in W$ together with a reduced expression $z = s_{i_r} \cdots s_{i_1}$, we defined the set $\Gamma(\overrightarrow{z}, w)$ of alcove walks of type $\overrightarrow{z} = (i_r, \ldots, i_1)$ beginning at wA. For an alcove walk $p = (p_0, \ldots, p_r) \in \Gamma(\overrightarrow{z}, w)$, the k-th step means the the transition from p_{k-1} to p_k, which is classified into the four types in Table 2.1.2.

Now we define $x^z \in \mathbb{H}$ for $z \in W$ with a chosen reduced expression $z = s_{i_r} \cdots s_{i_1}$. Let q be the alcove walk given by

$$q := (zA,\, zs_{i_1}A,\, zs_{i_1}s_{i_2}A,\, \ldots,\, zs_{i_1} \cdots s_{i_r}A = A) \in \Gamma(\overrightarrow{z}^{-1}, z).$$

Here $\overrightarrow{z}^{-1} := \overrightarrow{z^{-1}} = (i_1, \ldots, i_r)$. Then we define x^z by

$$x^z := (T_{i_r}^*)^{\epsilon_r} \cdots (T_{i_1}^*)^{\epsilon_1}, \tag{2.2.2}$$

where $T_i^* \in \mathbb{H}$ as in (1.3.24), and we set $\epsilon_k := 1$ if the k-th step is a positive crossing, and $\epsilon_k := -1$ if the k-th step is a negative crossing according to the classification in Table 2.1.2.

Proposition 2.2.1.3. Given $z, w \in W$ with a chosen reduced expression $z = s_{i_r} \cdots s_{i_1}$, we have

$$x^z S_w^Y = \sum_{p \in \Gamma(\overrightarrow{z}^{-1}, w^{-1})} S_{e(p)^{-1}}^Y g_p(Y) n_p(Y)$$

in $\widetilde{\mathbb{H}}$, where $e(p) \in W$ is the element (2.1.5), and the terms $g_p(Y)$ and $n_p(Y)$ are given by

$$\begin{aligned} g_p(Y) &:= \prod_{k \in \varphi_-(p)} (-\psi_{i_k}^-(Y^{-h_k(p)})) \prod_{k \in \varphi_+(p)} (-\psi_{i_k}^+(Y^{-h_k(p)})), \\ n_p(Y) &:= \prod_{k \in \xi_{\mathrm{des}}(p)} n_{i_k}(Y^{-h_k(p)}). \end{aligned}$$

Here $h_k(p)$ is given by (2.1.6), $\varphi_+(p)$ and $\varphi_-(p)$ are by (2.1.8), $\xi_{\mathrm{des}}(p)$ is by (2.1.9), $\psi_i^\pm(z) = (\varphi_i^\pm(z))^*$ is by (1.3.33), and $n_i(z)$ is given in Proposition 2.2.1.1.

Proof. We show the statement by induction on the length of $z \in W$. If $\ell(z) = 0$, that is $z = e$, then the right hand side consists only of the term for $p = (p_0 = wA)$, so that it is equal to S^Y_w, and we have the relation.

Next we assume $z \neq e$ and that the result holds for any element $w \in W$ such that $\ell(w) < \ell(z)$.

Fix a reduced expression of z, and write it as $z = s_i\zeta$, $\zeta = s_{i_r} \cdots s_{i_1}$. By the hypothesis, we can write

$$x^z S^Y_w = (T^*_i)^\epsilon x^\zeta S^Y_w = \sum_{p \in \Gamma(\overrightarrow{\zeta}^{-1}, w^{-1})} (T^*_i)^\epsilon S^Y_{e(p)^{-1}} g_p(Y) n_p(Y). \tag{2.2.3}$$

Here $\epsilon \in \{\pm 1\}$ is the sign determined by z. Let us calculate the rightmost side. Take an element

$$p = (w^{-1}A,\, w^{-1}s^{\epsilon_1}_{i_1}A,\, \ldots,\, w^{-1}s^{\epsilon_1}_{i_1} \cdots s^{\epsilon_r}_{i_r}A) \in \Gamma(\overrightarrow{\zeta}^{-1}, w^{-1}).$$

Since we have $(T^*_i)^{\pm 1} = S^Y_i - \psi^\pm_i(Y^{-a_i})$ by the definition (1.3.24) of T^*_i, the term contributed by p becomes

$$\begin{aligned}
(T^*_i)^\epsilon S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) &= (S^Y_i - \psi^\epsilon_i(Y^{-a_i})) S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) \\
&= S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) + (-\psi^\epsilon_i(Y^{-a_i})) S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) \\
&= S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) + S^Y_{e(p)^{-1}} (-\psi^\epsilon_i(Y^{-e(p)a_i})) g_p(Y) n_p(Y).
\end{aligned}$$

In the last equality we used (1.3.34). We treat the two terms in the last line separately.

For the first term $S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y)$, we further divide the argument into two cases according to the Bruhat order.

(i) The case $e(p)^{-1} \preccurlyeq_B s_i e(p)^{-1}$. By Proposition 2.2.1.1, we have $S^Y_i S^Y_{e(p)^{-1}} = S^Y_{s_i e(p)^{-1}} = S_{e(p_1)^{-1}}$, where the alcove walk

$$p_1 = (w^{-1}A,\, w^{-1}s^{\epsilon_1}_{i_1}A,\, \ldots,\, w^{-1}s^{\epsilon_1}_{i_1} \cdots s^{\epsilon_r}_{i_r}A,\, w^{-1}s^{\epsilon_1}_{i_1} \cdots s^{\epsilon_r}_{i_r} s^\epsilon_i A) \in \Gamma(\overrightarrow{z}, w^{-1}) \tag{2.2.4}$$

is an extension of p by a crossing (Table 2.1.2). By the hypothesis $e(p)^{-1} \preccurlyeq_B s_i e(p)^{-1}$, the last step of p_1 is an ascent, and we have $\varphi_+(p_1) = \varphi_+(p)$, $\varphi_-(p_1) = \varphi_-(p)$ and $\xi_{\mathrm{des}}(p_1) = \xi_{\mathrm{des}}(p)$. Thus we have $g_p(Y) n_p(Y) = g_{p_1}(Y) n_{p_1}(Y)$ and $S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) = S_{e(p_1)^{-1}} g_{p_1}(Y) n_{p_1}(Y)$.

(ii) The case $e(p)^{-1} \succcurlyeq_B s_i e(p)^{-1}$. By Proposition 2.2.1.1, we have

$$S^Y_i S^Y_{e(p)^{-1}} = n_i(Y^{-a_i}) S^Y_{s_i e(p)^{-1}} = n_i(Y^{-a_i}) S^Y_{e(p_1)^{-1}} = S^Y_{e(p_1)^{-1}} n_i(Y^{-e(p_1)a_i}).$$

Here $p_1 \in \Gamma(\overrightarrow{z}, w^{-1})$ is the same as (2.2.4), but in this case the last step is a descent crossing, and the hyperplane crossed by the last step is $H_{e(p_1)a_i}$ since

$$h_{r+1}(p_1) = -e(p)(a_i) = -e(p_1)s_i(a_i) = e(p_1)a_i.$$

We then have $\xi_{\mathrm{des}}(p_1) = \xi_{\mathrm{des}}(p) \cup \{r+1\}$ and $n_{p_1}(Y) = n_p(Y) n_i(Y^{-h_{r+1}(p_1)})$. Combining them with $\varphi_+(p_1) = \varphi_+(p)$ and $\varphi_-(p_1) = \varphi_-(p)$, we have $n_i(Y^{-e(p_1)a_i}) g_p(Y) n_p(Y) = g_{p_1}(Y) n_{p_1}(Y)$. Hence also in this case, we have $S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) = S_{e(p_1)^{-1}} g_{p_1}(Y) n_{p_1}(Y)$.

Taking the summation over p, we therefore have

$$\sum_{p \in \Gamma(\overrightarrow{\zeta}^{-1}, w^{-1})} S^Y_i S^Y_{e(p)^{-1}} g_p(Y) n_p(Y) = \sum_{\substack{p_1 \in \Gamma(\overrightarrow{z}^{-1}, w^{-1}), \\ \text{the last step is a crossing}}} S^Y_{e(p_1)^{-1}} g_{p_1}(Y) n_{p_1}(Y). \tag{2.2.5}$$

Next we consider the term $S^Y_{e(p)^{-1}}(-\psi^\epsilon_i(Y^{-e(p)a_i})) g_p(Y) n_p(Y)$. We make a similar argument as in the first term, and here we use the alcove walk $p_2 \in \Gamma(\overrightarrow{z}, w^{-1})$ which is an extension of p by a folding. We have $e(p_2) = e(p)$, $\varphi_+(p_2) = \varphi_+(p) \cup \{r+1\}$ and $\xi_{\mathrm{des}}(p_2) = \xi_{\mathrm{des}}(p)$. Using p_2 we have $S^Y_{e(p)^{-1}}(-\psi^\epsilon_i(Y^{-e(p)a_i})) g_p(Y) n_p(Y) = S^Y_{e(p_2)^{-1}} g_{p_2}(Y) n_{p_2}(Y)$. We therefore have

$$\sum_{p \in \Gamma(\overrightarrow{\zeta}^{-1}, w^{-1})} S^Y_{e(p)^{-1}}(-\psi^\epsilon_i(Y^{-e(p)a_i})) g_p(Y) n_p(Y) = \sum_{\substack{p_2 \in \Gamma(\overrightarrow{z}^{-1}, w^{-1}), \\ \text{the last step is a folding}}} S^Y_{e(p_2)^{-1}} g_{p_2}(Y) n_{p_2}(Y). \tag{2.2.6}$$

By (2.2.3) and (2.2.5), (2.2.6), we have $x^z S^Y_w = \sum_{p \in \Gamma(\overrightarrow{z}^{-1}, w^{-1})} S^Y_{e(p)^{-1}} g_p(Y) n_p(Y)$. Hence the induction step is proved. □

The definition (2.2.2) of x^z for $z \in W$ and the definition (1.3.23) of x^μ for $\mu \in \Lambda$ are consistent in the following sense. Recall that we denote by $\mathrm{t}(\mu) \in \mathrm{t}(\Lambda) \subset W$ the element associated to $\mu \in \Lambda$.

Lemma 2.2.1.4. We have $x^{\mathrm{t}(\mu)} = x^\mu$ for $\mu \in \Lambda$. In particular, we have $x^{\mathrm{t}(\epsilon_i)} = x_i$ for $i = 1, \ldots, n$.

Proof. It is enough to show the latter half. By (1.3.16), we have

$$Y_i^{-1} = T_i^{-1} \cdots T_{n-1}^{-1} T_n^{-1} T_{n-1}^{-1} \cdots T_1^{-1} T_0^{-1} T_1 \cdots T_{i-1} \quad (i = 1, \ldots, n).$$

Applying the anti-involution $*$ (1.3.24) to these.

$$x_i = (Y_i^{-1})^* = T_{i-1}^* \cdots T_1^* (T_0^*)^{-1} (T_1^*)^{-1} \cdots (T_{n-1}^*)^{-1} (T_n^*)^{-1} (T_{n-1}^*)^{-1} \cdots (T_i^*)^{-1}.$$

On the other hand, we can calculate $x^{\mathrm{t}(\epsilon_i)}$ directly by Definition (2.2.2), and can check $x_i = x^{\mathrm{t}(\epsilon_i)}$. □

We denote the *dominant chamber* for the weight lattice by

$$C := \{x \in V \mid \langle \alpha^\vee, x \rangle > 0, \ \alpha \in R_+\}.$$

As for the fundamental alcove A (2.1.1), we have $A \subset C$.

Let $v, w \in W$, and choose a reduced expression $v = s_{i_1} \cdots s_{i_r}$ of v. If an alcove walk $p \in \Gamma(\overrightarrow{v}, w)$ satisfies $e(p)^{-1} A \subset C$, where $e(p) \in W$ is the element (2.1.5), then using the W-valued function $w(\,)$ in (1.3.36), we define $\varpi(p) \in \Lambda_+$ by the relation

$$e(p)^{-1} = w(\varpi(p)). \tag{2.2.7}$$

Also we define $\Gamma^C(\overrightarrow{v}, w) \subset \Gamma(\overrightarrow{v}, w)$ by

$$\Gamma^C(\overrightarrow{v}, w) := \{p = (p_0, \ldots, p_r) \in \Gamma(\overrightarrow{v}, w) \mid p_i \in C, \ \forall\, i = 0, \ldots, r\}. \tag{2.2.8}$$

Using these symbols, we have the following corollary of Proposition 2.2.1.3.

Corollary 2.2.1.5 (c.f. [Yi12, Corollary 4.1])**.** Let λ and μ be elements in Λ, and fix a reduced expression $\mathrm{t}(\lambda) = s_{i_r} \cdots s_{i_1}$. Then we have

$$\begin{aligned}
x^\lambda E_\mu(x) &= \sum_{p \in \Gamma^C(\overrightarrow{\mathrm{t}(-\lambda)}, w(\mu)^{-1})} g_p n_p E_{\varpi(p)}(x), \\
g_p &:= \prod_{k \in \varphi_-(p)} \left(-\psi_{i_k}^-(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))})\right) \prod_{k \in \varphi_+(p)} \left(-\psi_{i_k}^+(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))})\right), \\
n_p &:= \prod_{k \in \xi_{\mathrm{des}}(p)} n_{i_k}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}).
\end{aligned}$$

Proof. We apply $x^z S_w^Y = \sum_{p \in \Gamma(\overrightarrow{z}^{-1}, w^{-1})} S^Y_{e(p)^{-1}} g_p(Y) n_p(Y)$ in Proposition 2.2.1.3 to $z = \mathrm{t}(\lambda)$ and $w = w(\mu)$. Since $x^{\mathrm{t}(\lambda)} = x^\lambda$ by Lemma 2.2.1.4, we have

$$x^\lambda S^Y_{w(\mu)} = \sum_{p \in \Gamma(\overrightarrow{\mathrm{t}(-\lambda)}, w(\mu)^{-1})} S^Y_{e(p)^{-1}} g_p(Y) n_p(Y).$$

Taking the product of each side with 1 and using the definition of the non-symmetric Koornwinder polynomial $E_\mu(x)$ (Fact 1.3.3.2) and the equality $Y^\beta 1 = q^{\mathrm{sh}(\beta)} t^{\mathrm{ht}(\beta)}$ in (1.3.22), we have

$$x^\lambda E_\mu(x) = \sum_{p \in \Gamma(\overrightarrow{\mathrm{t}(-\lambda)}, w(\mu)^{-1})} g_p n_p S^Y_{e(p)^{-1}} 1.$$

Next we consider the condition under which the factor $n_{i_k}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))})$ in n_p vanishes. By the definition of the factor (Proposition 2.2.1.1), the condition is $q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))} = t^{\pm 1}$ ($i_k = 1, \ldots, n-1$) and $q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))} = t_0^{\pm 1} t_n^{\pm 1}$ ($i_k = n$). Then by the definition (2.1.6) of $h_k(p)$, the alcove walk p that contributes to the summation is contained in the dominant chamber C. Now the consequence follows from the definition of $E_\mu(x)$ and and that (2.2.7) of $\varpi(p)$. □

2.2.2 Some lemmas

In this subsection we prepare some lemmas for the symmetrizer U and the Koornwinder polynomials $P_\lambda(x)$, which are $(C_n^\vee, C_n)$-type analogue of [Yi12, Proposition 3.6].

Lemma 2.2.2.1 (c.f. [Yi12, Proposition 3.6 (a)])**.** The symmetrizer U (1.3.42) has the following expression.

$$U = \sum_{w \in W_0} S_w^Y \prod_{a \in \mathcal{L}(w^{-1}, w_0^{-1})} b(Y^{-a}),$$
$$b(Y^{-a}) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} Y^{-a}}{1 - Y^{-a}} & (a \notin W_0.a_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} Y^{-\frac{a}{2}})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} Y^{-\frac{a}{2}})}{1 - Y^{-a}} & (a \in W_0.a_n) \end{cases}. \tag{2.2.9}$$

Here $\mathcal{L}(v, w) \subset S$ is given by (2.1.3), and $w_0 \in W_0$ is the longest element (1.3.39).

Proof. By the definition of U and the definition (1.3.35) of the Y-intertwiner S_w^Y, we can expand U as

$$U = \sum_{w \in W_0} S_w^Y b_w(Y), \quad b_w(Y) \in \mathbb{K}(Y).$$

For the longest element $w_0 \in W_0$, the coefficient of T_{w_0} in U is 1, and thus we have $b_{w_0}(Y) = 1$. We calculate the term $b_w(Y)$ for $w \in W_0 \setminus \{w_0\}$ by induction on the length $\ell(w)$. Assume $b_v(Y) = \prod_{a \in \mathcal{L}(v^{-1}, w_0^{-1})} b(Y^{-a})$ for any element $v \in W_0$ satisfying $\ell(v) > \ell(w)$. By the equality $UT_i = Ut_i^{\frac{1}{2}}$ $(i = 1, \ldots, n)$ in (1.3.43) and the definition (1.3.32) of S_i^Y, we have

$$\sum_{w \in W_0} S_w^Y b_w(Y) t_i^{\frac{1}{2}} = U t_i^{\frac{1}{2}} = UT_i = \sum_{w \in W_0} S_w^Y b_w(Y) T_i = \sum_{w \in W_0} S_w^Y b_w(Y) (S_i^Y - \psi_i^+(Y^{-a_i})). \tag{2.2.10}$$

Now note that for $w \neq w_0$ there exists an index $i = 1, \ldots, n$ such that $w \preccurlyeq_B v := ws_i$. Taking this index i and comparing the coefficients of S_w^Y in the equality (2.2.10) with the help of (1.3.34) and Corollary 2.2.1.2, we have $b_v(Y) t_i^{\frac{1}{2}} = b_w(s_i.Y) - b_v(Y)\psi_i^+(Y^{-a_i})$. Here $b_w(s_i.Y)$ is obtained from $b_w(Y)$ by replacing Y^λ with $Y^{s_i.\lambda}$. Then by the definition (1.3.33) of $\psi_i^+(z)$ we have

$$\begin{aligned} b_w(Y)/b_v(s_i.Y) &= t_i^{\frac{1}{2}} + \psi_i^+(Y^{-s_i.a_i}) = t_i^{\frac{1}{2}} + \psi_i^+(Y^{a_i}) \\ &= \begin{cases} t_i^{\frac{1}{2}} \frac{1 - t^{-1} Y^{-a_i}}{1 - Y^{-a_i}} & (0 < i < n) \\ t_n^{\frac{1}{2}} \frac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} Y^{-\frac{a_n}{2}})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} Y^{-\frac{a_n}{2}})}{1 - Y^{-a_n}} & (i = n) \end{cases}, \end{aligned}$$

so that it is equal to $b(Y^{-a_i})$. On the other hand, by (2.1.4) we have $\mathcal{L}(w^{-1}, w_0^{-1}) = s_i.\mathcal{L}(v^{-1}, w_0^{-1}) \sqcup \{a_i\}$. Thus we have $b_w(Y) = b_v(s_i.Y) b(Y^{-a_i}) = \prod_{a \in \mathcal{L}(w^{-1}, w_0^{-1})} b(Y^{-a})$.

□

We can apply the argument of the proof to the stabilizer $W_\mu \subset W_0$ for a dominant weight $\mu \in \Lambda_+$ instead of W_0. As a result, we have the following claim.

Corollary 2.2.2.2. For each $\mu \in \Lambda_+$, we have

$$\sum_{u \in W_\mu} t_{w_\mu u}^{-\frac{1}{2}} T_u = \sum_{w \in W_\mu} S_w^Y \prod_{a \in \mathcal{L}(w^{-1}, w_\mu^{-1})} b(Y^{-a}).$$

Here $b(Y^{-a}) \in \mathbb{K}(Y)$ is given by (2.2.9).

For a dominant weight $\mu \in \Lambda_+$, we denote by

$$W^\mu \subset W_0 \tag{2.2.11}$$

the complete system of representatives of the quotient set W_0/W_μ consisting of the shortest elements. We also denote by

$$v_\mu \in W^\mu \tag{2.2.12}$$

its longest element.

Now let us recall the element $w(\mu) \in \mathrm{t}(\mu)W_0 \subset W$ in the (1.3.36). We then have the following lemma for the Koornwinder polynomial $P_\mu(x)$ (Fact 1.3.3.4) and the non-symmetric Koornwinder polynomial $E_\mu(x)$ (Fact 1.3.3.2).

Lemma 2.2.2.3 (c.f. [Yi12, Proposition 3.6 (b)]). For $\lambda \in \Lambda_+$ we have

$$P_\lambda(x) = \sum_{v \in W^\lambda} \Big[\prod_{a \in w(\lambda)^{-1}\mathcal{L}(v^{-1}, v_\lambda^{-1})} \rho(a) \Big] E_{v.\lambda}(x),$$

$$\rho(a) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \notin W.a_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \in W.a_n) \end{cases},$$

where $\mathrm{sh}(\beta)$ and $\mathrm{ht}(\beta)$ for $\beta \in S$ are given by (1.3.21).

Proof. We write Lemma 2.2.2.1 as

$$U = \sum_{w \in W_0} S_w^Y b_{(w^{-1}, w_0^{-1})}(Y), \quad b_{(w^{-1}, w_0^{-1})}(Y) := \prod_{a \in \mathcal{L}(w^{-1}, w_0^{-1})} b(Y^{-a}).$$

Since W^λ consists of representatives of W_0/W_λ, there exist $v \in W^\lambda$ and $u \in W_\lambda$ uniquely such that $w = vu$. Using Corollary 2.2.2.2, we have

$$\begin{aligned} U &= \sum_{w \in W_0} S_w^Y b_{(w^{-1}, w_0^{-1})}(Y) = \Big[\sum_{v \in W^\lambda} S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) \Big] \Big[\sum_{u \in W_\lambda} S_u^Y b_{(u^{-1}, w_\lambda^{-1})}(Y) \Big] \\ &= \Big[\sum_{v \in W^\lambda} S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) \Big] \Big[\sum_{u \in W_\lambda} t_{w_\lambda u}^{-\frac{1}{2}} T_u \Big]. \end{aligned}$$

In the second equality, we used the factorization of b-function, which follows from the equation on the third line of [M03, §.5, p.122]. Note that our b-function is written as $c(x)$ in [M03, §.4, (4.2.2)]. The product with $S_{w(\lambda)}^Y 1$ gives

$$U S_{w(\lambda)}^Y 1 = \Big[\sum_{v \in W^\lambda} S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) \Big] \Big[\sum_{u \in W_\lambda} t_{w_\lambda u}^{-\frac{1}{2}} T_u \Big] S_{w(\lambda)}^Y 1 = t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t) \sum_{v \in W^\lambda} S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) S_{w(\lambda)}^Y 1, \tag{2.2.13}$$

where in the second equality we used the Poincaré polynomial (1.3.41) and the relation $(T_u f)1 = t_u^{\frac{1}{2}} f$ for $u \in W_0$ and $f \in \mathbb{K}[x^{\pm 1}]$ satisfying $u(f) = f$. The latter relation is shown as follows. If $s_i f = f$ for some $i = 1, \ldots, n$, then we have $(T_i - t_i^{\frac{1}{2}}) f = c_i(x^{a_i})(s_i - 1) f = 0$, and so $T_i f = t_i^{\frac{1}{2}} f$. Now the relation follows by induction on the length of $u \in W_0$.

Let us continue the calculation (2.2.13). Note that we have $v w(\lambda) = w(v.\lambda)$ for $v \in W^\lambda$. By this relation and (1.3.34), each term in the right hand side of (2.2.13) becomes

$$S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) S_{w(\lambda)}^Y 1 = S_v^Y S_{w(\lambda)}^Y b_{(v^{-1}, v_\lambda^{-1})}(w(\lambda)^{-1}.Y) 1 = \big(S_{w(v.\lambda)}^Y 1\big) \big(b_{(v^{-1}, v_\lambda^{-1})}(w(\lambda)^{-1}.Y) 1\big).$$

Here $b_{(v^{-1}, v_\lambda^{-1})}(w(\lambda)^{-1}.Y)$ is obtained from $b_{(v^{-1}, v_\lambda^{-1})}(Y)$ by replacing Y^μ with $Y^{w(\lambda)^{-1}.\mu}$. Now let us recall the equality $Y^a 1 = q^{\mathrm{sh}(a)} t^{\mathrm{ht}(a)}$ in (1.3.22). Then we have $b(Y^{-a})1 = \rho(a)$, and therefore

$$b_{(v^{-1}, v_\lambda^{-1})}(w(\lambda)^{-1}.Y) 1 = \prod_{a \in \mathcal{L}(v^{-1}, v_\lambda^{-1})} \big(b(Y^{-w(\lambda)^{-1}.a}) 1\big) = \prod_{a \in w(\lambda)^{-1}\mathcal{L}(v^{-1}, v_\lambda^{-1})} \rho(a).$$

By summing over $v \in W^\lambda$ we have

$$US^Y_{w(\lambda)}1 = t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t) \sum_{v \in W^\lambda} \Big[\prod_{a \in w(\lambda)^{-1}\mathcal{L}(v^{-1}, v_\lambda^{-1})} \rho(a) \Big] E_{v.\lambda}(x),$$

Now the result follows from the definition of $P_\lambda(x)$ (Fact 1.3.3.4). $\square$

2.2.3 Ram-Yip type formula and its application

In [Yi12, Theorem 4.2], Yip derived an expansion formula $E_\mu(x)P_\lambda(x) = \sum_\nu a^\nu_{\lambda,\mu} E_\nu(x)$ for the product of the non-symmetric Macdonald polynomial $E_\mu(x)$ and the Macdonald polynomial $P_\lambda(x)$ in the case of untwisted affine root systems. In this subsection, we give its $(C_n^\vee, C_n)$-type analogue, i.e., an expansion formula for the product of the non-symmetric Koornwinder polynomial and the Koornwinder polynomial (Proposition 2.2.3.2).

As a preparation, we explain the explicit formula of non-symmetric Macdonald polynomials via alcove walks, established by Ram and Yip [RY11] for reduced affine root systems. Their argument is designed to work in general systems, including $(C_n^\vee, C_n)$-type, and the details were clarified by Orr and Shimozono [OS18]. Let us call these alcove-walk explicit formulas *Ram-Yip type formulas.* We focus on the Ram-Yip type formula of non-symmetric Koornwinder polynomials.

We prepare the necessary notations for the explanation. Let us given $v, w \in W$ and a reduced expression of w. For an alcove walk $p \in \Gamma(\overrightarrow{w}, z)$, we denote the decomposition of the element $e(p) \in W$ (2.1.5) with respect to the presentation $W = \mathrm{t}(\Lambda) \rtimes W_0$ by

$$e(p) = \mathrm{t}(\mathrm{wt}(p))\, \mathrm{d}(p), \quad \mathrm{d}(p) \in W_0,\ \mathrm{wt}(p) \in \Lambda. \tag{2.2.14}$$

Fact 2.2.3.1 ([RY11, Theorem 3.1], [OS18, Theorem 3.13])**.** For $\mu \in \Lambda$, let $w(\mu)$ be the shortest element among $\mathrm{t}(\mu)W_0 \subset W$ (1.3.36), and fix its reduced expression $w(\mu) = s_{i_1} \cdots s_{i_r}$. Then we have

$$E_\mu(x) = \sum_{p \in \Gamma(\overrightarrow{w(\mu)}, e)} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)},$$
$$f_p := \prod_{k \in \varphi_+(p)} \psi^+_{i_k}(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}) \prod_{k \in \varphi_-(p)} \psi^-_{i_k}(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}),$$

where we set $\beta_k := s_{i_r} \cdots s_{i_{k+1}}(a_{i_r})$ for $k = 1, \ldots, r$.

Next we introduce some notations necessary for Proposition 2.2.3.2, which are basically the ones in [Yi12, §4.1]. Let us given $v, w \in W$ and a reduced expression $v = s_{i_1} \cdots s_{i_r}$. Recall the set $\Gamma^C(\overrightarrow{v}, w)$ of alcove walks belonging to the dominant chamber C as in (2.2.8). Consider an alcove walk in $\Gamma^C(\overrightarrow{v}, w)$ together with coloring of all the folding steps by either black or gray. We call such a data a *colored alcove walk*, and denote by

$$\Gamma_2^C(\overrightarrow{v}, w) \tag{2.2.15}$$

the set of colored alcove walks arising from alcove walks in $\Gamma^C(\overrightarrow{v}, w)$.

For a colored alcove walk $p \in \Gamma_2^C(\overrightarrow{v}, w)$, we denote by

$$p^* \in \Gamma(\overrightarrow{v}^{-1}, w^{-1}e(p)) \tag{2.2.16}$$

the uncolored alcove walk obtained by straightening all the gray folding steps of p, by reversing the order, and by translation so that it ends at $e(p^*) = e \in W$. More explicitly, for a colored positive walk $p \in \Gamma_2^C(\overrightarrow{v}, w)$ with

$$p = (wA, ws_{i_1}^{b_1}A, \ldots, ws_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A),$$

we define $\widetilde{p}_k$ for $k = 1, \ldots, r$ as follows, according to whether the k-th step $p_{k-1} = ws_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}}A \to p_k = ws_{i_1}^{b_1} \cdots s_{i_k}^{b_k}A$ is a gray folding step or not:

$$\widetilde{p}_k := \begin{cases} ws_{i_1}^{b_1} \cdots s_{i_{k-1}}^{b_{k-1}} s_{i_k} A & (p_{k-1} \to p_k \text{ is a gray folding step}) \\ p_k & (\text{otherwise}) \end{cases}.$$

Thus we obtain a new uncolored alcove walk $\widetilde{p} = (\widetilde{p}_0, \dots, \widetilde{p}_r) \in \Gamma(\overrightarrow{v}, w)$, which was called the one obtained "by straightening all the gray foldings". Next we denote by $(c_1, \dots, c_r) \in \{0,1\}^r$ the bit sequence corresponding to $\widetilde{p}$. In other words, we have $\widetilde{p} = (wA, \dots, ws_{i_1}^{c_1} \cdots s_{i_r}^{c_r} A)$. Now the alcove walk p^* is obtained by reversing the order of $\widetilde{p}$ and translating the start to $w^{-1}e(\widetilde{p})$. Explicitly, we have

$$p^* := (s_{i_1}^{c_1} \cdots s_{i_r}^{c_r} A, s_{i_1}^{c_1} \cdots s_{i_{r-1}}^{c_{r-1}} A, \dots, s_{i_1}^{c_1} A, A).$$

Proposition 2.2.3.2 (c.f. [Yi12, Theorem 4.2])**.** For a weight $\mu \in \Lambda$, we take a reduced expression $w(\mu) = s_{i_r} \cdots s_{i_1}$ of $w(\mu) \in \mathrm{t}(\mu)W_0 \subset W$. Then for any dominant weight $\lambda \in \Lambda_+$ we have

$$E_\mu(x) P_\lambda(x) = \sum_{v \in W^\lambda} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, (vw(\lambda))^{-1})} A_p C_p E_{\varpi(p)}(x).$$

Here W^λ is given by (2.2.11), and the term A_p is given with the help of $\rho(a)$ in Lemma 2.2.2.3 by

$$A_p := \prod_{a \in w(\lambda)^{-1} \mathcal{L}(v^{-1}, v_\lambda^{-1})} \rho(a),$$

$$\rho(a) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \notin W.a_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-a)} t^{\frac{1}{2}\mathrm{ht}(-a)})}{1 - q^{\mathrm{sh}(-a)} t^{\mathrm{ht}(-a)}} & (a \in W.a_n) \end{cases}.$$

The term C_p is given by $C_p := \prod_{k=1}^r C_{p,k}$, whose factor $C_{p,k}$ is determined by the k-th step of p as follows.

$$C_{p,k} := \begin{cases} 1 & \text{the } k\text{-th step of } p \text{ is a positive crossing} \\ \prod_{k \in \xi_{\mathrm{des}}(p)} n_{i_k}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a negative crossing} \\ -\psi_{i_k}^{+}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a gray positive folding} \\ -\psi_{i_k}^{-}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a gray negative folding} \\ \psi_{i_k}^{+}(q^{\mathrm{sh}(-b_k)} t^{\mathrm{ht}(-b_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is positive} \\ \psi_{i_k}^{-}(q^{\mathrm{sh}(-b_k)} t^{\mathrm{ht}(-b_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is negative} \end{cases},$$

where $n_i(Y^a)$ is given by Proposition 2.2.1.1, $\psi_{i_k}^{\pm}(z)$ is given by (1.3.33) and $h_k(p)$ is given by (2.1.6). We also used $b_k := s_{i_1} \cdots s_{i_{r-1}}(a_{i_r})$ for $k = 1, \dots r$. Finally $\varpi(p)$ is given by (2.2.7).

Note that the term A_p actually depends only on $v \in W^\mu$, which corresponds to the beginning of the colored alcove walk p.

Proof. On the Ram-Yip type formula $E_\mu(x) = \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h t_{\mathrm{d}(h)}^{\frac{1}{2}} x^{\mathrm{wt}(h)}$ (Fact 2.2.3.1), let us act $US_{w(\lambda)}^Y 1$ from the left. Then we have

$$E_\mu(x) US_{w(\lambda)}^Y 1 = \Big[\sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h t_{\mathrm{d}(h)}^{\frac{1}{2}} x^{\mathrm{wt}(h)} \Big] US_{w(\lambda)}^Y 1 = \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h x^{e(h)} US_{w(\lambda)}^Y 1.$$

Here the second equality follows from the definition (2.2.14) of $\mathrm{wt}(h)$ and $\mathrm{d}(h)$, as well as from the relation $T_i U = t_i^{\frac{1}{2}} U$ in (1.3.43). Moreover, by Lemma 2.2.2.3 and using the notation in its proof, we have

$$\begin{aligned} E_\mu(x) US_{w(\lambda)}^Y 1 &= \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h x^{e(h)} \big[t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t) \sum_{v \in W^\lambda} S_v^Y b_{(v^{-1}, v_\lambda^{-1})}(Y) S_{w(\lambda)}^Y 1 \big] \\ &= t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t) \sum_{v \in W^\lambda} \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h S_v^Y S_{w(\lambda)}^Y b_{(v^{-1}, v_\lambda^{-1})}(w(\lambda)^{-1}.Y) 1 \\ &= t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t) \sum_{v \in W^\lambda} A_p \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h x^{e(h)} S_{vw(\lambda)}^Y 1. \end{aligned} \tag{2.2.17}$$

Here we set $A_p := b_{((vw(\lambda))^{-1}, t(-w_0\lambda))} = \prod_{a \in w(\lambda)^{-1}\mathcal{L}(v^{-1}, v_\lambda^{-1})} \rho(a)$. As for the factor $f_h x^{e(h)} S^Y_{vw(\lambda)} 1$ in the final line of (2.2.17), denoting $z := (vw(\lambda))^{-1}$ and using Proposition 2.2.1.3 and Corollary 2.2.1.5, we have

$$f_h x^{e(h)} S^Y_{vw(\lambda)} 1 = f_h \sum_{q \in \Gamma(\overrightarrow{e(h)}^{-1}, z)} S^Y_{e(q)^{-1}} n_q(Y) g_q(Y) 1 = f_h \sum_{q \in \Gamma^C(\overrightarrow{e(h)}^{-1}, z)} n_q g_q E_{\varpi(q)}(x). \tag{2.2.18}$$

We will rewrite this sum over uncolored alcove walks in $\Gamma^C(\overrightarrow{e(h)}^{-1}, z)$ as a sum over colored alcove walks in $\Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, z)$.

Let us given an uncolored alcove walk $q \in \Gamma^C(\overrightarrow{e(h)}^{-1}, z)$. Since q is an alcove walk of type $\overrightarrow{e(h)}^{-1}$, which is one of type $\overrightarrow{w(\mu)}^{-1}$, we can compare the bit sequence of q with the bit sequence of h. In this comparison, if the k-th step of q is a folding and the k-th step of h is a crossing, then we color the k-th folding step of q by gray. Otherwise we color it by black. Thus we obtain a colored alcove walk, which is denoted by p. Note that we have $p \in \Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, z)$. Then each term of the right hand side in (2.2.18) is equal to

$$f_h n_q g_q E_{\varpi(q)}(x) = f_{p^*} n_p g_p E_{\varpi(p)}(x),$$

where p^* is given by (2.2.16). We can also express f_{p^*} using $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(a_{i_k})$ as

$$f_{p^*} = \prod_{k \in \varphi_+(p^*)} \psi^+_{i_k}(q^{\mathrm{sh}(-b_k)} t^{\mathrm{ht}(-b_k)}) \prod_{k \in \varphi_-(p^*)} \psi^-_{i_k}(q^{\mathrm{sh}(-b_k)} t^{\mathrm{ht}(-b_k)}).$$

As a result, the last line of (2.2.17) is rewritten by a sum over $p \in \Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, z)$.

Divided by the factor $t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)$, the left hand side of (2.2.17) is equal to $E_\mu(x) P_\lambda(x)$. Thus we have

$$\begin{aligned} E_\mu(x) P_\lambda(x) &= \sum_{v \in W^\lambda} A_p \sum_{h \in \Gamma(\overrightarrow{w(\mu)}, e)} f_h \sum_{q \in \Gamma^C(\overrightarrow{e(h)}^{-1}, z)} n_q g_q E_{\varpi(q)}(x) \\ &= \sum_{v \in W^\lambda} A_p \sum_{p \in \Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, z)} f_{p^*} g_p n_p E_{\varpi(p)}(x). \end{aligned}$$

We obtain the result by collecting the terms from f_{p^*}, g_p and n_p which depend only on the k-th step of $p \in \Gamma_2^C(\overrightarrow{w(\mu)}^{-1}, z)$ and denoting them by $C_{p,k}$. $\square$

2.2.4 Littlewood-Richardson coefficients for Koornwinder polynomials

In this subsection, we derive our main Theorem 2.2.4.2 on LR coefficients of Koornwinder polynomials.

We start with a preliminary lemma. Recall the complete system W^λ of representatives of W_0/W_λ in (2.2.11) and the element $w(\lambda) \in t(\lambda)W_0$ in (1.3.36).

Lemma 2.2.4.1 (c.f. [Yi12, Proposition 3.7])**.** Let $\lambda \in \Lambda_+$. If $v \in W^\lambda$ satisfies $vw(\lambda) \succcurlyeq_B w(\lambda)$, then we have

$$U S^Y_v S^Y_{w(\lambda)} 1 = \Big[\prod_{a \in \mathcal{L}(w(\lambda)^{-1}, (vw(\lambda))^{-1})} \rho(-a) \Big] U S^Y_{w(\lambda)} 1,$$

where $\rho(a)$ is defined in Lemma 2.2.2.3.

Proof. Recall the equality $UT_i = Ut_i^{\frac{1}{2}}$ for $i = 1, \dots, n$ in (1.3.43). Therefore we have

$$U S^Y_i S^Y_w 1 = U\big(T_i^* + \psi_i^+(Y^{-a_i})\big) S^Y_w 1 = U S^Y_w \big(t_i^{\frac{1}{2}} + \psi_i^+(q^{\mathrm{sh}(-w^{-1}a_i)} t^{\mathrm{ht}(-w^{-1}a_i)})\big) 1.$$

Assume that $v \in W^\lambda$ satisfies $vw(\lambda) \succcurlyeq_B w(\lambda)$, and take a reduced expression $v = s_{i_1} \cdots s_{i_r}$. Using the above relation, we expand the product $U S^Y_v S^Y_{w(\lambda)} 1 = U S^Y_{i_1} \cdots S^Y_{i_r} S^Y_{w(\lambda)} 1$ in order. We have

$$U S^Y_v S^Y_{w(\lambda)} 1 = U(t_{i_1}^{\frac{1}{2}} + \psi^+_{i_1}(Y^{-a_{i_1}})) S^Y_{s_{i_1} v} S^Y_{w(\lambda)} 1 = U S^Y_{s_{i_1} v}(t_{i_1}^{\frac{1}{2}} + \psi^+_{i_1}(Y^{-s_{i_r} \cdots s_{i_2}(a_{i_1})})) S^Y_{w(\lambda)} 1$$

$$
\begin{aligned}
&= U(t_{i_1}^{\frac{1}{2}} + \psi_{i_1}^{+}(Y^{-s_{i_r}\cdots s_{i_2}(a_{i_1})}))(t_{i_2}^{\frac{1}{2}} + \psi_{i_2}^{+}(Y^{-a_{i_2}}))S^Y_{s_{i_1}s_{i_2}v}S^Y_{w(\lambda)}1 \\
&= \cdots \\
&= U\Big[\prod_{j=1}^{r}(t_{i_j}^{\frac{1}{2}} + \psi_{i_j}^{+}(Y^{-w(\lambda)^{-1}s_{i_r}\cdots s_{i_{j+1}}a_{i_j}}))\Big]S^Y_{w(\lambda)}1 \\
&= US^Y_{w(\lambda)}\Big[\prod_{j=1}^{r}(t_{i_j}^{\frac{1}{2}} + \psi_{i_j}^{+}(Y^{-w(\lambda)^{-1}s_{i_r}\cdots s_{i_{j+1}}a_{i_j}}))\Big]1 = US^Y_{w(\lambda)}\Big[\prod_{a\in\mathcal{L}(w(\lambda)^{-1},(v.w(\lambda))^{-1})}\rho(-a)\Big]1.
\end{aligned}
$$

Therefore the claim is obtained. □

We prepare some symbols to state the main theorem. For $\mu \in \Lambda$, the orbit $W_0.\mu$ contains a unique dominant weight. We denote it by

$$\mu_+ \in W_0.\mu \cap \Lambda_+. \tag{2.2.19}$$

Let us also recall the set $\Gamma_2^C(\overrightarrow{v}, w)$ of colored alcove walks defined in (2.2.15).

Theorem 2.2.4.2. Let us given dominant weights $\lambda, \mu \in \Lambda_+$. Choose a reduced expression $w(\lambda) = s_{i_r}\cdots s_{i_1}$ of the shortest element $w(\lambda) \in \mathrm{t}(\lambda)W_0$ in (1.3.36). Then we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}}W_\lambda(t)}\sum_{v\in W^\mu}\sum_{p\in\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(vw(\mu))^{-1})} A_pB_pC_pP_{-w_0.\mathrm{wt}(p)}(x).$$

Here $A_p \coloneqq \prod_{a\in w(\mu)^{-1}\mathcal{L}(v^{-1},v_\mu^{-1})}\rho(a)$ with $\rho(a)$ given in Proposition 2.2.3.2. The term C_p is the same as that in Proposition 2.2.3.2, and $\mathrm{wt}(p) \in \Lambda$ is defined by (2.2.14). The term B_p is defined by

$$B_p \coloneqq \prod_{a\in\mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0,e(p))}\rho(-a).$$

Proof. The strategy is to calculate the product of Koornwinder polynomials by acting the symmetrizer U to each side of the equation in Proposition 2.2.3.2.

For a colored alcove walk $p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})$, let $z \in W_0$ be the shortest element among $\{z \in W_0 \mid z.\varpi(p)_+ = \varpi(p)\}$. Note that we have $w(\varpi(p)_+)^{-1} = w(-w_0\varpi(p)_+)$. Since $e(p) \in \mathrm{t}(\mathrm{wt}(p))W_0$ by the definition of $\mathrm{wt}(p)$, $w(-w_0\varpi(p)_+)$ is the shortest element among $\mathrm{t}(\mathrm{wt}(p))W_0$. By Lemma 2.2.4.1, we then have

$$UE_{\varpi(p)}(x)1 = US_z^YS^Y_{w(\varpi(p)_+)}1 = \Big[\prod_{a\in\mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0,e(p))}\rho(-a)\Big]P_{-w_0.\mathrm{wt}(p)}(x).$$

By Proposition 2.2.3.2 and this equality, we have

$$
\begin{aligned}
P_\lambda(x)P_\mu(x) &= \frac{1}{t_{w_\lambda}^{-\frac{1}{2}}W_\lambda(t)}UE_\lambda(x)P_\mu(x)1 \\
&= \frac{1}{t_{w_\lambda}^{-\frac{1}{2}}W_\lambda(t)}\sum_{v\in W^\mu}\sum_{p\in\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(vw(\mu))^{-1})} A_pB_pC_pP_{-w_0.\mathrm{wt}(p)}(x).
\end{aligned}
$$

Hence the claim is obtained. □

2.3 Special cases of Littlewood-Richardson coefficients

In Theorem 2.2.4.2, we derived an explicit formula of the LR coefficient $c^\nu_{\lambda,\mu}$ in the product $P_\lambda(x)P_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu}P_\nu(x)$ of Koornwinder polynomials using alcove walks. In this section, we discuss several specializations of the formula.

2.3.1 Askey-Wilson polynomials

As mentioned in § 1.3.1, Koornwinder polynomials in the rank one case are nothing but Askey-Wilson polynomials. In this case LR coefficients of Askey-Wilson polynomials are expected to be simpler than the general rank case in Theorem 2.2.4.2.

As a preparation, we summarize the data of the root system of rank 1. We consider the Euclid space $V = \mathbb{R}\epsilon^\vee$ of dimension 1 and its dual space $V^* = \mathbb{R}\epsilon$. The root system of type C_1 is $R = \{\pm 2\epsilon\} \subset V^*$, the simple root is $a_1 = 2\epsilon$, and the fundamental weight is $\omega = \epsilon$. The weight lattice is $\Lambda = \mathbb{Z}\epsilon \subset V^*$, and the set of dominant weights is $\Lambda_+ = \mathbb{N}\epsilon$. The finite Weyl group W_0 is the group of order two generated by $s_1 := s_{a_1}$, and the longest element of W_0 is $w_0 = s_1$. The affine root system of rank 1 is $S = \left\{\pm 2\epsilon + kc, \pm\epsilon + \frac{k}{2}c \mid k \in \mathbb{Z}\right\}$ with $a_0 = c - 2\epsilon$, and the extended affine Weyl group W is the group generated by s_1 and $s_0 := s_{a_0}$. The decomposition $W = \mathrm{t}(\Lambda) \rtimes W_0$ (1.3.8) is the semi-direct product of $\mathrm{t}(\Lambda) = \langle \mathrm{t}(\epsilon_1) = s_0 s_1 \rangle \simeq \mathbb{Z}^2$ and $W_0 = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

We denote by

$$P_l(x) = P_l(x; q, t_0, t_1, u_0, u_1)$$

the Askey-Wilson polynomial associated to the dominant weight $\lambda = l\omega = l\epsilon$ ($l \in \mathbb{N}$). Note that it has five parameters.

First, we consider the simplest case. Following the case of type A, we call the LR coefficients $c^\nu_{\lambda,\mu}$ with λ or μ equal to a minuscule weight *Pieri coefficients*. Since the weight ω_1 is the unique minuscule weight in the root system of type C_n, we consider the case $\lambda = \omega$ for the rank one case.

Let us write down explicitly the Askey-Wilson polynomial $P_1(x) = P_\omega(x)$. In the following calculation, we need an explicit form of the term $\rho(a)$ ($a \in S$) in Proposition 2.2.3.2 and Theorem 2.2.4.2. The result is:

$$\rho(a) := t_1^{\frac{1}{2}} \frac{(1 + q^{\frac{k}{2}} t_0^{\frac{1}{2}} t_1^{-\frac{1}{2}} (t_0 t_1)^{-\frac{j}{2}})(1 - q^{\frac{k}{2}} t_0^{-\frac{1}{2}} t_1^{-\frac{1}{2}} (t_0 t_1)^{-\frac{j}{2}})}{1 - q^k (t_0 t_1)^{-j}} \quad (a = 2j\epsilon + kc \in S). \tag{2.3.1}$$

Lemma 2.3.1.1. The Askey-Wilson polynomial associated to the minuscule weight ω is

$$P_1(x) = x + x^{-1} + \rho(2c - a_1)\psi_0^-(qt_0t_1) + t_1^{\frac{1}{2}}\psi_0^+(qt_0t_1) + \psi_1^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

Here $\psi_k^\pm(z)$ ($k = 0, 1$) is given by (1.3.33) with $n = 1$. Explicitly, we have

$$\psi_0^\pm(z) := \mp\frac{(u_1^{\frac{1}{2}} - u_1^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}, \quad \psi_1^\pm(z) := \mp\frac{(t_1^{\frac{1}{2}} - t_1^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}. \tag{2.3.2}$$

Proof. Below we use the word *non-symmetric Askey-Wilson polynomials* to mean non-symmetric Koornwinder polynomials (Fact 1.3.3.2) in the rank 1 case. By Lemma 2.2.2.3, we can rewrite $P_1(x)$ as a linear combination of non-symmetric Askey-Wilson polynomials $E_k(x) = E_{k\omega}(x)$, $k \in \mathbb{Z}$. The result is

$$P_1(x) = \rho(2c - a_1)E_1(x) + E_{-1}(x).$$

Next, using the Ram-Yip type formula (Fact 2.2.3.1), we can expand $E_1(x)$ and $E_{-1}(x)$ by monomials. The results are

$$E_1(x) = t_1^{\frac{1}{2}}x + \psi_0^-(qt_0t_1), \quad E_{-1}(x) = x^{-1} + t_1^{\frac{1}{2}}\psi_0^+(qt_0t_1) + t_1^{\frac{1}{2}}\psi_1^+(q^2t_0t_1)x + \psi_1^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

By these formulas, we have

$$P_1(x) = x^{-1} + (t_1^{\frac{1}{2}}\rho(2c - a_1) + t_1^{\frac{1}{2}}\psi_1^+(q^2t_0t_1))x + \psi_0^-(qt_0t_1)\rho(2c - a_1) + t_1^{\frac{1}{2}}\psi_0^+(qt_0t_1) + \psi_1^+(q^2t_0t_1)\psi_0^-(qt_0t_1).$$

By a direct calculation, the coefficients of x is shown to be

$$t_1^{\frac{1}{2}}\rho(2c - a_1) + t_1^{\frac{1}{2}}\psi_1^+(q^2t_0t_1) = 1.$$

Therefore the claim is obtained. □

Remark 2.3.1.2. Let us replace the Noumi parameters (q, t_0, t_1, u_0, u_1) with the original parameters (q, a, b, c, d) of Askey-Wilson polynomials in [AW85]. The correspondence of parameters can be rewritten as

$$(q, t_0, t_1, u_0, u_1) = (q, -q^{-1}ab, -cd, -a/b, -c/d).$$

Using this correspondence and the relation $abcd = qt_0t_n$, we can rewrite $P_1(x)$ as

$$P_1(x) = x + x^{-1} + \frac{\pi s - s'}{1-\pi}, \quad \pi := abcd, \ s := a+b+c+d, \ s' := a^{-1}+b^{-1}+c^{-1}+d^{-1}.$$

We can then compare $P_1(x)$ with the original Askey-Wilson polynomials $p_n(z)$ in [AW85, p.5]. By loc. cit., we have $p_1(z) = 2(1-\pi)z + \pi s - s'$, and thus

$$(1-\pi)P_1(x) = p_1\big((x+x^{-1})/2\big).$$

Therefore they coincide up to the normalization factor.

Proposition 2.3.1.3. For a dominant weight $\lambda = l\omega \in \Lambda_+$, $l \in \mathbb{N}$, we have

$$\begin{aligned} P_1(x)P_l(x) &= P_{l+1}(x) + F_l P_l(x) + G_l P_{l-1}(x), \\ F_l &:= \rho(-2lc+a_1)(-\psi_0^-(q^{2l+1}t_0t_1) + \psi_0^-(qt_0t_1)) + \rho(2lc-a_1)(-\psi_0^+(q^{2l-1}t_0t_1) + \psi_0^+(qt_0t_1)), \\ G_l &:= \rho(2lc-a_1)\rho(-2(l-1)c+a_1)n_0(q^{2l-1}t_0t_1). \end{aligned}$$

Here $\rho(a)$ is given by (2.3.1), $\psi_0^\pm(z)$ is given by (2.3.2), and $n_0(z)$ is given in Proposition 2.2.1.1 with $n=1$. Explicitly, the last one is given by

$$n_0(z) := \frac{(1-u_1^{\frac12}u_0^{\frac12}z^{\frac12})(1+u_1^{\frac12}u_0^{-\frac12}z^{\frac12})}{1-z}\,\frac{(1+u_1^{-\frac12}u_0^{\frac12}z^{\frac12})(1-u_1^{-\frac12}u_0^{-\frac12}z^{\frac12})}{1-z}.$$

Proof. By Theorem 2.2.4.2, we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac12}W_\lambda(t)} \sum_{v\in W^\mu} \sum_{p\in\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(vw(\mu))^{-1})} A_pB_pC_pP_{-w_0.\mathrm{wt}(p)}(x)$$

for dominant weights $\lambda, \mu \in \Lambda_+$. We apply this equation to the case $\lambda = \omega$ and $\mu = l\omega$. In this case the stabilizer $W_\mu \subset W_0$ in (1.3.38) is $W_\mu = \{e\}$, and the complete system W^μ (2.2.11) of representatives of W_0/W_λ is $W^{l\omega} = \{e, s_1\}$. As for the shortest element $w(\nu) \in \mathrm{t}(\nu)W_0$ given in (1.3.36), we have by $\mathrm{t}(\omega) = s_0s_1$ that $w(\omega) = s_0$ and $w(l\omega) = (s_0s_1)^{l-1}s_0$.

First, we calculate the denominator $t_{w_\omega}^{-\frac12}W_\omega(t)$. As for the longest element $w_\lambda \in W^\lambda$ in (1.3.39), we have $w_\omega = e$. Thus, by recalling the definition (1.3.40) of t_w $(w \in W)$, we have $t_{w_\omega}^{-\frac12}W_\omega(t) = t_e^{-\frac12}t_e = 1$.

Next, as for the sum in the right hand side, we calculate the case $v = s_1$. The set of alcove walks is then $\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1}) = \Gamma_2^C(\overrightarrow{s_0}, \mathrm{t}(l\omega))$. In the upper half of Table 2.3.1, we display the alcove walks p therein together with the corresponding terms A_p, B_p and C_p. In the table we denote by H_0 and H_1 the hyperplanes in the W-orbits of H_{a_0} and H_{a_1} respectively. We also denote a black folding by a solid line, and a gray folding by a dotted line.

Next we study the case $v = e$. The set of alcove walks is $\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1}) = \Gamma_2^C(\overrightarrow{s_0}, w(l\omega))$, and in the lower half of Table 2.3.1 we display the alcove walks p therein together with the corresponding terms A_p, B_p and C_p.

The claim is obtained by collecting the above calculations. □

Remark 2.3.1.4. Continuing Remark 2.3.1.2, we rewrite the result in Proposition 2.3.1.3 in terms of the original parameters (q, a, b, c, d) of Askey-Wilson polynomials. The result is

$$P_1(x)P_l(x) = P_{l+1}(x) + F_lP_l(x) + G_{l-1}P_{l-1}(x), \tag{2.3.3}$$

where the factors F_l and G_l are given by

$$F_l := \frac{f_l + (\pi s' - s)}{1-\pi}, \quad f_l := q^{l-1}\frac{(1+q^{2l-1}\pi)(qs+\pi s') - q^{l-1}(1+q)\pi(s+qs')}{(1-q^{2l-2}\pi)(1-q^{2l}\pi)},$$

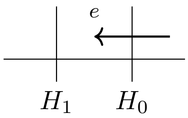
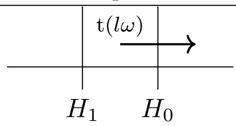
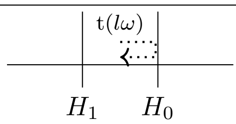
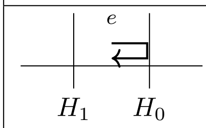
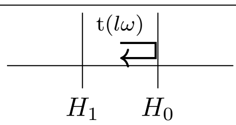
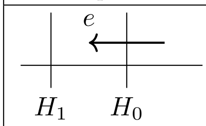
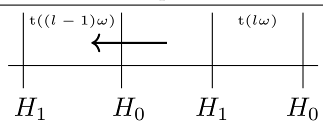
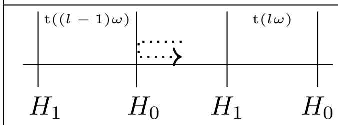
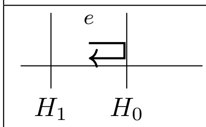
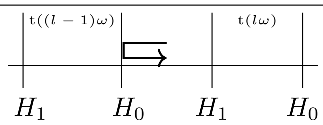
<table>
<tr><td colspan="5">$v = s_1$</td></tr>
<tr><td>$p*$</td><td>p</td><td>A_p</td><td>B_p</td><td>C_p</td></tr>
<tr><td rowspan="2">
</td><td>
</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>
</td><td>1</td><td>$\rho(-2lc + a_1)$</td><td>$-\psi_0^-(q^{2l+1}t_0t_1)$</td></tr>
<tr><td>
</td><td>
</td><td>1</td><td>$\rho(-2lc + a_1)$</td><td>$\psi_0^-(qt_0t_1)$</td></tr>
<tr><td colspan="5">$v = e$</td></tr>
<tr><td>$p*$</td><td>p</td><td>A_p</td><td>B_p</td><td>C_p</td></tr>
<tr><td rowspan="2">
</td><td>
</td><td>$\rho(2lc - a_1)$</td><td>$\rho(-(2l-2)c + a_1)$</td><td>$n_0(q^{2l-1}t_0t_1)$</td></tr>
<tr><td>
</td><td>$\rho(2lc - a_1)$</td><td>1</td><td>$-\psi_0^+(q^{2l-1}t_0t_1)$</td></tr>
<tr><td>
</td><td>
</td><td>$\rho(2lc - a_1)$</td><td>1</td><td>$\psi_0^-(qt_0t_1)$</td></tr>
</table>

Table 2.3.1: Colored alcove walks in Proposition 2.3.1.3

$$G_l := \frac{g_l \gamma_{l-1}}{\gamma_1 \gamma_l}, \quad g_l := (1-q^l)\frac{(1-q^{l-1}ab)(1-q^{l-1}ac)(1-q^{l-1}ad)(1-q^{l-1}bc)(1-q^{l-1}bd)(1-q^{l-1}cd)}{(1-q^{2l-2}\pi)(1-q^{2l-1}\pi)},$$
$$\pi := abcd, \quad s := a+b+c+d, \quad s' := a^{-1}+b^{-1}+c^{-1}+d^{-1},$$
$$\gamma_l := (q^{l-1}\pi;q)_l = (1-q^{l-1}\pi)(1-q^l\pi)\cdots(1-q^{2l-2}\pi).$$

In the case $l=0$, we have $\rho(-a_1)=0$, and thus $F_1=0$. If we define $p_l(z)$ by the relation $P_l(x) = \gamma_l^{-1} p_l((x+x^{-1})/2)$, then the relation (2.3.3) can be rewritten as

$$2zp_l(z) = h_l p_{l+1}(z) + f_l p_l(z) + g_l p_{l-1}(z), \quad h_l := \frac{1-q^{l-1}\pi}{(1-q^{2l-1}\pi)(1-q^{2l}\pi)}, \quad p_0(z)=1, \ p_{-1}(z)=0.$$

This recurrence formula is nothing but the one in [AW85, (1.24)–(1.27)]. Thus p_l coincides with the original Askey-Wilson polynomial in [AW85], and in particular, it can be expressed as a q-hypergeometric series.

So far we studied Pieri coefficients. Next we study the general LR coefficients for Askey-Wilson polynomials.

Corollary 2.3.1.5. For dominant weights $l\omega$ and $m\omega$ in $(\mathfrak{h}^*_{\mathbb{Z}})_+$, $l,m \in \mathbb{N}$, we have

$$P_{l\omega}(x)P_{m\omega}(x) = \sum_{v\in W_0} \sum_{p\in\Gamma_2^C(\overrightarrow{\mathrm{t}((l-1)\omega)s_0}, \mathrm{t}(m\omega)s_1 v)} A_p^{AW} B_p^{AW} C_p P_{\mathrm{wt}(p)}(x),$$

where the terms A_p^{AW} and B_p^{AW} are given by

$$A_p^{AW} := \begin{cases} \rho(2mc-a_1) & (v=e) \\ 1 & (v=s_1) \end{cases}, \quad B_p^{AW} := \begin{cases} \rho(-\ell(e(p))c+a_1) & (\ell(e(p))\in 2\mathbb{Z}) \\ 1 & (\ell(e(p))\notin 2\mathbb{Z}) \end{cases}$$

with $\rho(a)$ in Proposition 2.3.1.3, and C_p is given in Theorem 2.2.4.2.

Proof. We apply the formula

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v\in W^\mu} \sum_{p\in\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(vw(\mu))^{-1})} A_p B_p C_p P_{-w_0.\mathrm{wt}(p)}(x)$$

in Theorem 2.2.4.2 to the case $\lambda = l\omega$ and $\mu = m\omega$. Similarly as in Proposition 2.3.1.3, we have $W_{l\omega} = \{e\}$ and $W^{m\omega} = \{e, s_1\} = W_0$. Using $\mathrm{t}(l\omega) = (s_0s_1)^l$ and $\mathrm{t}(m\omega) = (s_0s_1)^m$, we have $w(l\omega) = \mathrm{t}(l\omega)s_1 = (s_0s_1)^{l-1}s_0$ and $w(m\omega) = (s_0s_1)^{m-1}s_0$. Therefore the range of the sum of alcove walks in the right hand side becomes

$$\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(vw(\mu))^{-1}) = \Gamma_2^C(\overrightarrow{\mathrm{t}((l-1)\omega)s_0}, \mathrm{t}(m\omega)s_1v) \quad (v\in W_0).$$

As for the denominator $t_{w_{l\omega}}^{-\frac{1}{2}} W_{l\omega}(t)$, we have by $w_{l\omega} = e$ that $t_{w_{l\omega}}^{-\frac{1}{2}} W_{l\omega}(t) = t_e^{-\frac{1}{2}} t_e = 1$.

Now we study the factors A_p and B_p, and want to reduce the ranges of the products. First, as for the product $A_p = \prod_{a\in\mathcal{L}((vw(\mu))^{-1},\mathrm{t}(-w_0\mu))} \rho(a)$, the longest element $w_0 \in W_0$ is s_1 and $\mathrm{t}(\mu) = (s_0s_1)^l = w(\mu)^{-1}s_0$. Thus, in the case $v=e$, we have

$$\mathcal{L}((vw(\mu))^{-1},\mathrm{t}(-w_0\mu)) = \mathcal{L}(w(\mu)^{-1},\mathrm{t}(\mu)) = \{2mc - a_1\}.$$

In the case $v = s_1$, we have

$$\mathcal{L}((vw(\mu))^{-1},\mathrm{t}(-w_0\mu)) = \mathcal{L}(w(\mu)s_1,\mathrm{t}(\mu)) = \mathcal{L}(\mathrm{t}(\mu),\mathrm{t}(\mu)) = \emptyset.$$

Hence A_p is equal to A_p^{AW} in the claim.

Next we consider the product $B_p = \prod_{a\in\mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0,e(p))} \rho(-a)$. we separate the argument according to whether the length $\ell(e(p))$ of $e(p)$ is even or odd. In the case $\ell(e(p))$ is even, there is $k\in\mathbb{N}$ such that $e(p) = (s_0s_1)^n = \mathrm{t}(k\omega)$, $0\le k\le m$. In this case, the range of the product is

$$\mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0, e(p)) = \mathcal{L}(\mathrm{t}(k\omega)s_1,\mathrm{t}(k\omega)) = \{2kc - a_1\}.$$

In the case $\ell(e(p))$ is odd, there is $k\in\mathbb{N}$ such that we can write $e(p) = (s_0s_1)^{k-1}s_0 = \mathrm{t}(k\omega)s_1$, $1\le k\le m$. Thus the range of the product is

$$\mathcal{L}(\mathrm{t}(\mathrm{wt}(p))w_0, e(p)) = \mathcal{L}(\mathrm{t}(k\omega)s_1,\mathrm{t}(k\omega)s_1) = \emptyset.$$

Therefore B_p is equal to B_p^{AW} in the claim. □

2.3.2 Hall-Littlewood limit

In the case of type A_n, the specialized Macdonald polynomials $P_\lambda^{A_n}(x; q = 0, t)$ coincide with Hall-Littlewood polynomials. Motivated by this fact, Yip calls in [Yi12, §4.5] the specialized Macdonald polynomials in the untwisted cases at $q = 0$ *Hall-Littlewood polynomials*, and derived a simplified formula of LR coefficients. Following Yip's terminology, let us call the specialized Koornwinder polynomials

$$P_\lambda(x;t) \coloneqq P_\lambda(x; q = 0, t_0, t, t_n, u_0, u_n)$$

the Hall-Littlewood limit.

Proposition 2.3.2.1 (c.f. [Yi12, Corollary 4.13])**.** Let us given dominant weights $\lambda, \mu \in \Lambda_+$ and a reduced expression $w(\lambda) = s_{i_r} \cdots s_{i_1}$ of the shortest element $w(\lambda)$ (1.3.36). Then we have

$$\begin{aligned} P_\lambda(x;t)P_\mu(x;t) &= \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_+^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})} F_p(t) P_{-w_0.\mathrm{wt}(p)}(x;t), \\ F_p(t) &\coloneqq \prod_{a \in \mathcal{L}((vw(\mu))^{-1}, t(-w_0\mu))} t_a^{\frac{1}{2}} \prod_{a \in \mathcal{L}(t(\mathrm{wt}(p))w_0, e(p))} t_a^{-\frac{1}{2}} \\ &\quad \times \prod_{k \in \varphi_+(p),\ a_{i_k} \notin W.a_0} (t_{a_{i_k}}^{-\frac{1}{2}} - t_{a_{i_k}}^{\frac{1}{2}}) \prod_{k \in \varphi_+(p),\ a_{i_k} \in W.a_0} (u_n^{-\frac{1}{2}} - u_n^{\frac{1}{2}}). \end{aligned}$$

Here $\Gamma_+^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})$ is the subset of $\Gamma^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})$ consisting of alcove walks whose foldings are positive.

Proof. We denote the coefficient in Theorem 2.2.4.2 by

$$a_p(q,t) \coloneqq A_p B_p C_p.$$

First, we show that if $a_p(0,t) \neq 0$ for a colored alcove walk $p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})$, then all the foldings of p are gray and positive. We assume that the k-th step of p is a gray negative folding. Then, as for the factor $C_{p,k} = -\psi_{i_k}^-(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))})$ we have $C_{p,k}|_{q=0} = 0$. In fact, we have

$$\psi_{i_k}^-(z) = \frac{(t_{i_k}^{\frac{1}{2}} - t_{i_k}^{-\frac{1}{2}}) + z^{-\frac{1}{2}}(u_{i_k}^{\frac{1}{2}} - u_{i_k}^{-\frac{1}{2}})}{1 - z^{-1}} = \frac{z(t_{i_k}^{\frac{1}{2}} - t_{i_k}^{-\frac{1}{2}}) + z^{\frac{1}{2}}(u_{i_k}^{\frac{1}{2}} - u_{i_k}^{-\frac{1}{2}})}{1 - z},$$

and by substituting $z = q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}$ and $q = 0$ we have $C_{p,k}|_{q=0} = 0$. Thus we showed that no gray negative folding contributes to $a_p(0,t)$.

Next we show that black foldings of p don't contribute to $a_p(0,t)$. Note that there exists an alcove walk $l \in \Gamma(\overrightarrow{w(\lambda)}, e)$ whose steps are crossings since we fixed a reduced expression of $w(\lambda)$. Moreover all the steps of l are positive. Then we find that any alcove walk in $\Gamma_2^C(\overrightarrow{w(\lambda)}, e) \setminus \{l\}$ has a negative folding. In other words, if an alcove walk $p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (vw(\mu))^{-1})$ has a black folding, then p^* in (2.2.16) has at least one negative folding. Then, as for the factor $C_{p,k} = -\psi_{i_k}^-(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)})$, we have $C_{p,k}|_{q=0} = 0$ by a direct calculation. Thus, no black folding contributes to $a_p(0,t)$.

By the discussion so far, we find that neither colored folding contributes to $a_p(0,t)$. Thus, the set of alcove walks effective to the sum is $\{p \in \Gamma^C(\overrightarrow{w(\mu)}^{-1}, (v.w(\lambda))^{-1}) \mid \varphi(p) = \varphi_+(p)\}$.

Specializing $q = 0$ in A_p, B_p and C_p, we have

$$\begin{aligned} A_p|_{q=0} &= \prod_{a \in \mathcal{L}((vw(\mu))^{-1}, t(-w_0\mu))} t_a^{\frac{1}{2}}, \quad B_p|_{q=0} = \prod_{a \in \mathcal{L}(t(\mathrm{wt}(p))w_0, e(p))} t_a^{-\frac{1}{2}}, \\ C_p|_{q=0} &= \prod_{k \in \varphi_+(p),\ a_{i_k} \notin W.a_0} (t_{a_{i_k}}^{-\frac{1}{2}} - t_{a_{i_k}}^{\frac{1}{2}}) \prod_{k \in \varphi_+(p),\ a_{i_k} \in W.a_0} (u_n^{-\frac{1}{2}} - u_n^{\frac{1}{2}}). \end{aligned}$$

Therefore the claim is obtained. ∎

2.3.3 Examples in rank 2

Finally, as explicit examples of LR coefficients in Theorem 2.2.4.2, we calculate the product $P_\lambda(x)P_\mu(x)$ of Koornwinder polynomials of rank 2.

We write down the root system of rank 2. The root system of type C_2 is

$$R := \{\pm\epsilon_1 \pm \epsilon_2\} \cup \{\pm 2\epsilon_1, \pm 2\epsilon_2\} \subset V^* := \mathbb{R}\epsilon_1 \oplus \mathbb{R}\epsilon_2,$$

the simple roots are $a_1 = \epsilon_1 - \epsilon_2$ and $a_2 = 2\epsilon_2$, and the fundamental weights are $\omega_1 = \epsilon_1$ and $\omega_2 = \epsilon_1 + \epsilon_2$. The weight lattice is $\Lambda = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 \subset V^*$, and the set of dominant weights is $\Lambda_+ = \{\lambda_1\epsilon_1 + \lambda_2\epsilon_2 \in \mathfrak{h}^*_{\mathbb{Z}} \mid \lambda_1 \geq \lambda_2 \geq 0\}$. The finite Weyl group W_0 is the hyper-octahedral group of order 8 generated by $s_1 := s_{a_1}$ and $s_2 := s_{a_2}$. The longest element of W_0 is $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$.

The affine root system of type $(C_2^\vee, C_2)$ is

$$S = \left\{\pm 2\epsilon_i + kc, \pm\epsilon_i + \frac{1}{2}kc \mid k \in \mathbb{Z}, i = 1, 2\right\} \cup \{\pm\epsilon_1 \pm \epsilon_2 + kc \mid k \in \mathbb{Z}\},$$

and the affine simple root is $a_0 = c - 2\epsilon_1$. The extended affine Weyl group W is generated by s_1, s_2 and $s_0 := s_{a_0}$, and the decomposition $W = \mathrm{t}(\Lambda) \rtimes W_0$ (1.3.8) is a semi-direct product of $\mathrm{t}(\Lambda) = \langle \mathrm{t}(\epsilon_1), \mathrm{t}(\epsilon_2)\rangle \simeq \mathbb{Z}^2$ and $W_0 = \langle s_1, s_2\rangle \simeq \{\pm 1\}^2 \rtimes \mathfrak{S}_2$. The elements $\mathrm{t}(\epsilon_1)$ and $\mathrm{t}(\epsilon_2)$ have reduced expressions $\mathrm{t}(\epsilon_1) = s_0s_1s_2s_1$ and $\mathrm{t}(\epsilon_2) = s_1s_0s_1s_2$ respectively.

In this setting we apply Theorem 2.2.4.2 to the case $\lambda = \omega_1$ and $\mu = \omega_2$. The result is as follows.

Proposition 2.3.3.1. For Koornwinder polynomials of rank 2, we have

$$\begin{aligned} P_{\omega_1}(x)P_{\omega_2}(x) &= P_{\omega_1+\omega_2}(x) + FP_{\omega_2}(x) + GP_{\omega_1}(x), \\ F &:= \rho(-2c + (\epsilon_1+\epsilon_2))\rho(-2c + 2\epsilon_1)\rho(-(\epsilon_1 - \epsilon_2))(-\psi_0^-(q^3t_0t_1) + \psi_0^-(qt_0t_1)) \\ G &:= \rho(2c - (\epsilon_1+\epsilon_2))\rho(2c - 2\epsilon_2)\rho(-2\epsilon_2)\rho(-c + (\epsilon_1+\epsilon_2))n_0(qt_0t_1) \end{aligned}$$

Proof. Applying Theorem 2.2.4.2 to $n = 2$, $\lambda = \omega_1$ and $\mu = \omega_2$, we have

$$P_{\omega_1}(x)P_{\omega_2}(x) = \frac{1}{t_{w_{\omega_1}}^{-\frac{1}{2}}W_{\omega_1}(t)} \sum_{v \in W^{\omega_2}} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\omega_1)}^{-1}, (vw(\omega_2))^{-1})} A_pB_pC_pP_{-w_0.\mathrm{wt}(p)}(x).$$

We have $W_{\omega_1} = \{e, s_2\}$, $W^{\omega_2} = \{e, s_2, s_1s_2, s_2s_1s_2\}$ and $w(\omega_1) = s_0$, $w(\omega_2) = s_0s_1s_0$. The denominator $t_{w_{\omega_1}}^{-\frac{1}{2}}W_{\omega_1}(t)$ can be calculated with the help of $w_{\omega_1} = s_2$ as $t_{w_{\omega_1}}^{-\frac{1}{2}}W_{\omega_1}(t) = t_{s_2}^{-\frac{1}{2}}(t_e + t_{s_2}) = t_2^{-\frac{1}{2}} + t_2^{\frac{1}{2}}$.

Next we consider the term $A_pB_pC_p$. The alcove walk p^* associated to $p \in \Gamma_2^C(\overrightarrow{w(\omega_1)}^{-1}, (vw(\omega_2))^{-1})$ is given by either p_1^* or p_2^* in Table 2.3.2.

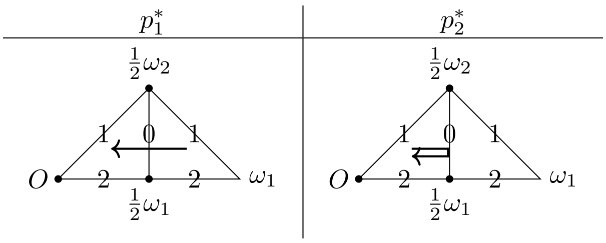


Table 2.3.2: Classification of p^*

Let us calculate the term $A_p = \prod_a \rho(a)$. The range of the product is

$$w(\mu)^{-1}\mathcal{L}(v^{-1}, v_\mu^{-1}) = \mathcal{L}\big((vw(\omega_2))^{-1}, \mathrm{t}(-w_0\omega_2)\big),$$

and according to $v \in W^{\omega_2} = \{e, s_2, s_1s_2, s_2s_1s_2\}$ it is given by

$$\mathcal{L}\big((vw(\omega_2))^{-1}, \mathrm{t}(-w_0\omega_2)\big) = \begin{cases} \{2c - 2\epsilon_1, 2c - (\epsilon_1+\epsilon_2), 2c - 2\epsilon_2\} & (v = e) \\ \{2c - (\epsilon_1+\epsilon_2), 2c - 2\epsilon_2\} & (v = s_2) \\ \{2c - 2\epsilon_2\} & (v = s_1s_2) \\ \emptyset & (v = s_2s_1s_2). \end{cases}$$

Then we have

$$A_p = \begin{cases} \rho(2c - 2\epsilon_1)\rho(2c - (\epsilon_1 + \epsilon_2))\rho(2c - 2\epsilon_2) & (v = e) \\ \rho(2c - (\epsilon_1 + \epsilon_2))\rho(2c - 2\epsilon_2) & (v = s_2) \\ \rho(2c - 2\epsilon_2) & (v = s_1 s_2) \\ 1 & (v = s_2 s_1 s_2) \end{cases}.$$

For each $v \in W^{\omega_2}$ and the corresponding colored alcove walks p, we calculate B_p and C_p. The results are shown in Tables 2.3.3–2.3.6. The symbol in the column of p such as X_{11} and X_{12} refers to the corresponding picture in Figure 2.3.1.

The claim is now obtained by summing the terms $A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x)$. □

p^*	p	B_p	C_p	$-w_0\mathrm{wt}(p)$
p_1^*	X_{11}	$\rho(-2\epsilon_2)\rho(-c + (\epsilon_1 + \epsilon_2))$	$n_0(qt_0t_1)$	ω_1
	X_{12}	$\rho(-(\epsilon_1 - \epsilon_2))$	$-\psi_0^+(qt_0t_1)$	ω_2
p_2^*	X_2	$\rho(-(\epsilon_1 - \epsilon_2))$	$\psi_0^+(qt_0t_1)$	ω_2

Table 2.3.3: Colored alcove walks in the case $v = e$

p^*	p	B_p	C_p	$-w_0\mathrm{wt}(p)$
p_1^*	Y_{11}	$\rho(-2\epsilon_2)\rho(-2c + 2\epsilon_1)\rho(-c + (\epsilon_1 + \epsilon_2))$	$n_0(qt_0t_1)$	ω_1
	Y_{12}	$\rho(-(\epsilon_1 - \epsilon_2))\rho(-2c + 2\epsilon_1)$	$-\psi_0^+(qt_0t_1)$	ω_2
p_2^*	Y_2	$\rho(-(\epsilon_1 - \epsilon_2))\rho(-2c + 2\epsilon_1)$	$\psi_0^+(qt_0t_1)$	ω_2

Table 2.3.4: Colored alcove walks in the case $v = s_2$

p^*	p	B_p	C_p	$-w_0\mathrm{wt}(p)$
p_1^*	Z_{11}	1	1	$\omega_1 + \omega_2$
	Z_{12}	$\rho(-2c + (\epsilon_1 + \epsilon_2))\rho(-2c + 2\epsilon_1)\rho(-(\epsilon_1 - \epsilon_2))$	$-\psi_0^-(q^3t_0t_1)$	ω_2
p_2^*	Z_2	$\rho(-2c + (\epsilon_1 + \epsilon_2))\rho(-2c + 2\epsilon_1)\rho(-(\epsilon_1 - \epsilon_2))$	$\psi_0^-(qt_0t_1)$	ω_2

Table 2.3.5: Colored alcove walks in the case $v = s_1 s_2$

p^*	p	B_p	C_p	$-w_0\mathrm{wt}(p)$
p_1^*	W_{11}	$\rho(2c - 2\epsilon_2)$	1	$\omega_1 + \omega_2$
	W_{12}	$\rho(-2c + 2\epsilon_2)\rho(-2c + (\epsilon_1 + \epsilon_2))\rho(-2c + 2\epsilon_1)\rho(-(\epsilon_1 - \epsilon_2))$	$-\psi_0^-(q^3t_0t_1)$	ω_2
p_2^*	W_2	$\rho(-2c + 2\epsilon_2)\rho(-2c + (\epsilon_1 + \epsilon_2))\rho(-2c + 2\epsilon_1)\rho(-(\epsilon_1 - \epsilon_2))$	$\psi_0^-(qt_0t_1)$	ω_2

Table 2.3.6: Colored alcove walks in the case $v = s_2 s_1 s_2$

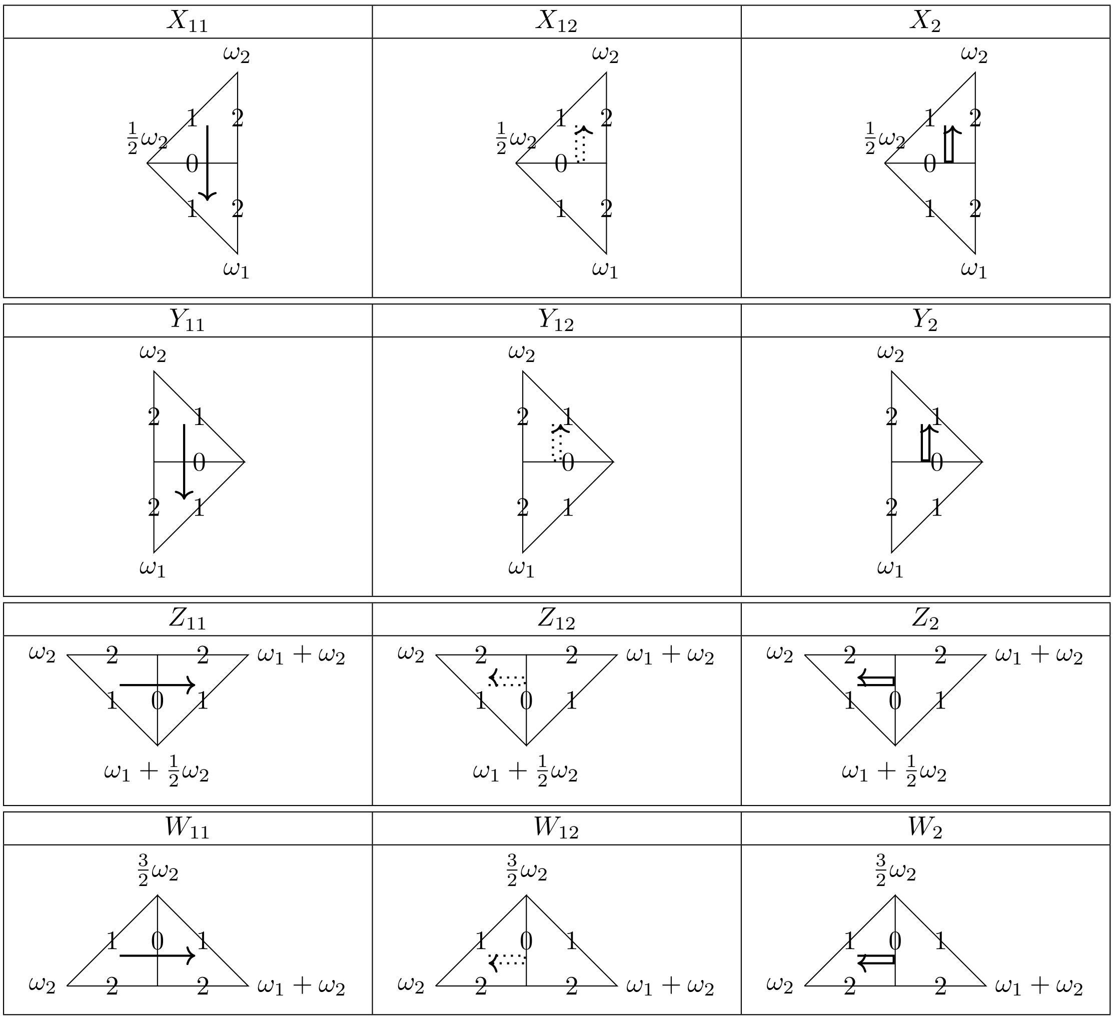


Figure 2.3.1: Colored alcove walks in Proposition 2.3.3.1

Chapter 3

Specialization of Koornwinder polynomials

Chapter 3 is based on the publication [YY22], co-authored with S. Yanagida.

3.0 Introduction

In [M03, p.12], Macdonald gives a comment that the affine root system of type $(C_n^\vee, C_n)$ has as its subsystem all the non-reduced affine root systems and the classical affine root systems of type B_n, $B_n^\vee$, C_n, $C_n^\vee$, BC_n and D_n. Also, at [M03, (5.17)], he comments that an appropriate specialization of parameters in the Koornwinder polynomials yields the Macdonald polynomials associated to the corresponding subsystem. Seemingly, the detailed explanation of such parameter specialization is not given in literature. The aim of this chapter is to clarify this point.

We will use the notation of parameters (other than q) of Koornwinder polynomials introduced by Noumi in [N95]:

$$t, t_0, t_n, u_0, u_n. \tag{3.0.1}$$

Let us call them *the Noumi parameters* for distinction. The details will be explained in § 3.1.2.

Now we can explain the main result of this chapter.

Theorem 3.0.0.1 (Propositions 3.1.3.1, 3.1.4.1–3.1.4.9)**.** For each type X listed in Table 3.0.1 and for each (not necessarily) dominant weight μ of type C_n, the specialization of the Noumi parameters in the (non-symmetric) Koornwinder polynomial with weight μ yields the (non-symmetric) Macdonald polynomial with μ of type X in the sense of Definition 1.3.1.1.

reduced		t	t_0	t_n	u_0	u_n	non-reduced		t	t_0	t_n	u_0	u_n
B_n	§3.1.4	t_l	1	t_s	1	t_s	(BC_n, C_n)	§3.1.4	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	§3.1.4	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	§3.1.4	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	§3.1.3	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	§3.1.4	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	§3.1.4	t_l	t_s	t_s	t_s	t_s							
BC_n	§3.1.4	t_m	t_l^2	t_s	1	t_s							
D_n	§3.1.4	t	1	1	1	1							

Table 3.0.1: Specialization table

Hereafter we refer Table 3.0.1 as the *specialization table*.

Let us explain how to read Theorem Theorem 3.0.0.1 and the specialization Table 3.0.1 in the case of type C_n. The associated Macdonald polynomial has the parameters q and two kinds of t's. The latter correspond to the two orbits of the extended affine Weyl group acting on the affine root system of type

C_n, and we denote them by t_s and t_l. Using them, we denote the symmetric Macdonald polynomial of type C_n by $P^C_\mu(x; q, t_s, t_l)$ with dominant weight μ. See § 3.1.3 for the detail of these symbols for type C_n. We also have the Koornwinder polynomial $P_\mu(x; q, t, t_0, t_n, u_0, u_n)$ with the same dominant weight, whose detail will be explained in § 3.1.2. Then, specializing the Noumi parameters as indicated in the type C_n row in Table 3.0.1, we obtain $P^C_\mu(x; q, t_s, t_l)$. In other words, the following identity holds.

$$P^C_\mu(x; q, t_s, t_l) = P_\mu(x; q, t_s, t_l^2, t_l^2, 1, 1). \tag{3.0.2}$$

See Proposition 3.1.3.1 for the detail of type C_n.

We derive each of the specializations in §3.1.3 and §3.1.4, as indicated in the specialization Table 3.0.1. Our argument is based on the fact that each family of Macdonald-Koornwinder polynomials is uniquely determined by the inner product. Thus, the desired specialization will be obtained by studying the degeneration of the weight function of the inner product, which is actually described in the formula [M03, (5.1.7)]. See (3.1.27) for the precise statement. As commented at [M03, (5.1.7)], all we have to do is to take care the correspondence of the orbits of the extended affine Weyl group.

In § 3.2, as a verification of the specializing Table 3.0.1, we check the obtained specializations by using explicit formulas of Macdonald-Koornwinder polynomials. We focus on *Ram-Yip type formulas* [RY11, OS18] which were mentioned before. These formulas give explicit description of the coefficients in the monomial expansion of non-symmetric Macdonald-Koornwinder polynomials as a summation of terms over the so-called *alcove walks*, the notion introduced by Ram [Ra06]. We do this check for Ram-Yip formulas of type B, C and D in the sense of [RY11]. The check is done just in case-by-case calculation, but since the situation is rather complicated due to the notational problem of affine root systems and parameters, we believe that it has some importance. The result is as follows.

Theorem 3.0.0.2 (Propositions 3.2.2.4, 3.2.1.5 and 3.2.3.5)**.** For each $\mu \in P_{C_n}$, we have

$$\begin{aligned} E_\mu(x; q, t_m^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}) &= E^{B,\mathrm{RY}}_\mu(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}}), \\ E_\mu(x; q, t_m^{\mathrm{RY}}, 1, t_s^{\mathrm{RY}}, 1, 1) &= E^{C,\mathrm{RY}}_\mu(x; q, t_s^{\mathrm{RY}}, t_m^{\mathrm{RY}}), \\ E_\mu(x; q, t, 1, 1, 1, 1) &= E^{D,\mathrm{RY}}_\mu(x; q, t). \end{aligned}$$

Here the left hand sides denote specializations of the non-symmetric Koornwinder polynomials $E_\mu(x)$, and the right hand side denotes the non-symmetric Macdonald polynomials of type B, C and D in the sense of [RY11]. For the detail, see the beginning of §3.2 for the explanation. Comparing these identities with the specialization Table 3.0.1, we find that $E^{B,\mathrm{RY}}_\mu(x)$ is equivalent to the polynomial of type B_n, $E^{C,\mathrm{RY}}_\mu(x)$ is to that of type $C_n^\vee$, and $E^{D,\mathrm{RY}}_\mu(x)$ is to that of type D_n in the sense of Definition 1.3.1.1.

3.1 Specialization table of Koornwinder polynomials

The aim of this section is to give the detail of the specialization Table 3.0.1. As explained in §3.0, we use the affine root systems in the sense of Macdonald [M71, M03]. Our main system is that of type $(C_n^\vee, C_n)$, which will be denoted by S. See (1.3.5) for the precise definition. According to the list of affine root systems in [M03, §1.3], those in Table 3.0.1 are subsystems of S. Explicitly, the following types are the subsystems of type $(C_n^\vee, C_n)$.

$$B_n,\ B_n^\vee,\ C_n,\ C_n^\vee,\ D_n\ BC_n,\ (BC_n, C_n),\ (C_n^\vee, BC_n),\ (B_n^\vee, B_n). \tag{3.1.1}$$

The details of these subsystems will be explained in § 3.1.3 and § 3.1.4.

3.1.1 Affine root system of type $(C_n^\vee, C_n)$

Let $n \in \mathbb{Z}_{\ge 2}$, and E be the n-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We take and fix an orthonormal basis $\{\epsilon_i \mid i = 1, 2, \ldots, n\}$ of E. Thus, we may identify $E = (V, \langle \cdot, \cdot \rangle)$ with $V = \oplus_{i=1}^n \mathbb{R}\epsilon_i$. Let F be the $\mathbb{R}$-linear space of affine linear functions $E \to \mathbb{R}$. The inner product $\langle \cdot, \cdot \rangle$ yields the isomorphism $F \xrightarrow{\sim} V \oplus \mathbb{R}c$, where c is the constant function $c(v) = 1$ for any $v \in V$. Hereafter we identify F and $V \oplus \mathbb{R}c$ by this isomorphism.

We denote by S the affine root system of type $(C_n^\vee, C_n)$ in the sense of [M03, §1.3, (1.3.18)]. Thus, S is a subset of $F = V \oplus \mathbb{R}c$ given by

$$\begin{aligned} &S = O_1 \sqcup O_2 \sqcup \cdots \sqcup O_5, \\ &O_1 := \{\pm\epsilon_i + rc \mid 1 \le i \le n,\, r \in \mathbb{Z}\},\ O_2 := 2O_1,\ O_3 := O_1 + \tfrac{1}{2}c,\ O_4 := 2O_3 = O_2 + c, \\ &O_5 := \{\pm\epsilon_i \pm \epsilon_j + rc \mid 1 \le i < j \le n,\, r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.2}$$

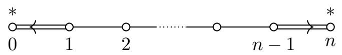
An element of S is called an affine root, or just a root. Following the choice of [Ya22], we consider the affine roots

$$a_0 := -2\epsilon_1 + c, \quad a_j := \epsilon_j - \epsilon_{j+1}\ (1 \le j \le n-1), \quad a_n := 2\epsilon_n. \tag{3.1.3}$$

They form a basis of S in the sense of [M03, §1.2]. Obviously we have

$$\tfrac{1}{2}a_0 \in O_3, \quad a_0 \in O_4, \quad a_j \in O_5 \quad (1 \le j \le n-1), \quad \tfrac{1}{2}a_n \in O_1, \quad a_n \in O_2.$$

Below is the Dynkin diagram cited from [M03, (1.3.18)]. The mark $*$ above the index i implies that $a_i, \frac{1}{2}a_i \in S$.



In fact, the description (3.1.2) gives the orbit decomposition of S by the action of the extended affine Weyl group. For the explanation, we need to introduce more symbols.

The inner product $\langle \cdot, \cdot \rangle$ on V is extended to $F = V \oplus \mathbb{R}c$ by

$$\langle v + rc, w + sc \rangle := \langle v, w \rangle, \quad v, w \in V, \quad r, s \in \mathbb{R}.$$

For a non-constant function $f \in F \setminus \mathbb{R}c$, we define $s_f \in \mathrm{GL}_{\mathbb{R}}(F)$ by

$$F \ni g \longmapsto s_f(g) := g - \langle g, f^\vee \rangle f, \quad f^\vee := \tfrac{2}{\langle f, f \rangle} f.$$

It is the reflection with respect to the hyperplane $H_f := f^{-1}(\{0\}) \subset V$. Now we consider the subset

$$R := \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i \mid 1 \le i \le n\} \subset S \cap V, \tag{3.1.4}$$

which is in fact the finite root system of type C_n. Among the affine roots a_i in (3.1.3), those except a_0 belong to R, which are the simple roots of type C_n. Then the finite Weyl group W_0 is the subgroup

$$W_0 := \langle s_i\ (i = 1, 2, \ldots, n) \rangle \subset \mathrm{GL}_{\mathbb{R}}(V), \quad s_i := s_{a_i}. \tag{3.1.5}$$

Note that each element in W_0 is an isometry for the inner product $\langle \cdot, \cdot \rangle$.

Next, for $v \in V$, we define $\mathrm{t}(v) \in \mathrm{GL}_{\mathbb{R}}(F)$ by

$$F \ni f \longmapsto \mathrm{t}(v)(f) := f - \langle f, v \rangle c. \tag{3.1.6}$$

Then, for $w \in W_0$, we have

$$w\, \mathrm{t}(v)\, w^{-1} = \mathrm{t}(wv). \tag{3.1.7}$$

Let $P_{C_n} \subset F$ be given by

$$P_{C_n} := \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \cdots \oplus \mathbb{Z}\epsilon_n, \tag{3.1.8}$$

which is in fact the weight lattice of the finite root system of type C_n. Then,

$$\mathrm{t}(P_{C_n}) := \{\mathrm{t}(\mu) \mid \mu \in P_{C_n}\} \subset \mathrm{GL}_{\mathbb{R}}(F) \tag{3.1.9}$$

is isomorphic to the additive group P_{C_n}. Viewing (3.1.7) as an action of W_0 on $\mathrm{t}(P_{C_n})$, we can take the semigroup of (3.1.5) and (3.1.9) to obtain the extended affine Weyl group W of type $(C_n^\vee, C_n)$:

$$W := \mathrm{t}(P_{C_n}) \rtimes W_0 \subset \mathrm{GL}_{\mathbb{R}}(F). \tag{3.1.10}$$

It acts on S by permutation [M03, (1.4.6), (1.4.7)], and the orbits are given in (3.1.2) [M03, 1.5].

Let us also give a description of W as an abstract group. We set

$$s_0 := \mathrm{t}(\epsilon_1)s_{2\epsilon_1} \in W. \tag{3.1.11}$$

Then W has a presentation with generators

$$W = \langle s_0, s_1, \ldots, s_n \rangle \tag{3.1.12}$$

and the following relations.

$$\begin{aligned} s_i^2 &= 1 \quad (0 \le i \le n), \\ s_i s_j &= s_j s_i \quad (|i-j|>1), \\ s_j s_{j+1} s_j &= s_{j+1} s_j s_{j+1} \quad (1 \le j \le n-2), \\ s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i \quad (i = 0, n-1). \end{aligned} \tag{3.1.13}$$

Hereafter the length $\ell(w)$ of $w \in W$ indicates that for a reduced expression in terms of the generators $\{s_i\}_{i=0}^n$. For later use, we write down a reduced expression of $\mathrm{t}(\epsilon_i)$:

$$\mathrm{t}(\epsilon_i) = s_{i-1}s_{i-2}\cdots s_1 s_0 s_1 s_2 \cdots s_n s_{n-1} s_{n-2} \cdots s_{i+1} s_i \quad (1 \le i \le n). \tag{3.1.14}$$

Let us also introduce $F_{\mathbb{Z}} \subset F$ by

$$F_{\mathbb{Z}} := P_{C_n} \oplus \tfrac{1}{2}\mathbb{Z}c. \tag{3.1.15}$$

Then we have $S \subset F_{\mathbb{Z}}$. We write down the action of W on $F_{\mathbb{Z}}$:

$$s_0(\epsilon_i) = \begin{cases} c - \epsilon_1 & (i=1) \\ \epsilon_i & (i \neq 1) \end{cases}, \qquad s_j(\epsilon_i) = \begin{cases} \epsilon_j & (i = j+1) \\ \epsilon_j + 1 & (i = j) \\ \epsilon_i & (i \neq j, j+1) \end{cases} \quad (1 \le j \le n-1),$$

$$s_n(\epsilon_i) = \begin{cases} -\epsilon_n & (i=n) \\ \epsilon_i & (i \neq n) \end{cases}, \qquad s_k(c) = c \quad (0 \le k \le n).$$

By these formulas, we can check the orbit decomposition (3.1.2) directly.

Closing this part, we recall the positive and negative parts of S. Let us write S as

$$S = \{\pm\epsilon_i + \tfrac{1}{2}rc,\ \pm 2\epsilon_i + rc \mid 1 \le i \le n,\ r \in \mathbb{Z}\} \cup \{\pm\epsilon_i \pm \epsilon_j + rc \mid 1 \le i < j \le n,\ r \in \mathbb{Z}\}.$$

It has the decomposition $S = S^+ \sqcup S^-$ with the sets $S^{\pm}$ of positive and negative roots, respectively. To describe $S^{\pm}$, let us recall the decomposition of the finite root system R of type C_n (see (3.1.4)) into positive and negative roots:

$$R = R_+ \sqcup R_-, \quad R_+ := \{2\epsilon_i \mid 1 \le i \le n\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\} \subset R, \quad R_- := -R_+.$$

Then, the sets $S_{\pm}$ are given by

$$S_+ := \{a + rc,\ a^{\vee} + \tfrac{1}{2}rc \mid a \in R_+,\ r \in \mathbb{N}\} \cup \{a + rc,\ a^{\vee} + \tfrac{1}{2}rc \mid a \in R_-,\ r \in \mathbb{N}\}, \quad S_- := -S_+.$$

Using (3.1.3), we have $a_i \in S^+$ for each $i = 0, 1, \ldots, n$. Moreover we have

$$S_+ = \sum_{i=0}^{n} \mathbb{N}a_i \setminus \{0\}. \tag{3.1.16}$$

We also define $\overline{S}, \overline{S}_{\pm} \subset S$ by

$$\overline{S} := \overline{S}_+ \sqcup \overline{S}_-, \quad \overline{S}_+ := \{\epsilon_i, 2\epsilon_i \mid 1 \le i \le n\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}, \quad \overline{S}_- := -\overline{S}_+. \tag{3.1.17}$$

Then, any $a \in S$ can be presented as $a = a + rc$ with $a \in \overline{S}$ and $r \in \frac{1}{2}\mathbb{Z}$, and we denote

$$\overline{a} := a \in \overline{S}. \tag{3.1.18}$$

3.1.2 Parameters, weight function and Koornwinder polynomials

In this subsection, we explain the parameters and the weight function for type $(C_n^\vee, C_n)$, and introduce the symmetric and non-symmetric Koornwinder polynomials. As for the parameters of Koornwinder polynomials, we mainly use *the Noumi parameters* in [N95], as mentioned in § 3.0. Due to the necessity in the specialization argument, we also give a summary of the comparison of the Noumi parameters with those given by Macdonald in [M03], which we will refer as *the Macdonald parameters*.

We begin with explanation on the parameters in [M03]. Let us write again the W-orbits (3.1.2) in $S = O_1 \sqcup \cdots \sqcup O_5$ and the affine roots a_i in (3.1.3):

$$O_1 \coloneqq \{\pm\epsilon_i + rc \mid 1 \le i \le n,\, r \in \mathbb{Z}\}, \quad O_2 \coloneqq 2O_1, \quad O_3 \coloneqq O_1 + \tfrac{1}{2}c, \quad O_4 \coloneqq 2O_3 = O_2 + c,$$
$$O_5 \coloneqq \{\pm\epsilon_i \pm \epsilon_j + rc \mid 1 \le i < j \le n,\, r \in \mathbb{Z}\}.$$
$$a_0 \coloneqq -2\epsilon_1 + c, \quad a_j \coloneqq \epsilon_j - \epsilon_{j+1}\ (1 \le j \le n-1), \quad a_n \coloneqq 2\epsilon_n,$$
$$\tfrac{1}{2}a_0 \in O_3, \quad a_0 \in O_4, \quad a_j \in O_5 \quad (1 \le j \le n-1), \quad \tfrac{1}{2}a_n \in O_1, \quad a_n \in O_2.$$

We attach a parameter $k_r \in \mathbb{R}$ to each W-orbit as

$$k_r \longleftrightarrow O_r \quad (r = 1, 2, \ldots, 5), \tag{3.1.19}$$

and define the label k [M03, §1.5] as a map on given by

$$k : S \longrightarrow \mathbb{R}, \quad k(a) \coloneqq k_r \quad \text{for } a \in O_r. \tag{3.1.20}$$

Let $q \in \mathbb{R}$ be chosen, and define the set of parameters as

$$\{q^{k(a)} \mid a \in S\} = \{q^{k_1}, q^{k_2}, \ldots, q^{k_5}\}. \tag{3.1.21}$$

We call q^{k_r}'s *the Macdonald parameters.* These are used in the formulation of Koornwinder polynomials in [M03].

As mentioned at (3.0.1) and in the beginning of this § 3.1.2, in the following argument, we will mainly use *the Noumi parameters*

$$t, t_0, t_n, u_0, u_n$$

introduced in [N95]. As will be shown in § 3.1.2 below, we have the following relation between the Macdonald parameters and the Noumi parameters.

$$(q^{2k_1}, q^{2k_2}, q^{2k_3}, q^{2k_4}, q^{k_5}) = (t_n u_n, \tfrac{t_n}{u_n}, t_0 u_0, \tfrac{t_0}{u_0}, t). \tag{3.1.22}$$

Restating by (3.1.19), the Noumi parameters and the W-orbits correspond in the way

$$t_n u_n \longleftrightarrow O_1, \quad t_n/u_n \longleftrightarrow O_2, \quad t_0 u_0 \longleftrightarrow O_3, \quad t_0/u_0 \longleftrightarrow O_4, \quad t \longleftrightarrow O_5. \tag{3.1.23}$$

Now we introduce the base field for (non-symmetric) Koornwinder polynomials. Adding the square $t^{1/2}, t_i^{1/2}, u_i^{1/2}$ of the Noumi parameters and the new parameter $q^{1/2}$, we define the base field $\mathbb{K}$ to be the rational function field

$$\mathbb{K} \coloneqq \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_n^{\frac{1}{2}}). \tag{3.1.24}$$

Next, following [M03, §5.1], we explain the weight function for (non-symmetric) Koornwinder polynomials. Using the exponent e in the sense of [M03, (1.4.5)], which is given by $e = 1$ in our $(C_n^\vee, C_n)$ case, we set $c_0 \coloneqq e^{-1} \cdot \sum_{i=0}^n a_i = \frac{1}{2}c$. Here we used the affine roots a_i in (3.1.3). Also, using $L \coloneqq P_{C^n}$, we set

$$\Lambda \coloneqq L \oplus \mathbb{Z}c_0 = P_{C^n} \oplus \tfrac{1}{2}\mathbb{Z} = \oplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \tfrac{1}{2}\mathbb{Z}.$$

Note that we have $S \subset \Lambda$. For each $f = \mu + rc_0 \in \Lambda$, we define

$$e^f \coloneqq e^\mu q^{r/e} = e^\mu q^r. \tag{3.1.25}$$

Then, for the label k in (3.1.20), we define the weight function $\Delta_{S,k}$ [M03, (5.1.7)] as

$$\Delta_{S,k} := \prod_{a\in S^+} \Delta_a = \prod_{a\in S^+} \frac{1-q^{k(2a)}e^a}{1-q^{k(a)}e^a}. \tag{3.1.26}$$

Here we used S^+ in (3.1.16) and set $k(2a) := 0$ if $2a \notin S$. As explained in [M03, (5.1.14)], we can rewrite $\Delta_{S,k}$ as

$$\Delta_{S,k} = \prod_{r=1}^{4} \prod_{a\in S^+\cap O_r} \Delta_\alpha \cdot \prod_{a\in S^+\cap O_5} \Delta_\alpha = \prod_{\alpha\in R_s^+} \frac{(e^{2\alpha}, qe^{-2\alpha}; q)_\infty}{\prod_{r=1}^{4}(v_r e^{\alpha}, v_r' e^{-\alpha}; q)_\infty} \cdot \prod_{\alpha\in R_l^+} \frac{(e^{\alpha}, qe^{-\alpha}; q)}{(q^{k_5}e^{\alpha}, q^{k_5+1}e^{-\alpha}; q)_\infty}.$$

Here R_s^+ and R_l^+ are the set of positive and short roots in the finite root system R of type C_n, respectively. Explicitly, we have

$$R_s^+ := \{\epsilon_i \mid 1 \le i \le n\}, \quad R_l^+ := \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}.$$

We also used the following 4×2 parameters $v_1, \ldots, v_4$ and $v_1', \ldots, v_4'$.

$$(v_1, \ldots, v_4) := (q^{k_1}, -q^{k_2}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}}). \quad (v_1', \ldots, v_4') := (q^{k_1+1}, -q^{k_2+1}, q^{k_3+\frac{1}{2}}, -q^{k_4+\frac{1}{2}}).$$

Finally, as mentioned in the last part of [M03, (5.1.7)], the following relation holds for each subsystem S^0 of the affine root system S.

$$\Delta_{S,k}|_{k(a)-k(2a)=0 \ (a\notin S^0)} = \Delta_{S^0,k}. \tag{3.1.27}$$

For the complete set of the subsystems S^0 in S, see the comment in the beginning of this § 3.1.

The weight function $\Delta_{S,k}$ defines an inner product on the space

$$\mathbb{K}[x^{\pm 1}] = \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}], \quad x_i := e^{\epsilon_i}$$

of the n-variable Laurent polynomials, where in the last part we used (3.1.25). Then, by [M03, §5.2], we have the family of non-symmetric Koornwinder polynomials

$$E_\mu(x) = E_\mu(x; q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}], \quad \mu \in P_{C_n}, \tag{3.1.28}$$

as a unique orthogonal basis of the inner product on $\mathbb{K}[X^{\pm 1}]$ satisfying the triangular property. Moreover, by [M03, §5.3], for a dominant weight μ in P_{C_n}, we have the symmetric Koornwinder polynomial

$$P_\mu(x) = P_\mu(x; q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}]^{W_0}. \tag{3.1.29}$$

Derivation of (3.1.22)

Let us derive the relation (3.1.22) between the Macdonald and Noumi parameters. We use the notation of the affine Hecke algebra given in [M03, Chapter 4]. We make one modification: The base field K is enlarged so that it contains τ_i's and τ_i''s defined blow, and $q^{1/2}$ (in the version of [M03], it contains q but doesn't $q^{1/2}$).

Let q be a real number such that $0 < q < 1$, and K be a subfield of $\mathbb{R}$ containing $q^{1/2}$. We denote by H the affine Hecke algebra associated to the extended affine Weyl group W of (3.1.10) in the sense of [M03, 4.1]. It is an associative K-algebra generated by

$$H = \langle T_0, T_1, \ldots, T_n \rangle \tag{3.1.30}$$

with certain defining relations, for which we refer [Ya22, (2.2.3)–(2.2.5)].

Remark 3.1.2.1. We give another description of the affine Hecke algebra H. As a K-linear space, it has the form

$$H = H_0 \otimes_K K[Y^{t(\epsilon_j)} \mid j = 1, 2, \ldots, n] \simeq H_0 \otimes_K KP_{C_n}, \tag{3.1.31}$$

where H_0 denotes the Hecke algebra associated to the finite Weyl group W_0 of type C_n (see (3.1.5)), and KP_{C_n} denotes the group algebra of the additive group P_{C_n}. The commuting elements $Y^{\mathrm{t}(\epsilon_j)}$'s are defined in [M03, §3.2], and using the reduced expressions (3.1.14), we have the following relations between $Y^{\mathrm{t}(\epsilon_j)}$'s and the generators T_i's in (3.1.30).

$$Y^{\mathrm{t}(\epsilon_j)} = T_{j-1}^{-1} \cdots T_1^{-1} T_0 \cdots T_{n-1} T_n T_{n-1} \cdots T_j.$$

Note that the ordering of T_i's is opposite of those in some literature, for example [Sa99, §2.2, p.399], [I03, §3.1, p.312] and [Chi21]. This discrepancy is reflected on the triangular property of the (non-symmetric) Koornwinder polynomials. Namely, the choices of the ordering on the space $\mathbb{K}[x^{\pm 1}]$ and $\mathbb{K}[x^{\pm 1}]^{W_0}$, where the (non-symmetric) Koornwinder polynomials live, are in opposite between ours and those other literature.

Recall the Macdonald parameters $q^{k_1}, q^{k_2}, \ldots, q^{k_5}$ in (3.1.20). Following [M03, (4.4.3)], we introduce the additional parameters $\kappa_i, \kappa_i' \in K$ for $i = 0, 1, \ldots, n$ as

$$\begin{aligned} &k_1 = k(a_n) = \tfrac{1}{2}(\kappa_n + \kappa_n'), \quad k_2 = k(2a_n) = \tfrac{1}{2}(\kappa_n - \kappa_n'), \\ &k_3 = k(a_0) = \tfrac{1}{2}(\kappa_0 + \kappa_0'), \quad k_4 = k(2a_0) = \tfrac{1}{2}(\kappa_0 - \kappa_0'), \quad k_5 = k(a_j) = \kappa_j = \kappa_j' \quad (1 \le j \le n-1). \end{aligned}$$

We also introduce $\tau_i, \tau_i' \in K$ for $i = 0, 1, \ldots, n$ by

$$\tau_i := q^{\kappa_i/2}, \quad \tau_i' := q^{\kappa_i'/2}.$$

By definition, we have

$$q^{k_1} = \tau_n \tau_n', \quad q^{k_2} = \tau_n/\tau_n', \quad q^{k_3} = \tau_0 \tau_0', \quad q^{k_4} = \tau_0/\tau_0', \quad q^{k_5} = \tau_j \tau_j' \quad (1 \le j \le n-1). \tag{3.1.32}$$

Using the parameters τ_i and τ_i', we explain the basic representation β of H [M03, (4.3.10)], which actually goes back to Lusztig [L89]. It is a faithful representation in the group algebra $A = KL$ of $L := Q_{C_n}^\vee = Q_{B_n} = P_{C_n}$ given by

$$\beta \colon H \longrightarrow \mathrm{End}_K(KP_{C_n}), \quad \beta(T_i) := \tau_i s_i + \mathbf{b}_i(1 - s_i) \quad (0 \le i \le n), \tag{3.1.33}$$

where, expressing the element of KP_{C_n} corresponding to $\alpha \in P_{C_n}$ as e^α, the function $\mathbf{b}_i$ is defined by

$$\mathbf{b}_i = \mathbf{b}(\tau_i, \tau_i'; x^{\alpha_i}) := \frac{\tau_i - \tau_i^{-1} + (\tau_i' - \tau_i'^{-1}) x^{\alpha_i/2}}{1 - x^{\alpha_i}}, \tag{3.1.34}$$

$$x^{\alpha_i} := \begin{cases} x_i/x_{i+1} = e^{\epsilon_i - \epsilon_{i+1}} & (1 \le i \le n-1) \\ q x_1^{-2} = q e^{-2\epsilon_1} & (i = 0) \\ x_n^2 = e^{2\epsilon_n} & (i = n) \end{cases}.$$

Here the symbol $\mathbf{b}(t, u; z)$ is borrowed from [M03, (4.2.1)], and the symbol x^{α_i} is from [Ya22]. Note that the representation β is well defined although the function $\mathbf{b}_i$ does not belong to the group algebra KP_{C_n}.

The relation of the Macdonald and Noumi parameters is obtained by the comparison between the realizations of the basic representation in [M03] and [N95]. Slightly extending the Noumi parameters as

$$t_0, \ t_n, \ u_0, \ u_n, \ t_j := t, \ u_j := 1 \ (j = 1, 2, \ldots, n-1),$$

we define the function $d_i(z)$ for $i = 0, 1, \ldots, n$ as

$$d_i(z) := \frac{t_i^{1/2} - t_i^{-1/2} + (u_i^{1/2} - u_i^{-1/2}) z^{1/2}}{1 - z}.$$

Then, comparing [N95, p.52] and [M03, §4.3] (see also [Ya22, (2.2.8)–(2.2.11)]), we have the relation

$$\mathbf{b}_i = d_i(x^{\alpha_i}). \tag{3.1.35}$$

This relation (3.1.35) yields the correspondence

$$(\tau_0, \tau_n, \tau_0', \tau_n', \tau_j = \tau_j') = (t_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_n^{\frac{1}{2}}, t^{\frac{1}{2}}). \tag{3.1.36}$$

Combining it with (3.1.32), we obtain the relation (3.1.22).

3.1.3 Specialization to affine root system of type C_n

As an illustration of deriving the specialization Table 3.0.1, we explain how to find the parameter specialization for type C_n:

	t	t_0	t_n	u_0	u_n
C_n	t_s	t_l^2	t_l^2	1	1

As mentioned in §3.0, we need to observe the correspondence of the orbits of extended affine Weyl groups of type $(C_n^\vee, C_n)$ and of type C_n. So we start with the explanation on the description of the type C_n as the affine root subsystem of the type $(C_n^\vee, C_n)$.

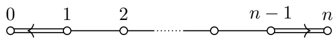
Using the description (3.1.2) of the affine root system S of type $(C_n^\vee, C_n)$, let us consider the following subset S^C of S.

$$\begin{aligned} S^C := O_s^C \sqcup O_l^C, \quad O_l^C &:= O_2 \sqcup O_4 = \{\pm 2\epsilon_i + r \mid 1 \le i \le n,\, r \in \mathbb{Z}\}, \\ O_s^C &:= O_5 = \{\pm \epsilon_i \pm \epsilon_j + r \mid 1 \le i < j \le n,\, r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.37}$$

It is the affine root system of type C_n in the sense of [M03, §1.3, (1.3.4)]. The following gives a basis $\{a_0^C, a_1^C, \ldots, a_n^C\}$ of S^C in the sense of [M03, §1.2].

$$a_0^C := 2a_0 = -2\epsilon_1 + 1, \quad a_j^C := a_j = \epsilon_j - \epsilon_{j+1} \quad (1 \le j \le n-1), \quad a_n^C := 2a_n = 2\epsilon_n,$$

Here is the Dynkin diagram cited from [M03, (1.3.4)]:



The description (3.1.37) gives the decomposition of the extended affine Weyl group W^C. To describe it, recall the finite Weyl group W_0 of type C_n in (3.1.5), which can be rewritten as $W_0 = \langle s_{a_1^C}, s_{a_2^C}, \ldots, s_{a_n^C} \rangle$. We also denote by

$$L' = P_{C_n}^\vee = P_{B_n} := \oplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) \tag{3.1.38}$$

the weight lattice of finite root system of type B_n. Then W^C is given by

$$W^C := W_0 \ltimes \mathrm{t}(L') = W_0 \ltimes \mathrm{t}(P_{B_n}), \tag{3.1.39}$$

and it acts on S^C [M03, §1.4, (1.4.6), (1.4.7)]. The corresponding W^C-orbits are given by the above O_s^C and O_l^C [M03, §1.5]. By (3.1.37), we have $a_0^C, a_n^C \in O_2 \sqcup O_4 = O_l^C$ and $a_j^C \in O_5 = O_s^C$ $(1 \le j \le n-1)$.

Next, we explain the parameters for S^C. Similarly as in § 3.1.2, we attach parameters $k_s^C, k_l^C \in \mathbb{R}$ to the W^C-orbits as

$$k_s^C \longleftrightarrow O_s^C, \quad k_l^C \longleftrightarrow O_l^C, \tag{3.1.40}$$

and define the label $k^C : S^C \to \mathbb{R}$ in the same way as $k : S \to \mathbb{R}$ in (3.1.20). We also denote

$$t_l^C := q^{k_l^C}, \quad t_s^C := q^{k_s^C} \quad (1 \le j \le n-1), \tag{3.1.41}$$

We now argue that under the specialization

$$(t, t_0, t_n, u_0, u_n) \longmapsto \big(t_s^C, (t_l^C)^2, (t_l^C)^2, 1, 1\big),$$

the non-symmetric Koornwinder polynomials degenerate into the non-symmetric Macdonald polynomials of type C_n. Recalling that both polynomials are determined uniquely by the inner products, or by the weight functions, we see that it is enough to check that the weight function $\Delta_{S,k}$ in (3.1.26) of type (C_n, C_n) degenerates to that of type C_n. The latter weight function is given by [M03, (5.1.7)]:

$$\Delta^C = \Delta_{S^C,k^C} := \prod_{a \in (S^C)^+} \frac{1 - q^{k^C(2a)} e^a}{1 - q^{k^C(a)} e^a}.$$

Here $(S^C)^+ \subset S^C$ is the set of positive roots with respect to the basis $\{a_0^C, a_1^C, \ldots, a_n^C\}$, i.e., $(S^C)^+ := \sum_{i=0}^n \mathbb{N}a_i^C \setminus \{0\}$, and $k^C : S^C \to \mathbb{R}$ is the extension of the label k^C (see (3.1.40)) by $k^C(2a) := 0$ $(a \notin S)$. Recalling (3.1.27), we have

$$\Delta_{S,k}|_{k(a)-k(2a)=0 \ (a \in S \setminus S^C)} = \Delta_{S^C,k}.$$

Thus, the desired specialization is given by

$$k(a) - k(2a) \longmapsto 0 \quad (a \in S \setminus S^C), \quad k(a) - k(2a) \longmapsto k^C(a) \quad (a \in S^C). \tag{3.1.42}$$

Since (3.1.37) yields $S \setminus S^C = O_1 \sqcup O_3$, $S^C = O_s^C \sqcup O_l^C$, $O_s^C = O_5$ and $O_l^C = O_2 \sqcup O_4$, the map (3.1.42) can be rewritten in terms of $k_1, k_2, \ldots, k_5$ and k_s^C, k_l^C as

$$k_1 - k_2,\, k_3 - k_4 \longmapsto 0, \quad k_2,\, k_4 \longmapsto k_l^C. \quad k_5 \longmapsto k_s^C.$$

Using (3.1.22) and (3.1.41), and assuming $u_0, u_n > 0$, we can further rewrite it as

$$\begin{aligned} &(t_n u_n)/\tfrac{t_n}{u_n},\, (t_0 u_0)/\tfrac{t_0}{u_0} \longmapsto 1, \quad \tfrac{t_0}{u_0},\, \tfrac{t_n}{u_n} \longmapsto (t_l^C)^2, \quad t \longmapsto t_s^C \\ \iff &\ (t, t_0, t_n, u_0, u_n) \longmapsto \big(t_s^C, (t_l^C)^2, (t_l^C)^2, 1, 1\big). \end{aligned} \tag{3.1.43}$$

Now we suppress the superscript C in t_s^C and t_l^C, and denote by

$$E_\mu^C(x; q, t_s, t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of type C_n (Definition 1.3.1.1). Similarly, for a dominant $\mu \in P_{C_n}$, we denote by $P_\mu^C(x; q, t_s, t_l)$ the symmetric Macdonald polynomials of type C_n . Then the conclusion of this § 3.1.3 is:

Proposition 3.1.3.1. For any $\mu \in P_{C_n}$, we have

$$E_\mu^C(x; q, t_s, t_l) = E_\mu(x; q, t_s, t_l^2, t_l^2, 1, 1).$$

Also, for a dominant weight μ, we have

$$P_\mu^C(x; q, t_s, t_l) = P_\mu(x; q, t_s, t_l^2, t_l^2, 1, 1).$$

The following table shows the comparison of the correspondence (3.1.23) between the Noumi parameters and the W-orbits with that (3.1.40) between the parameters of type C_n and the W^C-orbits.

Type $(C_n^\vee, C_n)$	Type C_n
$t_n u_n \longleftrightarrow O_1$	
$t_0 u_0 \longleftrightarrow O_3$	
$t_n/u_n \longleftrightarrow O_2$ $t_0/u_0 \longleftrightarrow O_4$	$t_l^C \longleftrightarrow O_l = O_2 \sqcup O_4$
$t \longleftrightarrow O_5$	$t_s^C \longleftrightarrow O_s = O_5$

Remark 3.1.3.2. One may wonder whether it is possible to see the specialization (3.1.43) on the level of affine Hecke algebras. To clarify the point, let us denote by H^C the affine Hecke algebra associated to the group W^C (3.1.39) in the sense of [M03, 4.1]. It is an associative algebra over $K \subset \mathbb{R}$ (see § 3.1.2), and as a K-linear space, it has the form $H^C = H_0 \otimes_K K[Y_C^{\lambda'} \mid \lambda' \in P_{B_n}] \simeq H_0 \otimes_K KP_{B_n}$ by [M03, (4.2.7), (4.3.1)]. Here we used similar notation as in (3.1.31). In particular, H_0 is the Hecke algebra associated to the finite Weyl group W_0 of type C_n, and the part $K[Y_C^{\lambda'} \mid \lambda' \in P_{B_n}]$ is a commutative subalgebra. Also, following [M03, (4.4.2)], we define $\tau_{C,i} = \tau'_{C,i} := q^{k_r^C/2}$, where $r := s$ $(a_i \in O_s^C)$ and $r := l$ $(a_i \in O_l^C)$. Then, using the function (3.1.34) with the parameters $\tau_{C,i}$ and $\tau'_{C,i}$ instead of τ_i and τ'_i, we have a faithful H^C-module

$$\beta^C \colon H^C \longhookrightarrow \mathrm{End}_K(KP_{C_n}).$$

which is the basic representation of type C_n. The basic representations β (3.1.33) and β^C sit in the following diagram.

$$
\begin{tikzcd}
H \arrow[d, hook, "\beta"] & H^C \arrow[d, hook, "\beta^C"] \\
\mathrm{End}_K(KP_{C_n}) \arrow[r, equal] & \mathrm{End}_K(KP_{C_n})
\end{tikzcd}
$$

One can see that the specialization (3.1.43) maps $\beta(T_j) \mapsto \beta^C(T_j) \quad (1 \le j \le n-1)$, but the images of $\beta(T_i)$ is not equal to $\beta^C(T_i)$ for $i = 0, n$. Thus, it is unclear whether we can see the specialization on the level of affine Hecke algebras H and H^C.

3.1.4 Specialization to other subsystems

For all the subsystems of the affine root system S of type $(C_n^\vee, C_n)$, we can make similar arguments as in § 3.1.3, which will yield the specialization Table 3.0.1. In this subsection, we list all the arguments except type C_n which is already done. Let us write again the specialization table:

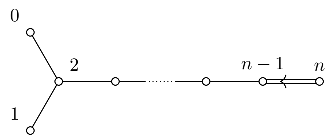
reduced		t	t_0	t_n	u_0	u_n	non-reduced		t	t_0	t_n	u_0	u_n
B_n	§ 3.1.4	t_l	1	t_s	1	t_s	(BC_n, C_n)	§ 3.1.4	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	§ 3.1.4	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	§ 3.1.4	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	§ 3.1.3	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	§ 3.1.4	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	§ 3.1.4	t_l	t_s	t_s	t_s	t_s							
BC_n	§ 3.1.4	t_m	t_l^2	t_s	1	t_s							
D_n	§ 3.1.4	t	1	1	1	1							

A remark is in order on the treatment of the type BC_n and the non-reduced systems. As we have seen in § 3.1.2, the argument on the specialization to type C_n used the extended affine Weyl group of of type C_n. In contrast, as commented at the beginning of §5.1 and (5.1.7) in [M03], we don't have the extended affine Weyl groups (or the affine Hecke algebras) associated to the type BC_n and the non-reduced systems, so we cannot follow the argument in § 3.1.2. However, the (non-symmetric) Macdonald polynomials for non-reduced systems are defined as the specialization of Koornwinder polynomials in [M03], and thus the situations are easier than reduced systems.

Type B_n

For $n \in \mathbb{Z}_{\ge 3}$, the following subset $S^B \subset S$ forms the affine root system of type B_n in the sense of [M03, §1.3, (1.3.2)].

$$
\begin{aligned}
S^B := O_s^B \sqcup O_l^B, \quad & O_s^B := O_1 = \{\pm\epsilon_i + r \mid 1 \le i \le n,\ r \in \mathbb{Z}\}, \\
& O_l^B := O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \le i < j \le n,\ r \in \mathbb{Z}\}.
\end{aligned}
\tag{3.1.44}
$$



Using the symbol $L' = P_{B_n}^\vee = P_{C_n} = \oplus_{i=1}^n \mathbb{Z}\epsilon_i$ in [M03, 1.4], the extended affine Weyl group is given by

$$W^B := W_0^B \ltimes \mathrm{t}(L') = W_0^B \ltimes \mathrm{t}(P_{C_n}) \simeq W.$$

Here W_0^B denotes the Weyl group of the finite root lattice B_n. The group W^B acts on S^B by permutation, and the W^B-orbits are given by O_s^B and O_l^B. We attach parameters k_s^B and k_l^B to the W^B-orbits as

$$O_s^B \longleftrightarrow k_s^B, \quad O_l^B \longleftrightarrow k_l^B,$$

and define the label k^B by

$$k^B : S^B \longrightarrow \mathbb{R}, \quad k^B(a) := k_s^B \quad (a \in O_s^B), \quad k^B(a) := k_l^B \quad (a \in O_l^B).$$

Mimicking the relation (3.1.22), we introduce the parameters of type B_n by

$$t_s^B := q^{k_s^B}, \quad t_l^B := q^{k_l^B}. \tag{3.1.45}$$

They correspond to the W^B-orbits as $t_s^B \leftrightarrow O_s^B$ and $t_l^B \leftrightarrow O_l^B$.

The weight function of type B_n is given by

$$\Delta^B = \Delta_{S^B,k^B} := \prod_{a\in(S^B)^+} \frac{1-q^{k^B(2a)}e^a}{1-q^{k^B(a)}e^a}.$$

Then, (3.1.27) yields

$$\Delta_{S,k}|_{k(a)-k(2a)=0\ (a\in S\setminus S^B)} = \Delta_{S^B,k}.$$

Thus the desired specialization is given by

$$k(a)-k(2a)\longmapsto 0 \quad (a\in S\setminus S^B), \quad k(a)-k(2a)\longmapsto k^B(a) \quad (a\in S^B).$$

By (3.1.44), we have $S\setminus S^B = O_2\sqcup O_3\sqcup O_4$, $S^B = O_s^B \sqcup O_l^B$, $O_s^B = O_1$ and $O_l^B = O_5$. Then, we can rewrite the specialization in terms of $k_1,\dots,k_5$ and k_s^B, k_l^B as

$$k_2-0,\ k_3-k_4,\ k_4-0\longmapsto 0, \quad k_1\longmapsto k_s^B, \quad k_5\longmapsto k_l^B.$$

Using (3.1.22) and (3.1.45), and assuming $t_0,u_0>0$, we have

$$\begin{aligned}&\tfrac{t_n}{u_n},\ (t_0u_0)/\tfrac{t_0}{u_0},\ \tfrac{t_0}{u_0}\longmapsto 1, \quad t_nu_n\longmapsto (t_s^B)^2, \quad t\longmapsto t_l^B\\ \iff\ &(t,t_0,t_n,u_0,u_n)\longmapsto \big(t_l^B,1,t_s^B,1,t_s^B\big).\end{aligned} \tag{3.1.46}$$

Now we suppress the superscript B in the parameters, and denote by

$$E_\mu^B(x;q,t_s,t_l), \quad \mu\in P_{B_n} := \oplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \tfrac{1}{2}(\epsilon_1+\epsilon_2+\cdots+\epsilon_n)$$

the non-symmetric Macdonald polynomial of type B_n (Definition 1.3.1.1). Having that $P_{C_n}\subset P_{B_n}$, we conclude:

Proposition 3.1.4.1. For any $\mu\in P_{C_n}$, we have

$$E_\mu^B(x;q,t_s,t_l) = E_\mu(x;q,t_l,1,t_s,1,t_s).$$

Also, for a dominant weight μ, we have

$$P_\mu^B(x;q,t_s,t_l) = P_\mu(x;q,t_l,1,t_s,1,t_s)$$

for the symmetric Macdonald polynomials of type B_n.

Remark 3.1.4.2. We can make a similar observation as in Remark 3.1.3.2. Let us denote by H^B the affine Hecke algebra for the extended Weyl group W^B in the sense of [M03, Chap. 4]. As a linear space over the base field K, we have $H^B\simeq H_0\otimes_K KP_{C_n}\simeq H$. Denoting by β^B the basic representation of H^B, we have the following diagram.

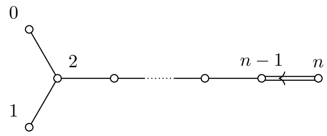
$$\begin{tikzcd} H \arrow[d, hook, "\beta"] & H^B \arrow[d, hook, "\beta^B"] \\ \mathrm{End}_K(KP_{C_n}) \arrow[r, hook] & \mathrm{End}_K(KP_{B_n}) \end{tikzcd}$$

As in Remark 3.1.3.2, we can that the specialization (3.1.46) maps $\beta(T_j)\mapsto\beta^B(T_j)$ for $j=1,2,\dots,n-1$, but the images of $\beta(T_i)$ is not equal to $\beta^B(T_i)$ for $i=0,n$.

Type $B_n^\vee$

For $n \in \mathbb{Z}_{\geq 3}$, the following subset $S^{B^\vee} \subset S$ forms the affine root system of type $B_n^\vee$ in the sense of [M03, §1.3, (1.3.3)].

$$\begin{aligned} S^{B^\vee} \coloneqq O_l^{B^\vee} \sqcup O_s^{B^\vee}, \quad O_l^{B^\vee} &\coloneqq O_2 = \{\pm 2\epsilon_i + 2r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\ O_s^{B^\vee} &\coloneqq O_5 = \{\pm \epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n,\ r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.47}$$



Using the symbol $L = L' = P_{B_n}^\vee = P_{C_n} = \oplus_{i=1}^n \mathbb{Z}\epsilon_i$ in [M03, 1.4], the extended affine Weyl group is given by

$$W^{B^\vee} \coloneqq W_0^B \ltimes \mathrm{t}(L') = W_0^B \ltimes \mathrm{t}(P_{C_n}) \simeq W.$$

It acts on $S^{B^\vee}$, and the $W^{B^\vee}$-orbits are $O_s^{B^\vee}$ and $O_l^{B^\vee}$. We attach parameters to these orbits as

$$k_s^{B^\vee} \longleftrightarrow O_s^{B^\vee}, \quad k_l^{B^\vee} \longleftrightarrow O_l^{B^\vee},$$

and define the label $k^{B^\vee} : S^{B^\vee} \to \mathbb{R}$ as before. We also introduce another set of parameters as

$$t_l^{B^\vee} \coloneqq \tau_{B^\vee,n}^2 = q^{k_n^{B^\vee}}, \quad t_s^{B^\vee} \coloneqq \tau_{B^\vee,j}^2 = q^{k_j^{B^\vee}} \quad (0 \leq i \leq n-1). \tag{3.1.48}$$

They correspond to the $W^{B^\vee}$-orbits as $t_s^{B^\vee} \leftrightarrow O_s^{B^\vee}$ and $t_l^{B^\vee} \leftrightarrow O_l^{B^\vee}$.

The weight function of type $B_n^\vee$ is given by

$$\Delta^{B^\vee} = \Delta_{S^{B^\vee}, k^{B^\vee}} \coloneqq \prod_{a \in (S^{B^\vee})^+} \frac{1 - q^{k^{B^\vee}(2a)} e^a}{1 - q^{k^{B^\vee}(a)} e^a}.$$

Then, (3.1.27) yields

$$\Delta_{S,k}|_{k(a)-k(2a)=0\ (a \in S \setminus S^{B^\vee})} = \Delta_{S^{B^\vee},k}.$$

Thus the specialization from type $(C_n^\vee, C_n)$ to type $B_n^\vee$ is given by

$$k(a) - k(2a) \longmapsto 0 \quad (a \in S \setminus S^{B^\vee}), \quad k(a) - k(2a) \longmapsto k^B(a) \quad (a \in S^B).$$

By (3.1.47), we have $S \setminus S^{B^\vee} = O_1 \sqcup O_3 \sqcup O_4$, $S^{B^\vee} = O_s^{B^\vee} \sqcup O_l^{B^\vee}$, $O_s^{B^\vee} = O_5$ and $O_l^{B^\vee} = O_2$. Then the above specialization can be written as

$$k_1 - k_2,\ k_3 - k_4,\ k_4 - 0 \longmapsto 0, \quad k_2 \longmapsto k_l^{B^\vee}, \quad k_5 \longmapsto k_s^{B^\vee}.$$

Using (3.1.22) and (3.1.48), and assuming $t_0, u_n, u_0 > 0$, we can further rewrite it as

$$\begin{aligned} &(t_n u_n)/\tfrac{t_n}{u_n},\ (t_0 u_0)/\tfrac{t_0}{u_0},\ \tfrac{t_0}{u_0} \longmapsto 1, \quad t_n/u_n \longmapsto (t_l^{B^\vee})^2, \quad t \longmapsto t_s^{B^\vee} \\ \iff\ &(t, t_0, t_n, u_0, u_n) \longmapsto \big(t_s^{B^\vee}, 1, (t_l^{B^\vee})^2, 1, 1\big). \end{aligned}$$

Now we suppress the superscript $B^\vee$ in the parameters, and denote by

$$E_\mu^{B^\vee}(x; q, t_s, t_l), \quad \mu \in P_{B_n}^\vee = P_{C_n}$$

the non-symmetric Macdonald polynomial of type $B_n^\vee$ (Definition 1.3.1.1). The conclusion of this § 3.1.4 is:

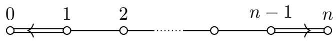
Proposition 3.1.4.3. For any $\mu \in P_{C_n}$, we have

$$E_\mu^{B^\vee}(x; q, t_s, t_l) = E_\mu(x; q, t_s, 1, t_l^2, 1, 1).$$

Type $C_n^\vee$

For $n \in \mathbb{Z}_{\geq 2}$, the following subset $S^{C^\vee} \subset S$ forms the affine root system of type $C_n^\vee$ in the sense of [M03, §1.3, (1.3.5)].

$$
\begin{aligned}
S^{C^\vee} := O_s^{C^\vee} \sqcup O_l^{C^\vee}, \quad O_s^{C^\vee} &:= O_1 \sqcup O_3 = \{\pm\epsilon_i + \tfrac{1}{2}r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\
O_l^{C^\vee} &:= O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n,\ r \in \mathbb{Z}\}.
\end{aligned} \tag{3.1.49}
$$



Using $L = L' = P_{C_n}^\vee = P_{B_n} = \oplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$, the extended affine Weyl group is given by

$$
W^{C^\vee} := W_0 \ltimes \mathrm{t}(L') = W_0 \ltimes \mathrm{t}(P_{B_n}) = W^C.
$$

The $W^{C^\vee}$-orbits on $S^{C^\vee}$ are $O_s^{C^\vee}$ and $O_l^{C^\vee}$. We define the label $k^{C^\vee} : S^{C^\vee} \to \mathbb{R}$ using the correspondence

$$
k_s^{C^\vee} \longleftrightarrow O_s^{C^\vee}, \quad k_l^{C^\vee} \longleftrightarrow O_l^{C^\vee}.
$$

Mimicking the relation (3.1.22), we define another set of parameters as

$$
t_s^{C^\vee} := q^{k_s^{C^\vee}}, \quad t_l^{C^\vee} := q^{k_l^{C^\vee}}. \tag{3.1.50}
$$

They correspond to the $W^{C^\vee}$-orbits as $t_s^{C^\vee} \leftrightarrow O_s^{C^\vee}$ and $t_l^{C^\vee} \leftrightarrow O_l^{C^\vee}$.

The weight function is given by

$$
\Delta^{C^\vee} = \Delta_{S^{C^\vee}, k^{C^\vee}} := \prod_{a \in (S^{C^\vee})^+} \frac{1 - q^{k^{C^\vee}(2a)} e^a}{1 - q^{k^{C^\vee}(a)} e^a}.
$$

Then (3.1.27) yields

$$
\Delta_{S,k}|_{k(a) - k(2a) = 0 \ (a \in S \setminus S^{C^\vee})} = \Delta_{S^{C^\vee}, k}.
$$

Thus the specialization to type $C_n^\vee$ is given by

$$
k(a) - k(2a) \longmapsto 0 \quad (a \in S \setminus S^{C^\vee}), \quad k(a) - k(2a) \longmapsto k^B(a) \quad (a \in S^B).
$$

By (3.1.49), we have $S \setminus S^{C^\vee} = O_2 \sqcup O_4$, $S^{C^\vee} = O_s^{C^\vee} \sqcup O_l^{C^\vee}$, $O_s^{C^\vee} = O_1 \sqcup O_3$ and $O_l^{C^\vee} = O_5$. Then we can rewrite the above specialization as

$$
k_2 - 0,\ k_4 - 0 \longmapsto 0, \quad k_1,\ k_3 \longmapsto k_s^{C^\vee}, \quad k_5 \longmapsto k_l^{C^\vee}.
$$

Using (3.1.22) and (3.1.50), we can rewrite it as

$$
\begin{aligned}
&\tfrac{t_n}{u_n},\ \tfrac{t_0}{u_0} \longmapsto 1, \quad t_n u_n,\ t_0 u_0 \longmapsto (t_s^{C^\vee})^2, \quad t \longmapsto t_l^{C^\vee} \\
\iff &(t, t_0, t_n, u_0, u_n) \longmapsto (t_l^{C^\vee}, t_s^{C^\vee}, t_s^{C^\vee}, t_s^{C^\vee}, t_s^{C^\vee}).
\end{aligned}
$$

We suppress the superscript $C^\vee$ in the parameters, and denote by

$$
E_\mu^{C^\vee}(x; q, t_s, t_l), \quad \mu \in P_{C_n}^\vee = P_{B_n}
$$

the non-symmetric Macdonald polynomial of type $C_n^\vee$ (Definition 1.3.1.1). Noting that $P_{C_n} \subset P_{B_n}$, we have the conclusion:

Proposition 3.1.4.4. For any $\mu \in P_{C_n}$, we have

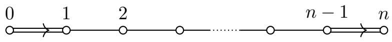
$$
E_\mu^{C^\vee}(x; q, t_s, t_l) = E_\mu(x; q, t_l, t_s, t_s, t_s, t_s).
$$

Type BC_n

For $n \in \mathbb{Z}_{\geq 1}$, the following subset $S^{BC} \subset S$ forms the affine root system of type BC_n in the sense of [M03, §1.3, (1.3.6)].

$$\begin{aligned} S^{BC} := O_s^{BC} \sqcup O_m^{BC} \sqcup O_l^{BC}, \quad & O_s^{BC} := O_1 = \{\pm\epsilon_i + r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\ & O_l^{BC} := O_4 = \{\pm 2\epsilon_i + 2r + 1 \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\ & O_m^{BC} := O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.51}$$

Hereafter we assume $n \geq 2$ to make the argument compatible with that so far. The Dynkin diagram is then given by



(3.1.52)

Recall the comment in the beginning of this § 3.1.4. We will not introduce a new extended affine Weyl group, but consider the group W of type $(C_n^\vee, C_n)$ (see (3.1.10)). It acts on S^{BC}, and the W-orbits are given by O_s^{BC}, O_m^{BC} and O_l^{BC}. Hence, we already have the correspondence between the Macdonald parameters of type $(C_n^\vee, C_n)$ and the W-orbits on S^{BC}. Let us denote

$$t_s^{BC} := q^{k_1}, \quad t_m^{BC} := q^{k_5}, \quad t_l^{BC} := q^{k_4}, \tag{3.1.53}$$

which correspond to the W-orbits O_s^{BC}, O_m^{BC} and O_l^{BC}, respectively.

Following [M03, (5.1.77)], we define the weight function $\Delta_{S^{BC},k}$ of type BC_n to be the specialization of $\Delta_{S,k}$ of type $(C_n^\vee, C_n)$. In other words, we take the right hand side of (3.1.27) as the definition:

$$\Delta^{BC} = \Delta_{S^{BC},k} := \Delta_{S,k}|_{k(a)-k(2a)=0\ (a \in S \setminus S^{BC})}.$$

By (3.1.51), we have $S \setminus S^{BC} = O_2 \sqcup O_3$ and $S^{BC} = O_s^{BC} \sqcup O_m^{BC} \sqcup O_l^{BC} = O_1 \sqcup O_5 \sqcup O_4$. Them, we can see that the specialization to type BC_n is given by

$$k_2 - 0,\ k_3 - k_4 \longmapsto 0.$$

Using (3.1.22) and (3.1.53), we can rewrite it as

$$\begin{aligned} & \tfrac{t_n}{u_n},\ (t_0 u_0)/\tfrac{t_0}{u_0} \longmapsto 1, \quad t_n u_n \longmapsto (t_s^{BC})^2, \quad \tfrac{t_0}{u_0} \longmapsto (t_l^{BC})^2, \quad t \longmapsto t_m^{BC} \\ \iff & \ (t, t_0, t_n, u_0, u_n) \longmapsto \left(t_m^{BC}, (t_l^{BC})^2, t_s^{BC}, 1, t_s^{BC}\right). \end{aligned}$$

Now we suppress the superscript BC in the parameters, and denote by

$$E_\mu^{BC}(x; q, t_s, t_m, t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of type BC_n (Definition 1.3.1.1). Then the conclusion is:

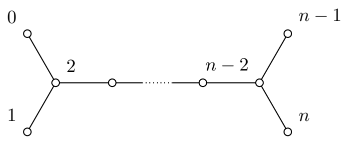
Proposition 3.1.4.5. For any $\mu \in P_{C_n}$, we have

$$E_\mu^{BC}(x; q, t_s, t_m, t_l) = E_\mu(x; q, t_m, t_l^2, t_s, 1, t_s).$$

Type D_n

For $n \in \mathbb{Z}_{\geq 4}$, the following subset $S^D \subset S$ forms the affine root system of type D_n in the sense of [M03, §1.3, (1.3.7)].

$$S^D := O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, r \in \mathbb{Z}\}. \tag{3.1.54}$$



Using the Weyl group W_0^D and the weight lattice

$$L' = P_{D_n} := \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n \oplus \mathbb{Z}\tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n) \tag{3.1.55}$$

of the finite root system of type D_n, the extended affine Weyl group is given by

$$W^D := W_0^D \ltimes \mathrm{t}(L') = W_0^D \ltimes \mathrm{t}(P_{D_n}). \tag{3.1.56}$$

It acts on S^D by permutation, and there is a unique orbit. Attaching $k^D \in \mathbb{R}$ to this unique orbit, we define the label by $k^D(a) := k^D$ $(a \in S^D)$, and introduce

$$t^D := q^{k^D}. \tag{3.1.57}$$

The weight function is given by

$$\Delta^D = \Delta_{S^D,k^D} := \prod_{a \in (S^D)^+} \frac{1 - q^{k^B(2a)} e^a}{1 - q^{k^B(a)} e^a}.$$

The relation (3.1.27) yields

$$\Delta_{S,k}|_{k(a)-k(2a)=0 \ (a \in S \setminus S^D)} = \Delta_{S^D,k}.$$

Thus, the specialization to type D_n is given by

$$k(a) - k(2a) \longmapsto 0 \quad (a \in S \setminus S^D), \quad k(a) - k(2a) \longmapsto k^D(a) \quad (a \in S^D).$$

By (3.1.54), we have $S \setminus S^D = O_1 \sqcup \cdots \sqcup O_4$ and $S^B = O_5$. Then, we can rewrite the specialization in terms of $k_1, \ldots, k_5$ and k^D as

$$k_2 - 0,\, k_3 - k_4,\, k_4 - 0 \longmapsto 0, \quad k_1 \longmapsto k_s^B, \quad k_5 \longmapsto k_l^B.$$

Using (3.1.22) and (3.1.57), we have

$$t_n u_n,\, t_n/u_n,\, t_0 u_0,\, t_0/u_0 \longmapsto 1, \quad t \longmapsto t^D \iff (t, t_0, t_n, u_0, u_n) \longmapsto \left(t^D, 1, 1, 1, 1\right).$$

We suppress the superscript D in the parameters, and denote by

$$E_\mu^D(x; q, t), \quad \mu \in P_{D_n}$$

the non-symmetric Macdonald polynomial of type D_n (Definition 1.3.1.1). Since $P_{C_n} \subset P_{D_n}$, we have:

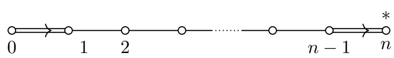
Proposition 3.1.4.6. For any $\mu \in P_{C_n}$, we have

$$E_\mu^D(x; q, t) = E_\mu(x; q, t, 1, 1, 1, 1).$$

Type (BC_n, C_n)

For $n \in \mathbb{Z}{\geq}\, 1$, the following subset $S^{BC,C} \subset S$ forms the affine root system of type (BC_n, C_n) in the sense of [M03, §1.3, (1.3.15)].

$$\begin{aligned} S^{BC,C} &:= O_s^{BC,C} \sqcup O_m^{BC,C} \sqcup O_l^{BC,C}, \\ O_s^{BC,C} &:= O_1 = \{\pm\epsilon_i + r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\ O_l^{BC,C} &:= O_2 \sqcup O_4 = \{\pm 2\epsilon_i + r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\ O_m^{BC,C} &:= O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n,\ r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.58}$$



The diagram is for $n \geq 2$, and hereafter we assume this condition. The mark $*$ above the index n implies that there is a basis $\{a_i^{BC,C}\}_{i=0}^n$ such that $a_n^{BC,C}, 2a_n^{BC,C} \in S^{BC,C}$. There are three W-orbits $O_s^{BC,C}$, $O_m^{BC,C}$ and $O_l^{BC,C}$. We introduce the parameters

$$t_s^{BC,C} := q^{k_1}, \quad t_m^{BC,C} := q^{k_5}, \quad t_l^{BC,C} := q^{k_2}, \tag{3.1.59}$$

which correspond to the W-orbit $O_s^{BC,C}$, $O_m^{BC,C}$ and $O_l^{BC,C}$, respectively.

Similarly as in the previous § 3.1.4, the weight function $\Delta_{S^{BC,C},k}$ of type (BC_n, C_n) is defined by the specialization of $\Delta_{S,k}$ as

$$\Delta^{BC,C} = \Delta_{S^{BC,C},k} := \Delta_{S,k}|_{k(a)-k(2a)=0 \ (a \in S \setminus S^{BC,C})}.$$

By (3.1.58), we have $S \setminus S^{BC,C} = O_3$, $O_l^{BC,C} = O_2 \sqcup O_4$, which implies that the specialization to type (BC_n, C_n) is given by

$$k_3 - k_4 \longmapsto 0, \quad k_2 \longmapsto k_4.$$

Using (3.1.22) and (3.1.59), and assuming $u_0 > 0$, we can rewrite it as

$$\begin{aligned} &(t_0u_0)/\tfrac{t_0}{u_0} \longmapsto 1, \quad t_nu_n \longmapsto (t_s^{BC,C})^2, \quad \tfrac{t_n}{u_n}, \ \tfrac{t_0}{u_0} \longmapsto (t_l^{BC,C})^2, \quad t \longmapsto t_m^{BC,C} \\ \iff \ &(t, t_0, t_n, u_0, u_n) \longmapsto \big(t_m^{BC,C}, (t_l^{BC,C})^2, t_s^{BC,C}t_l^{BC,C}, 1, t_s^{BC,C}/t_l^{BC,C}\big). \end{aligned}$$

We suppress the superscript BC, C in the parameters, and denote by

$$E_\mu^{BC,C}(x; q, t_s, t_m, t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of type (BC_n, B_n) (Definition 1.3.1.1). The conclusion of this § 3.1.4 is:

Proposition 3.1.4.7. For any $\mu \in P_{C_n}$, we have

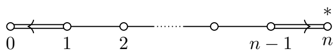
$$E_\mu^{BC,C}(x; q, t_s, t_m, t_l) = E_\mu(x; q, t_m, t_l^2, t_st_l, 1, t_s/t_l).$$

Type $(C_n^\vee, BC_n)$

For $n \in \mathbb{Z}_{\geq 1}$, the following subset $S^{C^\vee,BC} \subset S$ forms the affine root system of type (BC_n, C_n) in the sense of [M03, §1.3, (1.3.16)].

$$\begin{aligned} S^{C^\vee,BC} &:= O_s^{C^\vee,BC} \sqcup O_m^{C^\vee,BC} \sqcup O_l^{C^\vee,BC}, \\ O_s^{C^\vee,BC} &:= O_1 \sqcup O_3 = \{\pm\epsilon_i + \tfrac{1}{2}r \mid 1 \leq i \leq n, \ r \in \mathbb{Z}\}, \\ O_l^{C^\vee,BC} &:= O_2 = \{\pm 2\epsilon_i + 2r \mid 1 \leq i \leq n, \ r \in \mathbb{Z}\}, \\ O_m^{C^\vee,BC} &:= O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n, \ r \in \mathbb{Z}\}. \end{aligned} \tag{3.1.60}$$

Hereafter we assume $n \geq 2$. Then the Dynkin diagram is given by



There are three W-orbits $O_s^{C^\vee,BC}$, $O_m^{C^\vee,BC}$ and $O_l^{C^\vee,BC}$, and the parameters are defined to be

$$t_s^{C^\vee,BC} := q^{k_1}, \quad t_m^{C^\vee,BC} := q^{k_5}, \quad t_l^{C^\vee,BC} := q^{k_2}. \tag{3.1.61}$$

The weight function of type $(C_n^\vee, BC_n)$ is defined by

$$\Delta^{C^\vee,BC} = \Delta_{S^{C^\vee,BC},k} := \Delta_{S,k}|_{k(a)-k(2a)=0 \ (a \in S \setminus S^{C^\vee,BC})}.$$

By (3.1.60), we have $S \setminus S^{C^\vee,BC} = O_4$ and $O^{C^\vee,BC} = O_1 \sqcup O_3$, which implies that

$$k_4 - 0 \longmapsto 0, \quad k_1 \longmapsto k_3$$

give the desired specialization. Using (3.1.22) and (3.1.61), we can rewrite it as

$$
\begin{aligned}
&t_0/u_0 \longmapsto 1, \quad t_0u_0,\, t_nu_n \longmapsto (t_s^{C^\vee,BC})^2, \quad t_n/u_n \longmapsto (t_l^{C^\vee,BC})^2, \quad t \longmapsto t_m^{C^\vee,BC} \\
\iff &(t,t_0,t_n,u_0,u_n) \longmapsto \big(t_m^{C^\vee,BC}, t_s^{C^\vee,BC}, t_s^{C^\vee,BC}t_l^{C^\vee,BC}, t_s^{C^\vee,BC}, t_s^{C^\vee,BC}/t_l^{C^\vee,BC}\big).
\end{aligned}
$$

We suppress the superscript $C^\vee, BC$ in the parameters, and denote by

$$E_\mu^{C^\vee,BC}(x;q,t_s,t_m,t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of type $(C_n^\vee, BC_n)$ (Definition 1.3.1.1). The conclusion of this § 3.1.4 is:

Proposition 3.1.4.8. For any $\mu \in P_{C_n}$, we have

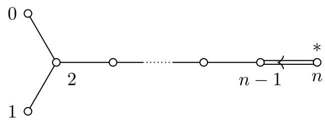
$$E_\mu^{C^\vee,BC}(x;q,t_s,t_m.t_l) = E_\mu(x;q,t_m,t_s,t_st_l,t_s,t_s/t_l).$$

Types $(C_2, C_2^\vee)$ and $(B_n^\vee, B_n)$

The affine root systems of type $(C_2, C_2^\vee)$ and of type $(B_n^\vee, B_n)$ with $n \in \mathbb{Z}_{\geq 3}$ in the sense of [M03, §1.3, (1.3.17)] are given by the following subset $S^{B^\vee,B} \subset S$.

$$
\begin{aligned}
S^{B^\vee,B} :=& O_s^{B^\vee,B} \sqcup O_m^{B^\vee,B} \sqcup O_l^{B^\vee,B}, \\
& O_s^{B^\vee,B} := O_1 = \{\pm\epsilon_i + r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\
& O_l^{B^\vee,B} := O_2 = \{\pm 2\epsilon_i + 2r \mid 1 \leq i \leq n,\ r \in \mathbb{Z}\}, \\
& O_m^{B^\vee,B} := O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \leq i < j \leq n,\, r \in \mathbb{Z}\}.
\end{aligned}
\tag{3.1.62}
$$

In the case $n \geq 3$, the Dynkin diagram is given by



The W-orbits are $O_s^{B^\vee,B}$, $O_m^{B^\vee,B}$ and $O_l^{B^\vee,B}$. The corresponding parameters are defined to be

$$t_s^{B^\vee,B} := q^{k_1}, \quad t_m^{B^\vee,B} := q^{k_5}, \quad t_l^{B^\vee,B} := q^{k_2}. \tag{3.1.63}$$

The weight function $\Delta_{S^{B^\vee,B},k}$ is defined by

$$\Delta^{B^\vee,B} = \Delta_{S^{B^\vee,B},k} := \Delta_{S,k}|_{k(a)-k(2a)=0\ (a \in S \setminus S^{B^\vee,B})}\,.$$

By (3.1.62), we have $S \setminus S^{B^\vee,B} = O_3 \sqcup O_4$, which implies that

$$k_3 - k_4,\, k_4 - 0 \longmapsto 0$$

gives the specialization to type $(C_2, C_2^\vee)$ and $(B_n^\vee, B_n)$. Using (3.1.22) and (3.1.63), we can rewrite it as

$$
\begin{aligned}
&t_0u_0,\, t_0/u_0 \longmapsto 1, \quad t_nu_n \longmapsto (t_s^{B^\vee,B})^2, \quad t_n/u_n \longmapsto (t_l^{B^\vee,B})^2, \quad t \longmapsto t_m^{B^\vee,B} \\
\iff &(t,t_0,t_n,u_0,u_n) \longmapsto \big(t_m^{B^\vee,B}, 1, t_s^{B^\vee,B}t_l^{B^\vee,B}, 1, t_s^{B^\vee,B}/t_l^{B^\vee,B}\big).
\end{aligned}
$$

We suppress the superscript $B^\vee, B$ in the parameters, and denote by

$$E_\mu^{B^\vee,B}(x;q,t_s,t_m,t_l), \quad \mu \in P_{C_n}$$

the non-symmetric Macdonald polynomial of types $(C_2, C_2^\vee)$ and $(B_n^\vee, B_n)$ (Definition 1.3.1.1). The conclusion of this § 3.1.4 is:

Proposition 3.1.4.9. For any $\mu \in P_{C_n}$, we have

$$E_\mu^{B^\vee,B}(x;q,t_s,t_m.t_l) = E_\mu(x;q,t_m,1,t_st_l,1,t_s/t_l).$$

3.1.5 Relation to Koornwinder's specializations in admissible pairs

As mentioned in § 3.0, in the original theory [M87], Macdonald used admissible pairs to formulate his family of multivariate orthogonal polynomials for general root systems. Here, an admissible pair means a pair (R, S) of root systems satisfying the following conditions.

- Both R and S span the common finite-dimensional Euclidean space V.
- S is a reduced.
- The Weyl groups are identical, i.e., $W_R = W_S$.

In [Ko92, §6.1], Koornwinder obtained Macdonald polynomials of the admissible pairs

$$(R, S) = (R_{BC_n}, S_{B_n}),\ (R_{BC_n}, S_{C_n})$$

by specializing the parameters in his polynomials. The parameters in [Ko92] are denoted as

$$a, b, c, d, t, q,$$

and we call them the Koornwinder parameters. The finite root systems R_{BC_n}, S_{B_n} and S_{C_n} are

$$\begin{aligned} R_{BC_n} &\coloneqq \{\pm\epsilon_i \mid 1 \le i \le n\} \sqcup \{\pm 2\epsilon_i \mid 1 \le i \le n\} \sqcup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}, \\ S_{B_n} &\coloneqq \{\pm\epsilon_i \mid 1 \le i \le n\} \sqcup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}, \\ S_{C_n} &\coloneqq \{\pm\epsilon_i \mid 1 \le i \le n\} \sqcup \{\tfrac{1}{2}(\pm\epsilon_i \pm \epsilon_j) \mid 1 \le i < j \le n\}. \end{aligned} \tag{3.1.64}$$

Using them, the specializations in [Ko92, §6.1] are described as

$$(R_{BC_n}, S_{B_n}) : (a, b, c, d, t, q) \longmapsto (q^{1/2}, -q^{1/2}, a_B b_B^{1/2}, -b_B^{1/2}, t_B, q), \tag{3.1.65}$$

$$(R_{BC_n}, S_{C_n}) : (a, b, c, d, t, q) \longmapsto (a_C b_C^{1/2}, q a_C b_C^{1/2}, -b_C^{1/2}, -q b_C^{1/2}, t_C, q^2). \tag{3.1.66}$$

There are only given these results in [Ko92, §6.1] . We guess that they are derived by the comparison of the weight functions of inner products, as we did in the previous § 3.1.3 and § 3.1.4.

In [N95, p.54], Noumi gave the correspondence between the Noumi parameters q, t, t_0, t_n, u_0, u_n and the Koornwinder parameters a, b, c, d, t, q. The correspondence is that q and t are common, and

$$(t_0, t_n, u_0, u_n) = (-cd/q, -ab, -c/d, -a/b).$$

We can then rewrite the specialization (3.1.65) to the admissible pair (R_{BC_n}, S_{B_n}) as

$$(t, t_0, t_n, u_0, u_n) \longmapsto (t_B, 1, a_B b_B, 1, a_B).$$

Thus, setting $t_B = t_m^{B^\vee, B}$, $a_B = t_s^{B^\vee, B}/t_l^{B^\vee, B}$ and $b_B = (t_l^{B^\vee, B})^2$, we see that it coincides with the specialization to type $(B_n^\vee, B_n)$ in § 3.1.4.

Let us remark that a similar rewriting of the specialization (3.1.66) to the admissible pair (R_{BC_n}, S_{C_n}) does not have a corresponding one in Table 3.0.1. It seems to be due to that the root system S_{C_n} in (3.1.64) cannot be treated in the formulation of [M03].

3.1.6 The rank one case

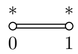
This subsection is added after the referee comments. We would like to appreciate the referees' suggestions.

As explained in the beginning of § 3.1.1, the argument so far assumes the rank $n \ge 2$. The purpose of this § 3.1.6 is to study the excluded case $n = 1$. As mentioned in the beginning of § 2.0, the Koornwinder polynomial is designed to give a multi-variable analogue of the Askey-Wilson polynomial. So it is natural to study what our specialization argument yields in the rank one case. The argument is similar to the previous one, so we only give an outline.

Let $E = (\mathbb{R}\epsilon, \langle\cdot,\cdot\rangle)$ be the 1-dimensional Euclidean space with basis ϵ, and F be the $\mathbb{R}$-linear space of affine linear functions on E. We identify $F \xrightarrow{\sim} \mathbb{R}\epsilon \oplus \mathbb{R}c$ by the inner product $\langle\cdot,\cdot\rangle$ as in the rank $n \ge 2$ case (§ 3.1.1). The affine root system of type $(C_1^\vee, C_1)$ is the subset $S = S^{C_1^\vee, C_1} \subset E$ given by

$$S = O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4, \quad O_1 \coloneqq \pm\epsilon + \mathbb{Z}c,\ O_2 \coloneqq 2O_1,\ O_3 \coloneqq O_1 + \tfrac{1}{2}c,\ O_4 \coloneqq 2O_3 = O_2 + c.$$

We take the basis $\{\frac{1}{2}a_0, a_0, \frac{1}{2}a_1, a_1\}$ of S with $a_0 := -2\epsilon + c$ and $a_1 := 2\epsilon$. The Dynkin diagram of S is shown in the next line, where the mark $*$ has the same meaning as in the rank $n \geq 2$ case.



Next, as in (3.1.5), we denote the simple reflections by $s_0 := s_{a_0}$ and $s_1 := s_{a_1}$, and define the finite Weyl group by $W_0 := \langle s_0 \rangle \subset \mathrm{GL}_{\mathbb{R}}(F)$. The extended affine Weyl group is defined by $W := \mathrm{t}(P_L) \rtimes W_0 \subset \mathrm{GL}_{\mathbb{R}}(F)$, where $P := \mathbb{Z}\epsilon \subset F$ is the weight lattice "of type C_1". Then we have $W = \langle s_0, s_1 \rangle$ as in (3.1.12), and the subsets $O_1, O_2, O_3, O_4 \subset S$ are W-orbits of $\frac{1}{2}a_1, a_1, \frac{1}{2}a_0, a_0$, respectively.

We attach parameters k_1, k_2, k_3, k_4 to these W-orbits under the correspondence $k_i \leftrightarrow O_i$ as in (3.1.19). Choosing $q \in \mathbb{R}$ with $0 < q < 1$, we set $k\colon S \to \mathbb{R}$ by $k(a) := k_i$ for $a \in O_i$ as in (3.1.20). We call

$$\{q^{k(a)} \mid a \in S\} = \{q^{k_1}, q^{k_2}, q^{k_3}, q^{k_4}\}$$

the set of *Macdonald parameters* as in (3.1.21). We also have *the Noumi parameters* t_0, t_1, u_0, u_1, which correspond to the Macdonald parameters by the relation

$$(q^{2k_1}, q^{2k_2}, q^{2k_3}, q^{2k_4}) = (t_1 u_1, \tfrac{t_1}{u_1}, t_0 u_0, \tfrac{t_0}{u_0}). \tag{3.1.67}$$

We define the base field to be $\mathbb{K} := \mathbb{Q}(q^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_1^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_1^{\frac{1}{2}})$ as in (3.1.24).

By the general theory [M03, §§5.2–5.3], we have the one-variable symmetric Laurent polynomial

$$P_l(x) = P_l(x; q, t_0, t_1, u_0, u_1) \in \mathbb{K}[x^{\pm 1}]^{W_0}$$

for each dominant weight $\lambda = l\epsilon \in \Lambda_+ := \mathbb{N}\epsilon$, where $W_0 = \langle s_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{K}[x^{\pm 1}]$ by $s_1(x) = x^{-1}$. The Laurent polynomial $P_l(x)$ $(l \in \mathbb{N})$ is equal to the *Askey-Wilson polynomial* [AW85]. Let us briefly explain the correspondence, referring to [N95, §3], [St00], [NS04] and [M03, §§6.4–6.6] for the detail. We use Gasper and Rahman's notation [GR04] for q-shifted factorials (1.1.1) and q-hypergeometric series

$${}_{s+1}\phi_s\left[\begin{matrix} a_1, & \cdots, & a_{s+1} \\ b_1, & \cdots, & b_s \end{matrix}; q,\ z\right] := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_{s+1}; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \frac{z^k}{(q; q)_k}.$$

The Askey-Wilson polynomial is now defined to be

$$p_l\big(\tfrac{1}{2}(x + x^{-1}); q, a, b, c, d\big) := \frac{a^{-l}(ab, ac, ad; q)_l}{(abcd; q)_l} \cdot {}_4\phi_3\left[\begin{matrix} q^{-l}, & q^{l-1}abcd, & ax, & a/x \\ & ab, & ac, & ad \end{matrix}; q,\ q\right]. \tag{3.1.68}$$

Although the form (3.1.68) is asymmetric with respect to the parameters a, b, c, d, the polynomial actually has the parameter symmetry, which can be seen from the recurrence relation [AW85, (1.24)–(1.27)]. See also [Ya22, §4, Remark 4.1.2] for the relation between the recurrence relation and the Yip-type formula of Littlewood-Richardson coefficients of Koornwinder polynomials and the reduction to the rank one Askey-Wilson case. Using $(x; q)_l := (x; q)_\infty / (q^l x; q)_\infty$ for $l \in \mathbb{N}$, we have the correspondence [AW85, p.5]

$$p_l\big(\tfrac{1}{2}(x + x^{-1}); q, a, b, c, d\big) = 2^l (abcdq^{l-1}; q)_l \cdot P_l(x; q, t_0, t_1, u_0, u_1),$$

where *the Askey-Wilson parameters* a, b, c, d correspond to the Macdonald parameters by

$$(a, b, c, d) = (q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, t_1^{\frac{1}{2}} u_1^{\frac{1}{2}}, -t_1^{\frac{1}{2}} u_1^{-\frac{1}{2}}).$$

Combining it with (3.1.67), we see that the Askey-Wilson parameters correspond to the W-orbits in S by $(a, b, c, d) \leftrightarrow (O_3, O_4, O_1, O_2)$.

We now turn to the specialization argument. We list up the subsystems of $S = S^{C_1^\vee, C_1}$ and the corresponding specialization rules in Table 3.1.1. The Dynkin diagrams are borrowed from [M03, §1.3].The "Noumi" column shows the specialization of the Noumi parameters t_0, t_1, u_0, u_1 in the same way as the specialization Table 3.0.1. The "Askey-Wilson" column shows the specialization of the Askey-Wilson parameters.

type	Dynkin	orbits	Noumi				Askey-Wilson			
$(C_1^\vee, C_1)$ Askey-Wilson	$*$ $*$ / 0 1	$O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$	t_0	t_1	u_0	u_1	a	b	c	d
$(C_1^\vee, BC_1)$	$*$ / 0 1	$O_1 \sqcup O_2 \sqcup O_3$	t_s	$t_s t_l$	t_s	t_s/t_l	$q^{\frac{1}{2}} t_s$	$-q^{\frac{1}{2}}$	t_s	$-t_l$
(BC_1, C_1) cont. q-Jacobi	$*$ / 0 1	$O_1 \sqcup O_2 \sqcup O_4$	t_l^2	$t_s t_l$	1	t_s/t_l	$q^{\frac{1}{2}} t_l$	$-q^{\frac{1}{2}} t_l$	t_s	$-t_l$
BC_1	0 1	$O_1 \sqcup O_4$	t_l^2	t_s	1	t_s	$q^{\frac{1}{2}} t_l$	$-q^{\frac{1}{2}} t_l$	t_s	-1
A_1	0 1	O_1	1	t	1	t	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	t	-1
		O_3	t	1	t	1	$q^{\frac{1}{2}} t$	$-q^{\frac{1}{2}}$	1	-1
Rogers		O_2	1	t^2	1	1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	t	$-t$
		O_4	t^2	1	1	1	$q^{\frac{1}{2}} t$	$-q^{\frac{1}{2}} t$	1	-1

Table 3.1.1: Subsystems of $(C_1^\vee, C_1)$ and parameter specializations

The specialization rules for the types $(C_1^\vee, BC_1), (BC_1, C_1)$ and BC_1 are obtained by making $n = 1$ and deleting the t column in the specialization Table 3.0.1. We can obtain the type A_1 by a similar argument as the reduced subsystems of $(C_n^\vee, C_n)$, noting that we have four embeddings $S^{A_1} \hookrightarrow S^{C_1^\vee, C_1}$ as indicated in the "orbits" column in Table 3.1.1.

Table 3.1.1 yields the degeneration scheme (Figure 3.1.1) of q-hypergeometric orthogonal polynomials which respects the embeddings of affine root systems into $(C_1^\vee, C_1)$. Our degeneration scheme seems to be new.
For comparison, let us recall the *Askey scheme of q-hypergeometric orthogonal polynomials* (also called the q-Askey scheme, see [KLS10, p.413] for example). It shows the classification and the behavior under parameter specializations of q-hypergeometric orthogonal polynomials. Among those polynomials, we could only find *the continuous q-Jacobi polynomial* and *the Rogers polynomial* in our Figure 3.1.1 at this moment. As we will explain below, the former appears naturally, but the appearance of the Rogers polynomial is tricky. It might be possible that all the polynomials in our Figure 3.1.1 can be expressed as those in the q-Askey scheme. However, according to the quite different forms of our scheme and the q-Askey scheme, we can say that the parameter specializations taken in the q-Askey scheme do not necessarily respect the affine root system structures.

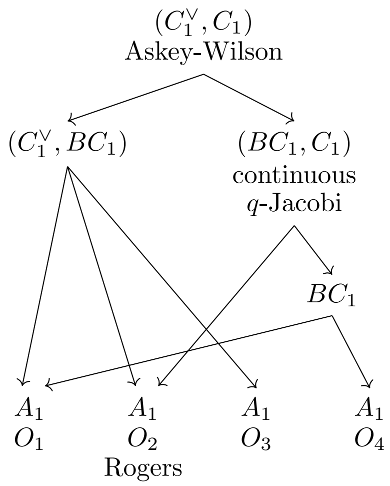


Figure 3.1.1: Root-theoretic degeneration scheme of Askey-Wilson polynomial

Remark 3.1.6.1. Recently, Koornwinder [Ko] proposed new degeneration schemes of q-hypergeometric orthogonal polynomials, called q-Verde-Star and q-Zhedanov schemes. These schemes looks quite different from ours, and the relation is unclear at this moment,

Among the specialized polynomials appearing in Table 3.1.1 and Figure 3.1.1, the type (BC_1, C_1) is essentially the same with the *continuous q-Jacobi polynomial* $P_l^{(\alpha,\beta)}(x;q)$ [KLS10, §14.10]. The relation with the Askey-Wilson polynomial is given by

$$P_l^{(\alpha,\beta)}(x;q) = (\text{const.}) \cdot p_n(x; q, q^{\frac{1}{2}\alpha+\frac{1}{4}}, q^{\frac{1}{2}\alpha+\frac{3}{4}}, -q^{\frac{1}{2}\beta+\frac{1}{4}}, -q^{\frac{1}{2}\beta+\frac{3}{4}}).$$

The appearance of $P_l^{(\alpha,\beta)}(x;q)$ is natural in view of the fact discovered by Koornwinder [Ko92, p.195] that the polynomial $P_l^{(\alpha,\beta)}(x;q)$ is the Macdonald symmetric polynomial of the admissible pair $R = S = BC_1$ (see § 3.1.5), which corresponds to the non-reduced affine root system (BC_1, C_1).

Let us also recall that the Macdonald symmetric polynomial of type A_1 is essentially equal to the *Rogers* or the *continuous q-ultraspherical polynomial* $C_n(x;a|q)$. See [M03, §6.3], [GR04, §7.4] and [KLS10, §14.10.1] for the detail. The generating function is given by

$$\sum_{l=0}^{\infty} C^l(x;a|q)y^l = \frac{(ayz, ay/z; q)_\infty}{(tz, t/z; q)_\infty}$$

with $x = (z + z^{-1})/2$. The Rogers polynomials are obtained by specializing parameters of the Askey-Wilson polynomials in several different ways. One of them is shown in [KLS10, p.420, (14.1.20)]:

$$C_l(x;a|q) = (\text{const.}) \cdot p_l(x; q, a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}),$$

which seems to be the most famous one, but does not appear in our Table 3.1.1. However, there is another one which we learned from [Ro21, (6.5b)]:

$$C_l(x;a^2|q^2) := (\text{const.}) \cdot p_l(x; q, a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}). \tag{3.1.69}$$

This relation appears in the third embedding $S^{A_1} \xrightarrow{\sim} O_2 \subset S^{C_1^\vee, C_1}$ of type A_1 in Table 3.1.1. Indeed, the embedded S^{A_1} is identified with the orbit O_2 of *long roots*, so the shift parameter q_A for the embedded system should be the square of the parameter q for the ambient system $S^{C_1^\vee, C_1}$, and we have the parameter q^2 in the Rogers polynomial and the parameter q in Askey-Wilson polynomial as in (3.1.69).

3.2 Specialization in Ram-Yip type formula

In this section, we give a partial check of the specialization Table 3.0.1 in the level of *Ram-Yip type formulas*. Precisely speaking, we show that the non-symmetric Koornwinder polynomial degenerates to the non-symmetric Macdonald polynomials of types B, C, D in the sense of [RY11] by the specializations given in Table 3.0.1, using explicit Ram-Yip type formulas of those polynomials. In this section, we use the notation in

Let us explain what we mean by the word *Ram-Yip type formulas*. In [RY11], Ram and Yip derived explicit formulas of non-symmetric Macdonald polynomials of reduced affine root systems using alcove walks. Their argument is designed to work in general setting, and the details are later given by Orr and Shimozono in [OS18], which derives among many results an explicit formula of the non-symmetric Koornwinder polynomial. We call all of these formulas Ram-Yip type formulas of non-symmetric Macdonald polynomials.

A caution is now in order. The realization of affine root systems in [RY11] is different from our default one in [M03]. For distinction, we denote by $S^{X,\mathrm{RY}}$ the affine root system of type X used in [RY11], and call the non-symmetric Macdonald polynomials of type X treated in loc. cit. the polynomial *of Ram-Yip type* X.

Let us summarize the results given in this § 3.2 in the following Table 3.2.1,

		t	t_0	t_n	u_0	u_n
B_n^{RY}	§ 3.2.2	t_m^{RY}	1	t_l^{RY}	1	t_l^{RY}
B_n		t_l	1	t_s	1	t_s
C_n^{RY}	§ 3.2.1	t_m^{RY}	1	t_s^{RY}	1	1
$B_n^\vee$		t_s	1	t_l^2	1	1
D_n	§ 3.2.3	t	1	1	1	1

Table 3.2.1: Specialization table for Ram-Yip formulas

As mentioned above, we treat the types B_n, C_n and D_n in the sense of [RY11], each in § 3.2.2, § 3.2.1 and § 3.2.3, respectively. Since the types B_n and C_n have discrepancy from those in our default [M03], we use the symbols B_n^{RY} and C_n^{RY} in Table 3.2.1. The type D_n has no discrepancy, and we use the symbol D_n. The B_n^{RY} row in Table 3.2.1 indicates the specialization of the Noumi parameters to obtain the non-symmetric polynomial of Ram-Yip type B_n. More explicitly, denoting the latter by $E_\mu^{B,\mathrm{RY}}(x)$, we have

$$E_\mu(x;q,t_m^{\mathrm{RY}},1,t_l^{\mathrm{RY}},1,t_l^{\mathrm{RY}}) = E_\mu^{B,\mathrm{RY}}(x;q,t_m^{\mathrm{RY}},t_l^{\mathrm{RY}}).$$

This equality will be shown in Proposition 3.2.2.4. The B_n row in Table 3.2.1 is a copy from the specialization Table 3.0.1, which we give in the intention of checking the specialization argument in §3.1.3 and §3.1.4. As for the other types, Table 3.2.1 claims that the type D_n is clean, but that the type C_n^{RY} (Ram-Yip type C_n) is a little confusing, which turns out to correspond to the type $B_n^\vee$ in the sense of [M03].

3.2.1 Ram-Yip type C_n

In this subsection, we show that the Ram-Yip formula of the non-symmetric Macdonald polynomial of type C_n in the sense of [RY11] can be obtained from the Ram-Yip type formula of type $(C_n^\vee, C_n)$ (Fact 2.2.3.1) by the corresponding specialization in Table 3.2.1:

$$t_0 = u_0 = u_n = 1.$$

See Proposition 3.2.1.5 for the precise statement.

A caution on the notation is in order. In [RY11], the Ram-Yip formula for what they call type C_n is derived using the affine root system of type $C_n^\vee$ in the sense of loc. cit. As mentioned before, it turns out that both the polynomial and the root system are different from those in [M03]. For distinction, we denote by $E_\mu^{C,\mathrm{RY}}(x)$ and $S^{C^\vee,\mathrm{RY}}$ the polynomial and the system treated in [RY11], and call them *the Macdonald polynomial of Ram-Yip type C_n* and *the affine root system of Ram-Yip type $C_n^\vee$*, respectively.

Affine root system of Ram-Yip type $C_n^\vee$

We start with the explanation on the system $S^{C^\vee,\mathrm{RY}}$. Let S be the affine root system of type $(C_n^\vee, C_n)$ in (3.1.2). The affine root system $S^{C^\vee,\mathrm{RY}}$ of Ram-Yip type $C_n^\vee$ is the subset of S given by

$$\begin{aligned} S^{C^\vee,\mathrm{RY}} :=& O_1 \sqcup O_5 \\ =& \{\pm\epsilon_i + rc \mid 1 \le i \le n,\ r \in \mathbb{Z}\} \sqcup \{\pm\epsilon_i \pm \epsilon_j + rc \mid 1 \le i < j \le n,\ r \in \mathbb{Z}\}, \end{aligned} \tag{3.2.1}$$

where we used the W-orbits in (3.1.2). The basis of $S^{C^\vee,\mathrm{RY}}$ in [RY11] is given by

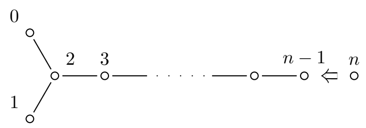
$$a_0^{C^\vee,\mathrm{RY}} := -(\epsilon_1+\epsilon_2)+c, \quad a_j^{C^\vee,\mathrm{RY}} := a_j = \epsilon_j - \epsilon_{j+1} \quad (j=1,\dots,n-1), \quad a_n^{C^\vee,\mathrm{RY}} := \epsilon_n.$$

Note that we have $a_j^{C^\vee,\mathrm{RY}} = a_j$ in (3.1.3), but the other two roots are different from those in (3.1.3).

Next, we turn to the extended affine Weyl group. The refections associated to the above basis are denoted by

$$s_0^{C^\vee} := s_{a_0^{C^\vee,\mathrm{RY}}}, \quad s_i = s_{a_i^{C^\vee,\mathrm{RY}}} \quad (i=1,\dots,n), \tag{3.2.2}$$

where we used $s_i \in W_0$ in (3.1.5). Note that we have the common s_n although $a_n^{C^\vee,\mathrm{RY}} \ne a_n$. We also consider the automorphism group $\Omega^{C^\vee,\mathrm{RY}}$ of the extended Dynkin diagram of type B_n:



Explicitly, using the weight lattice $P_{B_n} = \oplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)$ of type B_n in (3.1.38), we have

$$\Omega^{C^\vee,\mathrm{RY}} := P_{C_n}^\vee / Q_{C_n}^\vee = P_{B_n}/Q_{B_n} = \langle \pi^{C^\vee} \mid (\pi^{C^\vee})^2 = e \rangle.$$

The generator $\pi^{C^\vee}$ flips the diagram by transposing the vertices $0 \leftrightarrow 1$. Then, the extended affine Weyl group $W^{C^\vee,\mathrm{RY}}$ is defined to be the subgroup of $\mathrm{GL}_{\mathbb{R}}(V)$ generated by the reflections in (3.2.2) and $\pi^{C^\vee}$. In other words, we have

$$W^{C^\vee,\mathrm{RY}} := \langle s_0^{C^\vee}, s_1, \dots, s_n, \pi^{C^\vee} \rangle.$$

As an abstract group, $W^{C^\vee,\mathrm{RY}}$ is presented by these generators with the following relations.

$$
\begin{aligned}
&\pi^{C^\vee} s_0^{C^\vee} = s_1 \pi^{C^\vee}, && s_i{}^2 = (s_0^{C^\vee})^2 = (\pi^{C^\vee})^2 = e \quad (1 \le i \le n),\\
&s_0^{C^\vee} s_1 = s_1 s_0^{C^\vee}, && s_i s_j = s_j s_i \quad (|i-j|>1, (i,j) \notin \{(0,2),(2,0)\}),\\
&s_0^{C^\vee} s_2 s_0^{C^\vee} = s_2 s_0^{C^\vee} s_2, && s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \le i \le n-2),\\
&s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n.
\end{aligned}
\tag{3.2.3}
$$

In the second line, we abusively denoted $s_0 := s_0^{C^\vee}$. Let us write down the action of $W^{C^\vee,\mathrm{RY}}$ on $F_\mathbb{Z} = P_{C_n} \oplus \frac{1}{2}\mathbb{Z}$ in (3.1.15).

$$
s_0^{C^\vee}(\epsilon_i) = \begin{cases} c-\epsilon_2 & (i=1) \\ c-\epsilon_1 & (i=2) \\ \epsilon_i & (i \ne 1,2) \end{cases}, \qquad s_j(\epsilon_i) = \begin{cases} \epsilon_j & (i=j+1) \\ \epsilon_j+1 & (i=j) \\ \epsilon_i & (i \ne j, j+1) \end{cases} \qquad (1 \le j \le n-1),
$$
$$
s_n(\epsilon_i) = \begin{cases} -\epsilon_n & (i=n) \\ \epsilon_i & (i \ne n) \end{cases}, \qquad \pi^{C^\vee}(\epsilon_i) = \begin{cases} c-\epsilon_1 & (i=1) \\ \epsilon_i & (i \ne 1) \end{cases}.
$$

We can see from this action that $W^{C^\vee,\mathrm{RY}}$ preserves $S^{C^\vee,\mathrm{RY}} \subset S$, and the description $S^{C^\vee,\mathrm{RY}} = O_1 \sqcup O_5$ in (3.2.1) is actually the decomposition into $W^{C^\vee,\mathrm{RY}}$-orbits.

In fact, as the following lemma shows, the group $W^{C^\vee,\mathrm{RY}}$ is identical to W in (3.1.10).

Lemma 3.2.1.1. The following gives a group isomorphism $\varphi^C \colon W \xrightarrow{\sim} W^{C^\vee,\mathrm{RY}}$.

$$
\varphi^C(s_i) := s_i \quad (1 \le i \le n), \quad \varphi^C(s_0) := \pi^{C^\vee}.
$$

In particular, we have the following relations of subgroups in $\mathrm{GL}_\mathbb{R}(F_\mathbb{Z})$, $F_\mathbb{Z} = V \oplus \mathbb{R}c$.

$$
W = W^{C^\vee,\mathrm{RY}} = \mathrm{t}(P_{C_n}) \ltimes W_0.
$$

Proof. We regard W as the group with the presentation $\langle s_0, s_1, \ldots, s_n \rangle$ in (3.1.12). Since $\varphi^C(s_0 s_1 s_0) = \pi^{C^\vee} s_1 \pi^{C^\vee} = s_0^{C^\vee}$, we have the surjectivity of the homomorphism $\varphi^\vee$ up to well-definedness. Thus, it is enough to show that the defining relations (3.1.13) of W are mapped by φ^C to those (3.2.3) of $W^{C^\vee,\mathrm{RY}}$. The non-trivial parts are those containing $s_0 \in W$. As for the fourth relation $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$ in (3.1.13), the application of φ^C yields

$$
\varphi^C(s_0 s_1 s_0 s_1) = \varphi^C(s_1 s_0 s_1 s_0) \iff \pi^{C^\vee} s_1 \pi^{C^\vee} s_1 = s_1 \pi^{C^\vee} s_1 \pi^{C^\vee} \iff s_0^{C^\vee} s_1 = s_1 s_0^{C^\vee},
$$

which is in the third line of (3.2.3). The other relations are similarly checked. □

For later use, we write down the reduced expression of $\mathrm{t}(\epsilon_i) \in W^{C^\vee,\mathrm{RY}}$ for $i = 1, 2, \ldots, n$.

$$
\begin{aligned}
\mathrm{t}(\epsilon_1) &= \pi^{C^\vee} s_1 \cdots s_n s_{n-1} \cdots s_1 \\
\mathrm{t}(\epsilon_2) &= \pi^{C^\vee} s_0^{C^\vee} s_1 \cdots s_n s_{n-1} \cdots s_2 \\
\mathrm{t}(\epsilon_i) &= \pi^{C^\vee} s_{i-1} \cdots s_2 s_0^{C^\vee} s_1 \cdots s_n s_{n-1} \cdots s_i \quad (3 \le i \le n).
\end{aligned}
\tag{3.2.4}
$$

Ram-Yip formula of non-symmetric Macdonald polynomials of Ram-Yip type C_n

Recalling the $W^{C^\vee,\mathrm{RY}}$-orbit decomposition $S^{C^\vee,\mathrm{RY}} = O_1 \sqcup O_5$ in (3.2.1), we take parameters in the correspondence

$$
t_s^{\mathrm{RY}} \longleftrightarrow O_1, \quad t_m^{\mathrm{RY}} \longleftrightarrow O_5.
$$

For each $\mu \in P_{C_n}$, we have the non-symmetric Macdonald polynomial of Ram-Yip type C_n, which is then denoted by

$$
E_\mu^{C,\mathrm{RY}}(x) = E_\mu^{C,\mathrm{RY}}(x; q, t_s^{\mathrm{RY}}, t_m^{\mathrm{RY}}) \in \mathbb{K}_{C,\mathrm{RY}}[x^{\pm 1}], \quad \mathbb{K}_{C,\mathrm{RY}} := \mathbb{Q}(q^{\frac{1}{2}}, (t_s^{\mathrm{RY}})^{\frac{1}{2}}, (t_m^{\mathrm{RY}})^{\frac{1}{2}}).
$$

Below we explain the explicit formula of $E_\mu^{C,\mathrm{RY}}(x)$ given in [RY11].

For each $a = \alpha + rc \in S^{C^\vee,\mathrm{RY}} \subset P_{C_n} \oplus \mathbb{R}c$, we define $q^{\mathrm{sh}^C(a)}$ and $t^{\mathrm{ht}^C(a)}$ by

$$q^{\mathrm{sh}^C(\alpha+rc)} := q^{-r}, \quad t^{\mathrm{ht}^C(\alpha+rc)} := (t_s^{\mathrm{RY}})^{\langle \rho_s^C, \alpha\rangle} (t_m^{\mathrm{RY}})^{\langle \rho_m^C, \alpha\rangle}, \quad \rho_s^C := \sum_{i=1}^n \epsilon_i, \quad \rho_m^C = \sum_{i=1}^n (n-i)\epsilon_i. \tag{3.2.5}$$

We also denote the fundamental alcove of $S^{C^\vee,\mathrm{RY}}$ by

$$A^{C^\vee,\mathrm{RY}} := \left\{ x \in V \mid a_i^{C^\vee,\mathrm{RY}}(x) \geq 0, \ i = 0, 1, \ldots, n \right\}.$$

Then we have $A^{C^\vee,\mathrm{RY}} = A \cup s_0 A$, where A is the fundamental alcove (2.1.1) and s_0 is the 0-th reflection associated to $a_0 = -2\epsilon_1 + c \in S$ (3.1.3), both of type $(C_n^\vee, C_n)$. Note that $a_0 \neq a_0^{C^\vee,\mathrm{RY}}$, so that the corresponding hyperplanes and reflections are different. See Figure 3.2.1 for the case $n = 2$.

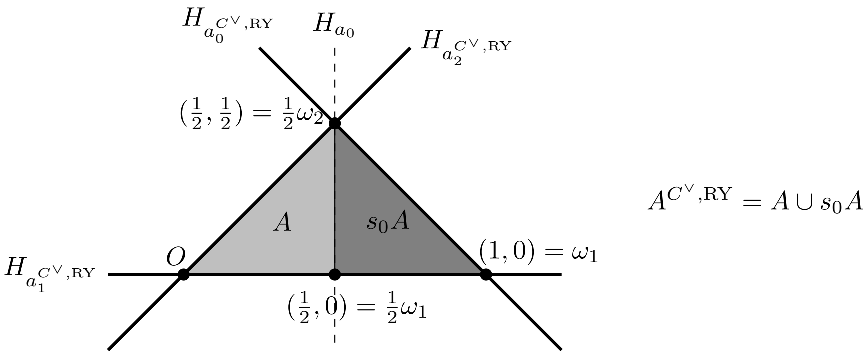


Figure 3.2.1: The fundamental alcove $A^{C^\vee,\mathrm{RY}}$ of Ram-Yip type C_2

Finally, for each $\mu \in P_{C_n}$, we denote the shortest element in the coset $\mathrm{t}(\mu)W_0$ by

$$w_C(\mu) \in W^{C^\vee,\mathrm{RY}}. \tag{3.2.6}$$

Finally, we denoted by $\Gamma_C(\overrightarrow{w}, z)$ the set of all alcove walks with start $z \in W^{C^\vee,\mathrm{RY}}$ of type $\overrightarrow{w}$.

Fact 3.2.1.2 ([RY11, Theorem 3.1])**.** Let $\mu \in P_{C_n}$ be arbitrary, and take a reduced expression $w_C(\mu) = (\pi^{C^\vee})^k s_{i_1} \cdots s_{i_r}$ with $k \in \{0, 1\}$, using the abbreviated symbols in (3.2.3). Then, we have

$$E_\mu^{C,\mathrm{RY}}(x) = \sum_{p \in \Gamma_C(\overrightarrow{w(\mu)}, e)} f_p^C t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)},$$
$$f_p^C := \prod_{k \in \varphi_+(p)} (\psi_{i_k}^C)^+ (q^{\mathrm{sh}^C(-\beta_k)} t^{\mathrm{ht}^C(-\beta_k)}) \prod_{k \in \varphi_-(p)} (\psi_{i_k}^C)^- (q^{\mathrm{sh}^C(-\beta_k)} t^{\mathrm{ht}^C(-\beta_k)}),$$

where $\beta_k := s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}} (a_{i_r}^{C^\vee,\mathrm{RY}})$ for $k = 1, 2, \ldots, r$, and $(\psi_i^C)^\pm(z)$ for $i = 0, 1, \ldots, n$ is given by

$$(\psi_i^C)^\pm(z) := \pm \frac{(t_m^{\mathrm{RY}})^{-\frac{1}{2}} - (t_m^{\mathrm{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (0 \leq i \leq n-1), \quad (\psi_n^C)^\pm(z) := \pm \frac{(t_s^{\mathrm{RY}})^{-\frac{1}{2}} - (t_s^{\mathrm{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}}. \tag{3.2.7}$$

Specialization to type C_n

In this part, we check that the specialization $t_0 = u_0 = u_n = 1$ of the Ram-Yip type formula for the non-symmetric Koornwinder polynomial $E_\mu(x)$ (Fact 2.2.3.1) is equal to the Ram-Yip formula for the non-symmetric Macdonald polynomial $E_\mu^{C,\mathrm{RY}}(x)$ of Ram-Yip type C_n (Fact 3.2.1.2). Using (3.1.28), we denote the specialized non-symmetric Koornwinder polynomial by

$$E_\mu^{\mathrm{sp},C}(x) = E_\mu^{\mathrm{sp},C}(x; q, t, t_n) := E_\mu(x; q, t, 1, t_n, 1, 1). \tag{3.2.8}$$

We denote by

$$\Gamma_0(\overrightarrow{w(\mu)}, e) \subset \Gamma(\overrightarrow{w(\mu)}, e)$$

the subset consisting of alcove walks without folding by s_0. We first show that under the specialization $t_0 = u_0 = u_n = 1$, the summation over $\Gamma(\overrightarrow{w(\mu)}, e)$ in Fact 2.2.3.1 reduces to that over $\Gamma_0(\overrightarrow{w(\mu)}, e)$.

Lemma 3.2.1.3. Let $\mu \in P_{C_n}$ be arbitrary, and take a reduced expression $w(\mu) = s_{i_1} \cdots s_{i_r}$ for the element $w(\mu) \in W$ given in (1.3.36). Then

$$\begin{aligned} E_\mu^{\mathrm{sp},C}(x) &= \sum_{p \in \Gamma_0(\overrightarrow{w(\mu)}, e)} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)}, \\ f_p &:= \prod_{k \in \varphi_+(p)} (\psi_{i_k}^{\mathrm{sp},C})^+ (q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}) \prod_{k \in \varphi_-(p)} (\psi_{i_k}^{\mathrm{sp},C})^- (q^{\mathrm{sh}(-\beta_k)}) t^{\mathrm{ht}(-\beta_k)}), \end{aligned}$$

where we used

$$(\psi_i^{\mathrm{sp},C})^\pm(z) := \pm \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 1, \ldots, n-1), \quad (\psi_n^{\mathrm{sp},C})^\pm(z) := \pm \frac{t_n^{-\frac{1}{2}} - t_n^{\frac{1}{2}}}{1 - z^{\pm 1}}. \tag{3.2.9}$$

Proof. The specialization $u_0 = u_n = 1$ yields $\psi_0^\pm(z) = 0$ by (1.3.33). Thus, no folding step by s_0 appear in the summation in Fact 2.2.3.1. Also, a direct calculation shows that under $t_0 = 1$, $\psi_i^\pm(z)$ is equal to $(\psi_i^{\mathrm{sp},C})^\pm(z)$ for $i = 1, \ldots, n$. $\square$

Comparing (3.2.7) and (3.2.9), we have

$$(\psi_i^{\mathrm{sp},C})^\pm(z)\Big|_{t = t_m^{\mathrm{RY}}} = (\psi_i^C)^\pm(z), \quad (\psi_n^{\mathrm{sp},C})^\pm(z)\Big|_{t_n = t_s^{\mathrm{RY}}} = (\psi_n^C)^\pm(z). \tag{3.2.10}$$

Hence, to check the identification of $E_\mu^{C,\mathrm{RY}}(x)$ with $E_\mu^{\mathrm{sp},C}(x)$, it is enough to construct a bijection

$$\Gamma_0(\overrightarrow{w(\mu)}, e) \longrightarrow \Gamma_C(\overrightarrow{w_C(\mu)}, e)$$

between the sets of alcove walks.

Lemma 3.2.1.4. For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_{i_1} \cdots s_{i_\ell}$ of the element $w(\mu) \in W$ in (1.3.36), and set

$$\begin{aligned} I &:= \{r \in \{1, 2, \ldots, \ell\} \mid i_r \neq 0\} = \{k_1 < k_2 < \cdots < k_s\} \quad (s \leq \ell), \\ J &:= \big\{(b_1, b_2, \ldots, b_\ell) \in \{0,1\}^\ell \mid b_i = 1 \ (i \notin I)\big\}. \end{aligned}$$

Also, define $\theta^C \colon J \to \{0,1\}^s$ by

$$J \ni (b_1, b_2, \ldots, b_\ell) \longmapsto (b_{k_1}, b_{k_2}, \ldots, b_{k_s}) \in \{0,1\}^s.$$

Then the following statements hold.

(1) The length of $w_C(\mu) \in W(C^{\vee,\mathrm{RY}})$ is $|I| = s$, and we can write $w_C(\mu)$ by

$$w_C(\mu) = \begin{cases} s_{j_1} s_{j_2} \cdots s_{j_s} & (s \in 2\mathbb{N}) \\ \pi^{C^\vee} s_{j_1} s_{j_2} \cdots s_{j_s} & (s \notin 2\mathbb{N}) \end{cases}$$

with some j_r's, where we used the abbreviation in (3.2.3).

(2) The map $\theta^C \colon J \to \{0,1\}^s$ induces a bijection

$$\begin{aligned} &\widetilde{\theta^C} \colon \Gamma_0(\overrightarrow{w(\mu)}, e) \longrightarrow \Gamma_C(\overrightarrow{w_C(\mu)}, e), \\ &p = (A, s_{i_1}^{b_1} A, \ldots, s_{i_1}^{b_1} \cdots s_{i_\ell}^{b_\ell} A) \longmapsto \begin{cases} (A_C, s_{j_1}^{b_{k_1}} A_C, \ldots, s_{j_1}^{b_{k_1}} \cdots s_{j_s}^{b_{k_s}} A_C) & (s \in 2\mathbb{N}) \\ (\pi^{C^\vee} A_C, \pi^{C^\vee} s_{j_1}^{b_{k_1}} A_C, \ldots, \pi^{C^\vee} s_{j_1}^{b_{k_1}} \cdots s_{j_s}^{b_{k_s}} A_C) & (s \notin 2\mathbb{N}) \end{cases}. \end{aligned}$$

(3) For any $p \in \Gamma_0(\overrightarrow{w(\mu)}, e)$, we have

$$\mathrm{wt}(p) = \mathrm{wt}\big(\widetilde{\theta^C}(p)\big), \quad \mathrm{d}(p) = \mathrm{d}\big(\widetilde{\theta^C}(p)\big).$$

Proof. (1) It is enough to show $\varphi^C(w(\mu)) = w_C(\mu)$ for any $\mu \in P_{C_n}$. First, we can see $\varphi^C(w(\epsilon_i)) = w_C(\epsilon_i)$ by the comparison between the reduced expressions (1.3.9) and (3.2.4). Since φ^C is a group isomorphism by Lemma 3.2.1.1, we see that $\varphi^C(w(\mu)) = w_C(\mu)$ for any $\mu \in P_{C_n}$.

(2) It is an immediate consequence of the item (1) and the bijectivity of θ.

(3) We want to show that for any $p \in \Gamma_0(\overrightarrow{w(\mu)}, e)$, expressing $e(p) = \mathrm{t}(\mathrm{wt}(p))\,\mathrm{d}(p)$, $\mathrm{wt}(p) \in P_{C_n}$, $\mathrm{d}(p) \in W_0$, we would have $\varphi^C(e(p)) = t\big(\mathrm{wt}(\widetilde{\theta^C}(p))\big)\,\mathrm{d}\big(\widetilde{\theta^C}(p)\big)$. For any $i = 1, 2, \dots, n$, we have $\varphi^C(\mathrm{t}(\epsilon_i)) = \mathrm{t}(\epsilon_i)$ by the comparison of (3.1.14) with (3.2.4). Thus we have $\mathrm{t}(\mathrm{wt}(p)) = \mathrm{t}(\mathrm{wt}(\widetilde{\theta^C}(p)))$ for any p, which means $\mathrm{wt}(p) = \mathrm{wt}(\widetilde{\theta^C}(p))$. On the other hand, since $\varphi^C|_{W_0} = \mathrm{id}_{W_0}$, we have $\mathrm{d}(p) = \mathrm{d}(\widetilde{\theta^C}(p))$ for any p. Thus the statement is proved.

□

Combining this lemma with (3.2.10), we obtain the desired identification

$$E^{\mathrm{sp},C}_\mu(x; q, t = t^{\mathrm{RY}}_m, t_n = t^{\mathrm{RY}}_s) = E^{C,\mathrm{RY}}_\mu(x; q, t^{\mathrm{RY}}_s, t^{\mathrm{RY}}_m).$$

The definition (3.2.8) of $E^{\mathrm{sp},C}_\mu(x)$ yields:

Proposition 3.2.1.5. For any $\mu \in P_{C_n}$, we have

$$E_\mu(x; q, t^{\mathrm{RY}}_m, 1, t^{\mathrm{RY}}_s, 1, 1) = E^{C,\mathrm{RY}}_\mu(x; q, t^{\mathrm{RY}}_s, t^{\mathrm{RY}}_m).$$

Comparing this result with the specialization Table 3.0.1, we see that it corresponds to type $B^\vee_n$. Thus, the Macdonald polynomial of Ram-Yip type C_n is the Macdonald polynomial of type $B^\vee_n$ in the sense of Definition 1.3.1.1.

3.2.2 Ram-Yip type B_n

The Ram-Yip formula of non-symmetric Macdonald polynomial of type B_n is derived in [RY11] using the affine root system of type $B^\vee_n$ in the sense of loc. cit. In this subsection, we give a similar argument as in the previous § 3.2.1 to type B_n, and show that under the specialization

$$t_n = u_n, \quad t_0 = u_0 = 1$$

we can recover the non-symmetric Macdonald polynomial of type B_n in the sense of [RY11] from the non-symmetric Koornwinder polynomial.

We will use similar terminologies on the affine root system and the non-symmetric Macdonald polynomials as in § 3.2.1. We denote by $S^{B^\vee,\mathrm{RY}}$ and $E^{B,\mathrm{RY}}_\mu(x)$ those considered in [RY11] for type B, and call them *the affine root system of Ram-Yip type* $B^\vee_n$ and *the Macdonald polynomial of Ram-Yip type* B_n, respectively.

Affine root system of Ram-Yip type $B^\vee_n$

Using the symbols in (3.1.2), the affine root system $S^{B^\vee,\mathrm{RY}}$ of Ram-Yip type is given by

$$\begin{aligned} S^{B^\vee,\mathrm{RY}} :=& (O_2 \sqcup O_4) \sqcup O_5 \\ =& \{\pm 2\epsilon_i + r \mid 1 \le i \le n,\ r \in \mathbb{Z}\} \sqcup \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \le i < j \le n,\ r \in \mathbb{Z}\}. \end{aligned} \tag{3.2.11}$$

The choice of the basis in [RY11] is given by

$$a^{B^\vee,\mathrm{RY}}_0 := a_0 = -2\epsilon_1 + c, \quad a^{B^\vee,\mathrm{RY}}_j := a_j = \epsilon_j - \epsilon_{j+1} \quad (j = 1, \dots, n-1), \quad a^{B^\vee,\mathrm{RY}}_n := a_n = 2\epsilon_n,$$

where a_i's are in (3.1.3). Thus, the associated reflections are $s_{a^{B^\vee}_i} = s_i$ in (3.1.5) and (3.1.11).

We turn to the explanation of the extended affine Weyl group. Let $\Omega_{B^\vee}$ be the automorphism group of the extended Dynkin diagram of type C_n:

$$\overset{1}{\circ} \Leftarrow \overset{2}{\circ}\!\!-\!\!-\!\!\overset{2}{\circ}\!\!-\!\!-\cdots\cdots-\!\!-\!\!\overset{2}{\circ}\!\!-\!\!-\!\!\overset{2}{\circ} \Rightarrow \overset{1}{\circ}$$

Explicitly, we have

$$\Omega_{B^\vee} := P^\vee_{B_n}/Q^\vee_{B_n} = P_{C_n}/Q_{C_n} = \langle \pi^{B^\vee} \mid (\pi^{B^\vee})^2 = e\rangle.$$

Then, the extended affine Weyl group $W^{B^\vee,\mathrm{RY}}$ is the subgroup of $\mathrm{GL}_{\mathbb{R}}(V)$, $V = \oplus_{i=1}^n \mathbb{R}\epsilon_i$ given by

$$W^{B^\vee,\mathrm{RY}} := \bigl\langle s_0, s_1, \ldots, s_n, \pi^{B^\vee} \bigr\rangle.$$

As an abstract group, $W^{B^\vee,\mathrm{RY}}$ has a presentation with these generators and the following relations.

$$\begin{aligned}
s_i^2 &= 1 && (i = 0, \ldots, n),\\
s_i s_j &= s_j s_i && (|i-j| > 1),\\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && (i = 1, \ldots, n-2),\\
s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i && (i = 0, n-1),\\
\pi^{B^\vee} s_i &= s_{n-i+1} \pi^{B^\vee} && (i = 0, 1, \ldots, n).
\end{aligned}$$

Let us write down the action of $W^{B^\vee,\mathrm{RY}}$ on $F_{\mathbb{Z}} = P_{C_n} \oplus \frac{1}{2}\mathbb{Z}c$ (3.1.15).

$$s_0(\epsilon_i) = \begin{cases} c - \epsilon_1 & (i = 1) \\ \epsilon_i & (i \neq 1) \end{cases}, \qquad s_j(\epsilon_i) = \begin{cases} \epsilon_j & (i = j+1) \\ \epsilon_j + 1 & (i = j) \\ \epsilon_i & (i \neq j, j+1) \end{cases} \qquad (j = 1, \ldots, n-1),$$

$$s_n(\epsilon_i) = \begin{cases} -\epsilon_n & (i = n) \\ \epsilon_i & (i \neq n) \end{cases}, \qquad \pi^{B^\vee}(\epsilon_i) = \frac{1}{2}c - \epsilon_{n-i+1}$$

We can see from this action that $W^{B^\vee,\mathrm{RY}}$ acts on $S^{B^\vee,\mathrm{RY}}$, and the description $S^{B^\vee,\mathrm{RY}} = O_1 \sqcup O_5$ in (3.2.11) is actually the decomposition into $W^{B^\vee,\mathrm{RY}}$-orbits.

The group $W^{B^\vee,\mathrm{RY}}$ also has the following descriptions.

$$W^{B^\vee,\mathrm{RY}} = \Omega_{B^\vee} \ltimes W = \mathrm{t}(P_{B_n}) \rtimes W_0, \quad P_{B_n} := \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n \oplus \mathbb{Z}\tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n), \tag{3.2.12}$$

where we used t in (3.1.6). For later use, we write down reduced expressions of $\mathrm{t}(\epsilon_i)$'s.

$$\begin{aligned}
\mathrm{t}(\epsilon_i) &= s_{i-1} \cdots s_1 s_0 s_1 \cdots s_n s_{n-1} \cdots s_i \quad (i = 1, 2, \ldots, n),\\
t\bigl(\tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)\bigr) &= \pi^{B^\vee}(s_n \cdots s_1) \cdots (s_n s_{n-1}) s_n.
\end{aligned} \tag{3.2.13}$$

Ram-Yip formula of non-symmetric Macdonald polynomial of type B_n

Next we consider the parameters for Macdonald polynomials. Recalling the $W^{B^\vee,\mathrm{RY}}$-orbit decomposition $S^{B^\vee,\mathrm{RY}} = O_5 \sqcup (O_2 \sqcup O_4)$ in (3.2.11), we take parameters t_m^{RY} and t_l^{RY} in the correspondence

$$t_m^{\mathrm{RY}} \longleftrightarrow O_5, \quad t_l^{\mathrm{RY}} \longleftrightarrow O_2 \sqcup O_4.$$

We have the non-symmetric Macdonald polynomial of Ram-Yip type B_n for $\mu \in P_{B_n}$ in (3.1.38), which is then denoted by

$$E_\mu^{B,\mathrm{RY}}(x) = E_\mu^{B,\mathrm{RY}}(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}}) \in \mathbb{K}_{B,\mathrm{RY}}[x^{\pm 1}], \quad \mathbb{K}_{B,\mathrm{RY}} := \mathbb{Q}\bigl(q^{\frac{1}{2}}, (t_m^{\mathrm{RY}})^{\frac{1}{2}}, (t_l^{\mathrm{RY}})^{\frac{1}{2}}\bigr).$$

For each $a = \alpha + rc \in S^{B^\vee,\mathrm{RY}}$, we define $q^{\mathrm{sh}^B(a)}$ and $t^{\mathrm{ht}^B(a)}$ by

$$q^{\mathrm{sh}^B(\alpha+rc)} := q^{-r}, \quad t^{\mathrm{ht}^B(\alpha+rc)} := t_m^{\langle \rho_m^B, \alpha\rangle} t_l^{\langle \rho_l^B, \alpha\rangle}, \quad \rho_m^B := \sum_{i=1}^n (n-i)\epsilon_i, \quad \rho_l^B := \tfrac{1}{2}\sum_{i=1}^n \epsilon_i. \tag{3.2.14}$$

We denote the fundamental alcove of Ram-Yip type $B_n^\vee$ by

$$A^{B^\vee,\mathrm{RY}} := \{x \in V \mid a_i^{B^\vee}(x) \geq 0,\ i = 0, 1, \ldots, n\}.$$

See Figure 3.2.2 for the case $n = 3$. We have $A^{B^\vee,\mathrm{RY}} = A$ in (2.1.1).

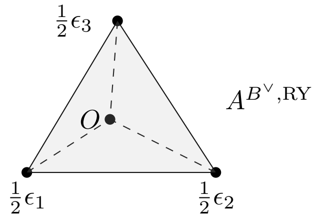


Figure 3.2.2: The fundamental alcove of Ram-Yip type $B_3^\vee$

We also denote by $w_B(\mu) \in W^{B^\vee,\mathrm{RY}}$ the shortest element of the coset $\mathrm{t}(\mu)W^{B^\vee,\mathrm{RY}}$. Finally, $\Gamma_B(\overrightarrow{w}, z)$ denotes the set of alcove walks with start $z \in W^{B^\vee,\mathrm{RY}}$ of type $\overrightarrow{w}$,

Fact 3.2.2.1 ([RY11, Theorem 3.1])**.** Let $\mu \in P_{B_n}$ be arbitrary, and take a reduced expression $w_B(\mu) = (\pi^{B^\vee})^k s_{i_1} s_{i_2} \cdots s_{i_r}$, $k \in \{0, 1\}$. Then we have

$$E_\mu^{B,\mathrm{RY}}(x) = \sum_{p \in \Gamma_B(\overrightarrow{w(\mu)}, e)} f_p^B t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)},$$

$$f_p^B := \prod_{k \in \varphi_+(p)} (\psi_{i_k}^B)^+(q^{\mathrm{sh}^B(-\beta_k)} t^{\mathrm{ht}^B(-\beta_k)}) \prod_{k \in \varphi_-(p)} (\psi_{i_k}^B)^-(q^{\mathrm{sh}^B(-\beta_k)} t^{\mathrm{ht}^B(-\beta_k)}),$$

where $\beta_k := s_{i_r} \cdots s_{i_{k+1}}(a_{i_r}^B)$ for $k = 1, 2, \ldots, r$, and $(\psi_i^B)^\pm(z)$ for $i = 0, 1, \ldots, n$ is given by

$$\begin{aligned} (\psi_j^B)^\pm(z) &:= \pm \frac{(t_m^{\mathrm{RY}})^{-\frac{1}{2}} - (t_m^{\mathrm{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (1 \leq j \leq n-1), \\ (\psi_i^B)^\pm(z) &:= \pm \frac{(t_l^{\mathrm{RY}})^{-\frac{1}{2}} - (t_l^{\mathrm{RY}})^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 0, n). \end{aligned} \tag{3.2.15}$$

Specialization to type B_n

In this part, we check that the specialization $t_n = u_n$, $t_0 = u_0 = 1$ of $E_\mu(x)$ in Fact 2.2.3.1 is equal to $E_\mu^{B,\mathrm{RY}}(x)$ in Fact 3.2.2.1. Using (3.1.28), we denote the specialized non-symmetric Koornwinder polynomial by

$$E_\mu^{\mathrm{sp},B}(x) = E_\mu^{\mathrm{sp},B}(x; q, t, t_n) := E_\mu(x; q, t, 1, t_n, 1, t_n). \tag{3.2.16}$$

Lemma 3.2.2.2. The map $s_i \mapsto s_i$ $(i = 0, \ldots, n)$ defines an injective group homomorphism $W \hookrightarrow W^{B^\vee,\mathrm{RY}}$.

Proof. Obvious from the structure $W^{B^\vee,\mathrm{RY}} = \Omega_{B^\vee} \ltimes W$ in (3.2.12). □

Lemma 3.2.2.3. For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_{i_1} s_{i_2} \cdots s_{i_r}$ of the element $w(\mu) \in W$ in (1.3.36). Then, we have

$$E_\mu^{\mathrm{sp},B}(x) = \sum_{p \in \Gamma(\overrightarrow{w(\mu)}, e)} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)},$$

$$f_p := \prod_{k \in \varphi_+(p)} (\psi_{i_k}^{\mathrm{sp},B})^+(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}) \prod_{k \in \varphi_-(p)} (\psi_{i_k}^{\mathrm{sp},B})^-(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}),$$

where $\beta_k := s_{i_r} s_{i_{r-1}} \cdots s_{i_{k+1}}(a_{i_r})$ for $k = 1, 2, \ldots, r$, and $(\psi_i^{\mathrm{sp},B})^\pm(z)$ for $i = 0, 1, \ldots, n$ is given by

$$(\psi_j^{\mathrm{sp},B})^\pm(z) := \pm \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (1 \leq j \leq n-1), \qquad (\psi_i^{\mathrm{sp},B})^\pm(z) := \pm \frac{t_n^{-\frac{1}{2}} - t_n^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 0, n). \tag{3.2.17}$$

Proof. A direct calculation with $t_n = u_n$ and $t_0 = u_0 = 1$ yields the result. $\square$

Since $P_{C_n} \subset P_{B_n}$, we have:

Proposition 3.2.2.4. For any $\mu \in P_{C_n}$, the following equality holds.

$$E_\mu(x; q, t_m^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}, 1, t_l^{\mathrm{RY}}) = E_\mu^{B,\mathrm{RY}}(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}}). \tag{3.2.18}$$

Proof. By (3.2.16), it is enough to show $E_\mu^{\mathrm{sp},B}(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}}) = E_\mu^{B}(x; q, t_m^{\mathrm{RY}}, t_l^{\mathrm{RY}})$. Comparing (3.2.15) and (3.2.17), we have

$$(\psi_j^{\mathrm{sp},B})^{\pm}(z)\Big|_{t=t_m^{\mathrm{RY}}} = (\psi_j^{B})^{\pm}(z), \quad (\psi_i^{\mathrm{sp},B})^{\pm}(z)\Big|_{t_n=t_l^{\mathrm{RY}}} = (\psi_i^{B})^{\pm}(z).$$

The embedding $W \hookrightarrow W^{B^\vee,\mathrm{RY}}$ in Lemma 3.2.2.2 implies that we have $\Gamma(\overrightarrow{w(\mu)}, e) = \Gamma_B(\overrightarrow{w(\mu)}, e)$ for any $\mu \in P_{B_n} \cap P_{C_n} = P_{C_n}$. Then, the result follows from Lemma 3.2.2.3. $\square$

Comparing this result with the specialization Table 3.0.1, we see that the specialization (3.2.18) corresponds to type B_n. Thus, the Macdonald polynomial of Ram-Yip type B_n is the Macdonald polynomial of type B_n in the sense of Definition 1.3.1.1.

3.2.3 Ram-Yip type D_n

By Proposition 3.1.4.6, we know that the specialization

$$t_n = u_n = t_0 = u_0 = 1$$

yields the non-symmetric Macdonald polynomial $E_\mu^D(x)$ of type D_n. In this subsection, we reprove it by using the Ram-Yip formula of type D_n, in which case there is no discrepancy between [M03] and [RY11], so we use our default notation for the affine root system and the non-symmetric Macdonald polynomials based on [M03].

Ram-Yip affine root system of type D

Recall the affine root system S^D of type D_n given in (3.1.54):

$$S^D := O_5 = \{\pm\epsilon_i \pm \epsilon_j + r \mid 1 \le i < j \le n,\ r \in \mathbb{Z}\}.$$

A basis given by

$$a_0^D := -\epsilon_1 - \epsilon_2 + c, \quad a_j^D := a_j = \epsilon_j - \epsilon_{j+1} \quad (1 \le j \le n-1), \quad a_n^D := \epsilon_{n-1} + \epsilon_n.$$

Denoting $s_n^D := s_{a_n^D}$, the finite Weyl group is given by $W_0^D := \langle s_1, \dots, s_{n-1}, s_n^D \rangle \simeq \{\pm 1\}^{n-1} \rtimes \mathfrak{S}_n$. Also, recall the weight lattice P_{D_n} in (3.1.55):

$$P_{D_n} := \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n \oplus \mathbb{Z}\tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$$

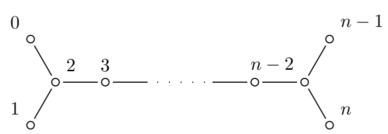
and the extended affine Weyl group $W^D = W_0^D \ltimes \mathrm{t}(P_{D_n})$ in (3.1.56). The group W^D has another description:

$$W^D = \langle s_0^D, s_1, \dots, s_{n-1}, s_n^D, \pi_1^D, \pi_{n-1}^D, \pi_n^D \rangle. \tag{3.2.19}$$

Here π_1^D, π_{n-1}^D and π_n^D denotes the generators of the automorphic group

$$\Omega_D := P_{D_n}/Q_{D_n} = \langle \pi_0^D = e, \pi_1^D, \pi_{n-1}^D, \pi_n^D \rangle$$

of the extended Dynkin diagram of type D_n:



As an abstract group, W^D is presented by the generators (3.2.19) and the following relations.

$$
\begin{aligned}
&s_i^2 = (s_0^D)^2 = e, && && \\
&s_0^D s_1 = s_1 s_0^D, && s_{n-1} s_n^D = s_n^D s_{n-1}, && s_i s_j = s_j s_i \quad (|i-j|>1), \\
&s_0^D s_2 s_0^D = s_2 s_0^D s_2, && s_n^D s_{n-2} s_n^D = s_{n-2} s_n^D s_{n-2}, && s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i=1,\dots,n-2), \\
&\pi_1^D s_0 = s_1 \pi_1^D, && \pi_{n-1}^D s_0^D = s_{n-1}\pi_{n-1}^D, && \pi_n^D s_0^D = s_n^D \pi_n^D, \\
&\pi_{n-1}^D s_1 = s_n \pi_{n-1}^D, && \pi_n^D s_1 = s_n \pi_n^D, && \pi_{n-1}^D s_i = s_{n-i}\pi_{n-1}^D \quad (i=2,\dots,n-2), \\
&\pi_n^D s_i = s_{n-i}\pi_n^D && (i=2,\dots,n-2), && (\pi_1^D)^2 = (\pi^D)^2 = (\pi_n^D)^2 = e \quad (i=1,\dots,n).
\end{aligned} \tag{3.2.20}
$$

Although it will not be used explicitly, let us write down the action of W^D on $F_{\mathbb{Z}}$ (3.1.15).

$$
s_0^D(\epsilon_i) = \begin{cases} c-\epsilon_2 & (i=1) \\ c-\epsilon_1 & (i=2) \\ \epsilon_i & (i\neq 1,2) \end{cases}, \qquad s_j(\epsilon_i) = \begin{cases} \epsilon_j & (i=j+1) \\ \epsilon_j+1 & (i=j) \\ \epsilon_i & (i\neq j,j+1) \end{cases} \qquad (j=1,\dots,n-1),
$$

$$
s_n^D(\epsilon_i) = \begin{cases} -\epsilon_n & (i=n-1) \\ -\epsilon_{n-1} & (i=n) \\ \epsilon_i & (i\neq n-1,n) \end{cases}, \qquad \pi_n^D(\epsilon_i) = \frac{1}{2}c - \epsilon_{n-i+1} \quad (i=0,\dots,n),
$$

$$
\pi_1^D(\epsilon_i) = \begin{cases} c-\epsilon_1 & (i=1) \\ \epsilon_i & (i\neq 1) \end{cases}, \qquad \pi_{n-1}^D(\epsilon_i) = \begin{cases} \frac{1}{2}c+\epsilon_n & (i=1) \\ \frac{1}{2}c-\epsilon_{n-i+1} & (i\neq 1) \end{cases}.
$$

We also write down reduced expressions of $\mathrm{t}(\epsilon_i)\in W^D$:

$$
\mathrm{t}(\epsilon_1) = \pi_1^D, \quad \mathrm{t}(\epsilon_2) = \pi_1^D s_0^D s_1, \quad \mathrm{t}(\epsilon_i) = \pi_1^D s_{i-1}\cdots s_2 s_0^D s_1 \cdots s_{i-1} \quad (i=3,\dots,n). \tag{3.2.21}
$$

Ram-Yip formula of non-symmetric Macdonald polynomial of type D

There is a unique W^D-orbit on the affine root system S^D, i.e., O_5, and correspondingly we set the parameter

$$
t \longleftrightarrow O_5.
$$

See also (3.1.57). For $\mu\in P_{D_n}$, the non-symmetric Macdonald polynomial of type D_n is denoted by

$$
E_\mu^D(x) = E_\mu^D(x;q,t).
$$

For each $a=\alpha+rc\in S^D$, we define $\mathrm{sh}^D(a)$ and $\mathrm{ht}^D(a)$ by

$$
q^{\mathrm{sh}^D(\alpha+rc)} := q^{-r}, \quad t^{\mathrm{ht}^D(\alpha+rc)} := t^{\langle\rho^D,\alpha\rangle}, \quad \rho^D = \sum_{i=1}^n (n-i)\epsilon_i. \tag{3.2.22}
$$

We also denote by $w_D(\mu)\in W^D$ the shortest element in the coset $\mathrm{t}(\mu)W_0^D$. For $\mu=\epsilon_i$, $i=1,2,\dots,n$, they are given by

$$
w_D(\epsilon_1) = \pi_1^D, \quad w_D(\epsilon_2) = \pi_1^D s_0^D, \quad w_D(\epsilon_i) = \pi_1^D s_{i-1}\cdots s_2 s_0^D \quad (3\le i\le n). \tag{3.2.23}
$$

The fundamental alcove of type D_n is denoted by

$$
A^D := \left\{x\in V \mid a_i^D(x)\ge 0,\ i=0,1,\dots,n\right\}.
$$

Finally, we denote by $\Gamma_D(\overrightarrow{w},z)$ the set of all alcove walks with start $z\in W^D$ of type $\overrightarrow{w}$,

Fact 3.2.3.1 ([RY11, Theorem 3.1]). For $\mu\in P_{D_n}$, take a reduced expression $w_D(\mu)=\pi_j^D s_{i_1}\cdots s_{i_r}$ of the element $w_D(\mu)\in W^D$ with some $j\in\{0,1,n-1,n\}$. Then we have

$$
E_\mu^D(x) = \sum_{p\in\Gamma_D(\overrightarrow{w(\mu)},e)} f_p^D t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)}, \tag{3.2.24}
$$

$$f_p^D := \prod_{k\in\varphi_+(p)} (\psi_{i_k}^D)^+(q^{\mathrm{sh}^D(-\beta_k)}t^{\mathrm{ht}^D(-\beta_k)})) \prod_{k\in\varphi_-(p)} (\psi_{i_k}^D)^-(q^{\mathrm{sh}^D(-\beta_k))}t^{\mathrm{ht}^D(-\beta_k)}),$$

where $\beta_k := s_{i_r}\cdots s_{i_{k+1}}(a_{i_r}^D)$ for $k = 1, 2, \dots, r$, and $(\psi_i^D)^\pm(z)$ for $i = 0, 1, \dots, n$ is given by

$$(\psi_i^D)^\pm(z) := \pm\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - z^{\pm 1}}.$$

For distinction, we denote by $E_\mu^{D,\mathrm{RY}}(x; q, t)$ the right hand side of (3.2.24).

Specialization to type D_n

In this part, we specialize $t_n = u_n = t_0 = u_0 = 1$ in $E_\mu(x)$ in Fact Fact 2.2.3.1, and show that it is equal to $E_\mu^{D,\mathrm{RY}}(x)$ in Fact Fact 3.2.3.1. We denote the specialized Koornwinder polynomial by

$$E_\mu^{\mathrm{sp},D}(x; q, t) := E_\mu(x; q, t, 1, 1, 1, 1). \tag{3.2.25}$$

Let $\Gamma_{0,n}(\overrightarrow{w(\mu)}, e) \subset \Gamma(\overrightarrow{w(\mu)}, e)$ be the subset consisting of alcove walks without folding by s_0 or s_n.

Lemma 3.2.3.2. For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_{i_1}s_{i_2}\cdots s_{i_r}$ of the element $w(\mu) \in W$ in (1.3.36). Then we have

$$E_\mu^{\mathrm{sp},D}(x) = \sum_{p\in\Gamma_{0,n}(\overrightarrow{w(\mu)},e)} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)},$$

$$f_p := \prod_{k\in\varphi_+(p)} (\psi_{i_k}^{\mathrm{sp},D})^+(q^{\mathrm{sh}(-\beta_k)}t^{\mathrm{ht}(-\beta_k)}) \prod_{k\in\varphi_-(p)} (\psi_{i_k}^{\mathrm{sp},D})^-(q^{\mathrm{sh}(-\beta_k)}t^{\mathrm{ht}(-\beta_k)}),$$

$$(\psi_i^{\mathrm{sp},D})^\pm(z) := \pm\frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 1, 2, \dots, n-1).$$

Proof. The specialization $t_0 = t_n = u_0 = u_n = 1$ in (1.3.33) yields $\psi_0^\pm(z) = \psi_n^\pm(z) = 0$. Thus the folding steps by s_0 or s_n does not appear in the summation of Fact Fact 2.2.3.1. □

Thus, it is enough to construct a bijection $\Gamma_{0,n}(\overrightarrow{w(\mu)}, e) \to \Gamma_D(\overrightarrow{w_D(\mu)}, e)$.

Lemma 3.2.3.3. The following gives an injective group homomorphism $\varphi^D : W \to W^D$.

$$\varphi^D(s_0) = \pi_1^D, \quad \varphi^D(s_i) = s_i \quad (1 \le i \le n-1), \quad \varphi^D(s_n) = e.$$

Proof. We can check that the relations (3.1.13) of W are mapped by φ^D to those (3.2.20) of W^D. Indeed, as for the final relation $s_0s_1s_0s_1 = s_1s_0s_1s_0$ in (3.1.13), we have

$$\varphi^D(s_0s_1s_0s_1) = \varphi^D(s_1s_0s_1s_0) \iff \pi_1^D s_1\pi_1^D s_1 = s_1\pi_1^D s_1\pi_1^D \iff s_0^D s_1 = s_1 s_0^D,$$

which is in the second line of (3.2.20). The other relations can be checked similarly. □

Lemma 3.2.3.4. For any $\mu \in P_{C_n}$, take a reduced expression $w(\mu) = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ of the element $w(\mu) \in W$ in (1.3.36), and set

$$I_0 := \{r \in \{1, 2, \dots, \ell\} \mid i_r \ne 0\}, \quad I_n := \{r \in \{1, 2, \dots, \ell\} \mid i_r \ne n\},$$
$$I := I_0 \cup I_n = \{k_1 < k_2 < \cdots < k_s\} \quad (s \le \ell), \quad J := \{(b_1, b_2, \dots, b_\ell) \in \{0, 1\}^\ell \mid b_i = 1 \ (i \notin I)\}.$$

Using them, define $\theta^D \colon J \to \{0, 1\}^s$ by

$$J \ni (b_1, b_2, \dots, b_\ell) \longmapsto (b_{k_1}, b_{k_2}, \dots, b_{k_s}) \in \{0, 1\}^s.$$

(1) The length of $w_D(\mu) \in W^D$ is equal to $|I| = s$, and

$$w_D(\mu) = \begin{cases} s_{j_1}s_{j_2}\cdots s_{j_s} & (|I_0| \in 2\mathbb{Z}) \\ \pi_1^D s_{j_1}s_{j_2}\cdots s_{j_s} & (|I_0| \notin 2\mathbb{Z}) \end{cases}.$$

(2) The map $\theta^D : J \to \{0,1\}^s$ induces a bijection

$$\widetilde{\theta^D} : \Gamma_{0,n}(\overrightarrow{w(\mu)}, e) \to \Gamma_D(\overrightarrow{w_D(\mu)}, e),$$
$$(A, s_{i_1}^{b_1} A, \ldots, s_{i_1}^{b_1} \cdots s_{i_\ell}^{b_\ell} A) \longmapsto \begin{cases} (A_D, s_{j_1}^{b_{k_1}} A_D, \ldots, s_{j_1}^{b_{k_1}} \cdots s_{j_s}^{b_{k_s}} A_D) & (|I_0| \in 2\mathbb{Z}) \\ (\pi_1^D A_D, \pi_1^D s_{j_1}^{b_{k_1}} A_D, \ldots, \pi_1^D s_{j_1}^{b_{k_1}} \cdots s_{j_s}^{b_{k_s}} A_D) & (|I_0| \notin 2\mathbb{Z}) \end{cases}.$$

(3) For any $p \in \Gamma_{0,n}(\overrightarrow{w(\mu)}, e)$, we have $\mathrm{wt}(p) = \mathrm{wt}(\widetilde{\theta^D}(p))$, $\mathrm{d}(p) = \mathrm{d}(\widetilde{\theta^D}(p))$.

Proof. (1) It is enough to show $\varphi^D(w(\mu)) = w_D(\mu)$ for any $\mu \in P_{C_n}$. By the reduced expressions (1.3.37) and (3.2.23), we have $\varphi^D(w(\epsilon_i)) = w_D(\epsilon_i)$ for each $i = 1, 2, \ldots, n$. Then, since φ^D is a group homomorphism by Lemma Lemma 3.2.3.3, we find the desired equality.

(2) It is an immediate consequence of (1) and the bijectivity of $\widetilde{\theta^D}$.

(3) Similarly as (1), we have $\varphi^D(\mathrm{t}(\epsilon_i)) = \mathrm{t}(\epsilon_i)$ for each $i = 1, 2, \ldots, n$, and thus $\mathrm{t}(\mathrm{wt}(p)) = \mathrm{t}(\mathrm{wt}(\widetilde{\theta^D}(p)))$ for each $p \in \Gamma_{0,n}(\overrightarrow{w(\mu)}, e)$, which implies $\mathrm{wt}(p) = \mathrm{wt}(\widetilde{\theta^D}(p))$. As for the remaining $\varphi^D(\mathrm{d}(p)) = \mathrm{d}(\widetilde{\theta^D}(p))$, since $\varphi^D(s_n) = e$ and φ^D preserves $s_1, s_2, \ldots, s_{n-1}$, the specialization $t_n = 1$ yields $t_{\mathrm{d}(p)} = t_{\mathrm{d}(\widetilde{\theta^D}(p))}$, which givers the consequence.

□

Thus we have $E_\mu^{\mathrm{sp},D}(x; q, t) = E_\mu^{D,\mathrm{RY}}(x; q, t)$ for any $\mu \in P_{C_n} \subset P_{D_n}$. Using (3.2.25), we have the conclusion:

Proposition 3.2.3.5. For any $\mu \in P_{C_n}$, the following equality holds.

$$E_\mu(x; q, t, 1, 1, 1, 1) = E_\mu^{D,\mathrm{RY}}(x; q, t).$$

3.3 Concluding remarks

The original motivation of our study on specialization is to find some explicit formula of symmetric Macdonald-Koornwinder polynomials, bearing in mind the Macdonald tableau formula [Ma95, Chap. VI, (7.13), (7.13')] for type GL_n. Certain progress has been developed for such tableau formulas of type B, C, D and $(C_n^\vee, C_n)$ by the recent papers [FH+15, HS18, HS20], although the connection to Ram-Yip type formulas seems to be still unclear.

Another interesting theme is the $t = \infty$ limit. By Sanderson [San00] and Ion [I03], it is known that the graded character of the level one (thin) Demazure module of an affine Lie algebra of type $X_l^{(r)}$, $X = A, D, E$, is equal to the non-symmetric Macdonald polynomial of the corresponding type specialized at $t = \infty$ if $X_l^{(r)} \neq A_{2l}^{(2)}$, and equal to non-symmetric Koornwinder polynomial specialized at $t = \infty$ in $A_{2l}^{(2)}$. There are vast amount of literature on this topic from representation-theoretic, combinatoric, and geometric points of view. For example, Orr and Shimozono [OS18] studied the relation of the limits and quantum Bruhat graphs. Let us also mention the article [Chi21] by Chihara, where the Demazure specialization for type $A_{2l}^{(2)}$ is identified with the graded character of a Demazure slice of the same type $A_{2l}^{(2)}$.

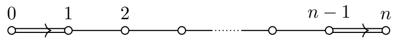
Returning to out study, it would be interesting to find a concrete connection between our argument and the argument given in [Chi21]. Let us close this paper by a naive explanation on why the non-symmetric Koornwinder polynomial is related to the representation theory of the affine Lie algebra of type $A_{2l}^{(2)}$. According to [I03, §3.2] and [Chi21, §1.5], one considers the specialization of the Noumi parameters

$$(t, t_0, t_n, u_0, u_n) = (t, t, t, 1, t). \tag{3.3.1}$$

Here we exchanged the specialized value of (t_0, u_0) and (t_n, u_n) in loc. cit., due to the numbering of roots explained below. Comparing (3.3.1) and the specialization Table 3.0.1, we find that (3.3.1) is included as the case $t_m = t_l^2 = t_s = t$ in the BC_n specialization of § 3.1.4:

	t	t_0	t_n	u_0	u_n
BC_n	t_m	t_l^2	t_s	1	t_s

Let us write again the Dynkin diagram (3.1.52) of the affine root system of type BC_n:



This is in fact the Dynkin diagram for the affine Lie algebra of type $A_n^{(2)}$ for even n [Ka90, p.55, §4.8, Table Aff 2], with the numbering of the roots $0, 1, \ldots, n$ reversed. Thus, very naively speaking, we can read the result of Ion on the Koornwinder specialization [I03, §3] from our specialization Table 3.0.1.

Chapter 4

Bispectral correspondence of QAKZ equations and Macdonald-type eigenvalue problems

Chapter 4 is based on the proceeding draft [YY] of the author's talk in the conference "Recent developments in Combinatorial Representation Theory" at RIMS, Kyoto University held in November 7th–11th, 2022, written with S. Yanagida.

4.0 Introduction

As mentioned in Preface, Abstract of Chapter 4, the purpose of this chapter is to give a review of the bispectral correspondence between QAKZ (quantum affine Knizhnik-Zamolodchikov) equations and the eigenvalue problems of Macdonald type, and to study the relation of the bispectral correspondence and the parameter specialization explained in Chapter 3.

Rank one review of bispectral correspondence

The first part (§ 4.1, § 4.2) is devoted to the review of the bispectral correspondence between QAKZ solutions and Macdonald-type eigenvalue problems, established by the works [vM11, vM11, St14].

Let us begin with the recollection on the original Cherednik's correspondence. We refer to [C05, §1.3] for an exposition of this correspondence. In [C92b], Cherednik introduced his QAKZ equations for arbitrary reduced root systems and for the type GL_n. Let $H = H(k, q)$ be the affine Hecke algebra of the concerning root systems, and let $T := \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$ be the algebraic torus associated to the weight lattice Λ. Then the QAKZ equations are q-difference equations for functions of torus variable $t \in T$ valued in a (left) H-module M satisfying certain conditions. In [C92a], Cherednik constructed a correspondence between solutions of the QAKZ equations for the principal series representation M_γ with central character $\gamma \in T$, and eigenfunctions of the q-difference operators of Macdonald type.

Below we explain the correspondence for the type GL_n. In this case, we can identify $\Lambda = \mathbb{Z}^n$ and put $t = (t_1, \ldots, t_n), \gamma = (\gamma_1, \ldots, \gamma_n) \in T$. Let $\mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma$ be the eigenspace of the Macdonald-Ruijsenaars q-difference operators of type GL_n, i.e.

$$\mathrm{SOL}_{\mathrm{Mac}}(k, q)_\gamma := \left\{ f(t) \in \mathcal{M}(T) \mid L_p^t f(t) = p(\gamma) f(t), \ \forall p \in \mathbb{C}[T]^{\mathfrak{S}_n} \right\},$$

where $\mathcal{M}(T)$ is the set of meromorphic functions on T, and L_p^t denotes the Macdonald-Ruijsenaars q-difference operator [R87, Ma95] associated to each symmetric polynomial p which acts on functions of t. For example, to the first elementary symmetric polynomial $e(z) = z_1 + \cdots + z_n$, the operator L_e^t is given by

$$L_e^t := \sum_{i=1}^n \prod_{j\neq i} \frac{kt_i - k^{-1}t_j}{t_i - t_j} T_{q,t_i}. \tag{4.0.1}$$

Here we used the q-shift operator T_{q,t_i} for $i = 1, \dots, n$:

$$(T_{q,t_i} f)(t_1, \dots, t_n) = f(t_1, \dots, qt_i, \dots, t_n), \quad f(t) \in \mathcal{M}(T).$$

Moreover, let $\mathrm{SOL}_{\mathrm{qKZ}}(k,q)_\gamma$ be the QAKZ equations of type GL_n, i.e.

$$\mathrm{SOL}_{\mathrm{qKZ}}(k,q)_\gamma := \left\{ f(t) \in H_0^{\mathcal{M}(T)} \mid C^\gamma_{\mathrm{t}(\lambda)}(t) f(q^{-\lambda} t) = f(t), \ \lambda \in \Lambda \right\},$$

where $H_0 = H_0(k)$ is the finite Hecke algebra of type A_{n-1} and $H_0^{\mathcal{M}(T)} := \mathcal{M}(T) \otimes_{\mathbb{C}} H_0$. We omit the precise definition of the q-difference operators $C^\gamma_{\mathrm{t}(\lambda)}(t)$. We will explain in detail the case of type A_1 and $(C_1^\vee, C_1)$ in § 4.1 and § 4.2, respectively.

Cherednik's correspondence for the type GL_n is now described as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{qKZ}}(k,q)_\gamma \longrightarrow \mathrm{SOL}_{\mathrm{Mac}}(k,q)_\gamma. \tag{$*$}$$

A bispectral analogue of Cherednik's correspondence is investigated by van Meer and Stokman [vMS09] for type GL, who introduced the bispectral QAKZ equations using Cherednik's duality anti-involution $* \colon \mathbb{H} \to \mathbb{H}$ of the double affine Hecke algebra $\mathbb{H}$ (see (1.3.24)). The bispectral QAKZ equations are consistent systems of q-difference equations for functions on the product torus $T \times T$, and splits up into two subsystems. Denoting by $(t, \gamma) \in T \times T$ the variable, we have:

- The first subsystem only acts on t, and for a fixed γ, the equations in t are Cherednik's QAKZ equations for the principal series representation M_γ of the affine Hecke algebra $H \subset \mathbb{H}$.
- For a fixed $t \in T$, the equations in γ are essentially the QAKZ equations for $M_{t^{-1}}$ of the image $H^* \subset \mathbb{H}$.

This argument can be extended to arbitrary reduced and non-reduced root systems, as done by van Meer [vM11] for reduced types and by Takeyama [T10] for the non-reduced type $(C_n^\vee, C_n)$.

After the build-up of bispectral QAKZ equations, it is rather straightforward, except for one issue, to make an analogue of Cherednik's construction of correspondence to the bispectral eigenvalue problems of Macdonald-type. Below we explain the case of type GL_n again. Let $\mathrm{SOL}_{\mathrm{bMac}}(k,q)$ be the bispectral eigenspace of the Macdonald-Ruijsenaars q-difference operators of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{bMac}}(k,q) := \left\{ f(t,\gamma) \in \mathcal{M}(T \times T) \;\middle|\; \begin{array}{l} L_p^t f(t,\gamma) = p(\gamma) f(t,\gamma) \\ L_e^\gamma f(t,\gamma) = p(t) f(t,\gamma) \end{array} \ \forall p \in \mathbb{C}[T]^{\mathfrak{S}_n} \right\}$$

where $\mathcal{M}(T \times T)$ is the set of meromorphic function on $T \times T$, and L_p^t, L_p^γ denote the Macdonald-Ruijsenaars q-difference operators attached to each symmetric polynomial p, acting on functions of t and γ, respectively. For the first elementary symmetric polynomial $e(z) = z_1 + \cdots + z_n$, they are given by

$$L_e^t := \sum_{i=1}^n \prod_{j\neq i} \frac{kt_i - k^{-1}t_j}{t_i - t_j} T_{q,t_i}, \quad L_e^\gamma := \sum_{i=1}^n \prod_{j\neq i} \frac{k^{-1}\gamma_i - k\gamma_j}{\gamma_i - \gamma_j} T_{q,\gamma_i}^{-1}.$$

Note that L_e^t is the same as (4.0.1), and the parameters q^{-1}, k^{-1} in L_p^γ are the reciprocal of those in L_p^t.

Next, let $\mathrm{SOL}_{\mathrm{bqKZ}}(k,q)$ be the solution space of the bispectral QAKZ equations of type GL_n, i.e.,

$$\mathrm{SOL}_{\mathrm{bqKZ}}(k,q) := \left\{ f(t,\gamma) \in H_0^{\mathcal{M}(T\times T)} \;\middle|\; \begin{array}{l} C_{(\mathrm{t}(\lambda),e)}(t,\gamma) f(q^{-\lambda} t, \gamma) = f(t,\gamma) \\ C_{(e,\mathrm{t}(\mu))}(t,\gamma) f(t, q^{\mu}\gamma) = f(t,\gamma) \end{array} \ \forall \lambda, \mu \in \Lambda \right\},$$

where $H_0^{\mathcal{M}(T\times T)} := \mathcal{M}(T \times T) \otimes_{\mathbb{C}} H_0$. We omit the precise definitions of the q-difference operators $C_{(\mathrm{t}(\lambda),e)}(t,\gamma)$ and $C_{(e,\mathrm{t}(\mu))}(t,\gamma)$, and refer to § 4.1 and § 4.2 for the explanation for type A_1 and $(C_1^\vee, C_1)$.

Mimicking $(*)$, the resulting bispectral correspondence is depicted as

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(k,q) \longrightarrow \mathrm{SOL}_{\mathrm{bMac}}(k,q).$$

type	Dynkin	orbits	Hecke parameters			
$(C_1^\vee, C_1)$ Askey-Wilson	$\overset{*}{\underset{0}{\circ}} = \overset{*}{\underset{1}{\circ}}$	$O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$	k_0	k_1	l_0	l_1
A_1	$\underset{0}{\circ} = \underset{1}{\circ}$	O_1	1	t	1	t
		O_3	t	1	t	1
Rogers		O_2	1	t^2	1	1
		O_4	t^2	1	1	1

Table 4.0.1: Type A_1 subsystems in $(C_1^\vee, C_1)$ and parameter specializations

The issue here is the existence of (some nice) asymptotic free solutions of the bispectral QAKZ equations, i.e., non-emptiness of the source, which was carefully done for type GL_n in [vM11, §5, Appendix]. The same argument works with minor modification for reduced and non-reduced root types (see [St14, §3]).

In Chapter 4, we give a review of the bispectral correspondence explained so far. Since the correspondence itself looks rather abstract, we decided to concentrate on the rank one cases and give detailed computations.

- In § 4.1, we treat the reduced root system of type A_1. The corresponding Macdonald-Koornwinder polynomials are the Rogers polynomials.
- In § 4.2, we treat the non-reduced root system of type $(C_1^\vee, C_1)$. The corresponding polynomials are the Askey-Wilson polynomials.

The GL_2 case could be included, but it is essentially the same with A_1, and we will not treat it.

Specializing parameters in the rank one bispectral problems

The second part (§ 4.3) is a complement of the first part, and is also a continuation of the paper [YY22] on the parameter specialization of Macdonald-Koornwinder polynomials. There we classify all the specializations based on the affine root systems appearing as subsystems of the type $(C_n^\vee, C_n)$ system. The obtained parameter specializations are compatible with degenerations of the Macdonald-Koornwinder inner product to the subsystem inner products.

In the rank one case [YY22, §2.6], where the concerned polynomials are Askey-Wilson polynomials, we discovered four ways of specialization of the type $(C_1^\vee, C_1)$ parameters to recover the type A_1. Table 4.0.1 is the excerpt from [YY22, §2.6, Table 2].

In § 4.3, we study the relation between our parameter specializations and the bispectral correspondence. To begin with, let us recall that the bispectral correspondence is built using the duality anti-involution $*$ of the DAHA $\mathbb{H}$. As reviewed in § 4.2.1 (4.2.16), the duality anti-involution $*$ of $\mathbb{H}$ affects on the Hecke parameters in the way

$$(k_1^*, k_0^*, l_1^*, l_0^*) = (k_1, l_1, k_0, l_0).$$

Then, we see from Table 4.0.1 that the specialization corresponding to the orbit O_2 is the only one which is compatible with the bispectral correspondence reviewed in the first part. Under this specialization, we establish the following commutative diagram (Theorem 4.3.1.2).

$$\begin{tikzcd}
\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)} \arrow[r, hook, "\chi_+^{(C_1^\vee, C_1)}"] \arrow[d, hook, "\mathrm{sp}"'] & \mathrm{SOL}_{\mathrm{bAW}} \arrow[d, hook, "\mathrm{sp}"] \\
\mathrm{SOL}_{\mathrm{bqKZ}}^{A_1} \arrow[r, hook, "\chi_+^{A_1}"'] & \mathrm{SOL}_{\mathrm{bMR}}
\end{tikzcd}$$

Notation and terminology

We use the notation in § 1.0, and Gasper-Rahman basic hypergeometric notation explained in § 1.1.1. Let us write down the latter again. Using q-shifted factorials (1.1.1):

$$(x; q)_\infty := \prod_{n=0}^{\infty} (1 - xq^n), \quad (x_1, \ldots, x_r; q)_\infty := \prod_{i=1}^{r} (x_i; q)_\infty,$$

the basic hypergeometric series is given by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_1, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix}; q, \ z\right] := \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. \tag{4.0.2} $$

We will also use the q-binomial coefficient

$$ \begin{bmatrix} \beta \\ n \end{bmatrix}_q := \frac{(q^{\beta-n+1}; q)_n}{(q; q)_n} \tag{4.0.3} $$

for $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$. Note that we have $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q = \frac{(q;q)_m}{(q;q)_n\,(q;q)_{m-n}}$ for $m, n \in \mathbb{N}$ with $m \geq n$.

4.1 Type A_1

4.1.1 Extended affine Hecke algebra

Here we recall the extended affine Hecke algebra of type A_1 and the basic representation.

The extended affine Weyl group of type A_1

We start with the recollection of the extended affine Weyl group of the affine root system of type A_1. For the detail, see [M03, §1, §2, §6.1], [vMS09, §2.1] and [vM11, §2.1].

Remark 4.1.1.1. Let us remark in advance that we work in the untwisted affine root system [M03, (1.4.1)], although [vM11] works in the twisted affine system [M03, (1.4.2)]. Since we only consider the type A_1, there is no essential difference, but there are a few notational differences. For instance, we define the extended affine Weyl group W to be the semi-direct product $W_0 \ltimes \mathrm{t}(P)$ using the weight lattice P, although in [vM11] it is defined to be $W_0 \ltimes \mathrm{t}(P^\vee)$ using the coweight lattice $P^\vee$.

We consider the one-dimensional real Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with

$$ V = \mathbb{R}\alpha, \quad \langle \alpha, \alpha \rangle = 2. \tag{4.1.1} $$

Let F be the space of affine real functions on V, which is identified with real vector space $V \oplus \mathbb{R}c$ by the map $(u \mapsto \langle v, u \rangle + r) \mapsto v + rc$ for $u, v \in V$ and $r \in \mathbb{R}$. Using the gradient map $D\colon F \to V$, $v + rc \mapsto v$, we extend the inner product $\langle \cdot, \cdot \rangle$ on V to a positive semi-definite bilinear form on F by $\langle f, g \rangle := \langle D(f), D(g) \rangle$ for $f, g \in F$.

Let $S(A_1) := \{\pm\alpha + nc \mid n \in \mathbb{Z}\} \subset F$ be the affine root system $S(A_1)$ in the sense of Macdonald [M03]. A basis of $S(A_1)$ is given by $\{a_1 := \alpha, a_0 := c - \alpha\}$, and the associated simple reflections $s_i\colon V \to V$ for $i = 0, 1$ are given by

$$ s_i(v) := v - a_i(v) D(a_i^\vee) \quad (v \in V), \tag{4.1.2} $$

where $a_i^\vee := 2a_i/\langle a_i, a_i \rangle = a_i \in F$. Explicitly, we have

$$ s_1(r\alpha) = -r\alpha, \quad s_0(r\alpha) = (1-r)\alpha \qquad (r \in \mathbb{R}). \tag{4.1.3} $$

We denote by $W_0 \subset O(V, \langle \cdot, \cdot \rangle)$ the subgroup generated by s_1. It is the Weyl group of the irreducible root system $R(A_1) = \{\pm\alpha\}$ of type A_1 in the sense of Bourbaki, and as an abstract group, we have $W_0 = \langle s_1 \mid s_1^2 \rangle \cong \mathfrak{S}_2$, the symmetric group of degree 2. Let us also denote the fundamental weight and the weight lattice of the root system $R(A_1)$ by

$$ \varpi := \tfrac{1}{2}\alpha, \quad \Lambda := \mathbb{Z}\varpi \subset V. $$

Then the W_0-action (4.1.3) preserves Λ.

We denote by $\mathrm{t}(\Lambda) := \{\mathrm{t}(\lambda) \mid \lambda \in \Lambda\}$ the abelian group with relations $\mathrm{t}(\lambda)\,\mathrm{t}(\mu) = \mathrm{t}(\lambda+\mu)$ for $\lambda, \mu \in \Lambda$. The group $\mathrm{t}(\Lambda)$ acts on V by translation:

$$ \mathrm{t}(\lambda)v = v + \lambda \quad (\lambda \in L, \ v \in V). \tag{4.1.4} $$

Then *the extended affine Weyl group* W *of* $S(A_1)$ is defined to be the semi-direct product group

$$W := W_0 \ltimes \mathrm{t}(\Lambda) \tag{4.1.5}$$

which acts on V faithfully. In other words, the group W is determined by W_0 and $\mathrm{t}(\Lambda)$, and by the additional relations

$$s_1\,\mathrm{t}(\lambda)s_1 = \mathrm{t}(s_1(\lambda)) \quad (\lambda \in \Lambda) \tag{4.1.6}$$

with $s_1(\lambda)$ given by (4.1.3).

The group W is generated by s_1, s_0 and $\mathrm{t}(\varpi)$. It is convenient to introduce $u := \mathrm{t}(\varpi)s_1$. By (4.1.6), we have $u^2 = \mathrm{t}(\varpi)\,\mathrm{t}(s_1(\varpi)) = \mathrm{t}(\varpi)\,\mathrm{t}(-\varpi) = e$. Also, by (4.1.6) and (4.1.3), we can check $s_0(v) = us_1u(v)$ for any $v \in V$. Thus, as an abstract group, W is generated by s_1, s_0, u with defining relations

$$s_1^2 = s_0^2 = u^2 = e, \quad us_1 = s_0u. \tag{4.1.7}$$

For later use, we write down a few relations in W.

$$\mathrm{t}(\varpi) = us_1 = s_0u, \quad \mathrm{t}(-\varpi) = s_1u = us_0. \tag{4.1.8}$$

$$\mathrm{t}(\alpha) = \mathrm{t}(2\varpi) = us_1us_1 = s_0s_1. \tag{4.1.9}$$

The extended affine Hecke algebra of type A_1

Here we recall the extended affine Hecke algebra H associated to the affine root system $S(A_1)$. For the detail, see [M03, §4, §6.1] and [vM11, §2.2, §2.3]. Hereafter we fix nonzero complex numbers $k \in \mathbb{C}^\times$.

Remark 4.1.1.2. Our parameter k correspond to τ in [M03].

Definition 4.1.1.3. *The extended affine Hecke algebra of type* A_1, denoted by

$$H = H(k) = H^{A_1}(k),$$

is the $\mathbb{C}$-algebra generated by T_1, T_0 and U with fundamental relations

$$(T_i - k)(T_i + k^{-1}) = 0 \quad (i = 1, 0), \quad U^2 = 1, \quad UT_1 = T_0U. \tag{4.1.10}$$

By comparing (4.1.7) and (4.1.10), we see that H is a deformation of the group ring $\mathbb{C}[W]$ of the extended affine Weyl group W of $S(A_1)$ explained above.

In order to attach an element $T_w \in H$ to each $w \in W$, let us recall from [M03, §2.2] that we have the length function and reduced expressions in W. The group W is an extension of the affine Weyl group $W_S := \langle s_1, s_0 \mid s_1^2, s_0^2 \rangle$ of $S(A_1)$ by the automorphism u of the Dynkin diagram of $S(A_1)$, so that any element $w \in W$ can be written as $w = w'u^r$ with $w' \in W_S$ and $r \in \{0, 1\}$. The group W_S is a Coxeter group, so that it has the length function $\ell(\cdot)$ and reduced expression of each element. Now, let $w' = s_{i_1} \cdots s_{i_l}$ be a reduced expression in W_S with $l = \ell(w')$. Then we define the length of $w \in W$ to be $\ell(w) := \ell(w') = l$, and call the expression $w = s_{i_1} \cdots s_{i_l}u^r \in W$ a reduced expression of w.

Now, for $w \in W$, take a reduced expression $w = s_{i_1} \cdots s_{i_l}u^r$ and define

$$T_w := T_{i_1} \cdots T_{i_l}U^r \in H.$$

Then T_w is independent of the choice of reduced expression. By convention we have $T_e = 1$, the unit of the ring H.

Next we introduce *the Dunkl operator* to be

$$Y := UT_1 \in H. \tag{4.1.11}$$

By (4.1.10), Y is invertible and

$$Y^{-1} = T_1^{-1}U = (T_1 - k + k^{-1})U.$$

Also note that these can be regarded as deformations of the translations $\mathrm{t}(\pm\varpi) \in W$ given in (4.1.8). Let us also define

$$Y^{\lambda} := Y^{l} \in H \quad (\lambda = l\varpi \in \Lambda,\ l \in \mathbb{Z}).$$

In particular, we have

$$Y^{\alpha} = Y^{2\varpi} = Y^{2} = UT_1UT_1 = T_0T_1, \tag{4.1.12}$$

which corresponds to (4.1.9). We denote by $\mathbb{C}[Y^{\pm 1}] \subset H$ the ring of Laurent polynomials in Y. We have an isomorphism of $\mathbb{C}$-linear spaces

$$H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}], \tag{4.1.13}$$

where

$$H_0 = H_0(k) := \mathbb{C}T_e + \mathbb{C}T_{s_1} = \mathbb{C} + \mathbb{C}T_1 \tag{4.1.14}$$

is the subalgebra of H generated by T_1. We call H_0 *the finite Hecke algebra of type* A_1.

The basic representation and the double affine Hecke algebra of type A_1

Next, we review the basic representation of the extended affine Hecke algebra $H = H(k)$, mainly following [M03, §6.1]. See also [C05, Theorem 3.2.1] and references therein.

Below we choose and fix a parameter $q^{1/2} \in \mathbb{C}^{\times}$. The extended affine Weyl group W acts on the ring of Laurent polynomials

$$\mathbb{C}[x^{\pm 1}], \quad x := e^{\alpha/2} \tag{4.1.15}$$

by letting the generators s_1, s_0, u operate as

$$(s_{1,q}f)(x) = f(x^{-1}), \quad (s_{0,q}f)(x) = f(qx^{-1}), \quad (u_q f)(x) = f(q^{1/2}x^{-1}), \tag{4.1.16}$$

where we indicated the dependence on q explicitly.

Now, using the parameter $k \in \mathbb{C}^{\times}$, and define $b(x;k), c(x;k) \in \mathbb{C}(x)$ by

$$c(x;k) := \frac{k^{-1} - kx}{1-x}, \quad b(x;k) := k - c(x;k) = \frac{k - k^{-1}}{1-x}. \tag{4.1.17}$$

Then, denoting $x_1 := x^2$ and $x_0 := qx^{-2}$, we have an algebra embedding

$$\rho_{k,q} \colon H(k) \longrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}]), \tag{4.1.18}$$

$$\rho_{k,q}(T_i) := c(x_i;k)s_{i,q} + b(x_i;k) = k + c(x_i;k)(s_{i,q} - 1), \quad \rho_{k,q}(U) := u_q. \tag{4.1.19}$$

Note that the image is in $\mathrm{End}(\mathbb{C}[x^{\pm 1}]) \subsetneq \mathrm{End}(\mathbb{C}(x))$. We call $\rho_{k,q}$ *the basic representation of* $H(k)$.

Using the basic representation $\rho_{k,q}$, we introduce:

Definition 4.1.1.4. *The double affine Hecke algebra (DAHA) of type* A_1, denoted as

$$\mathbb{H} = \mathbb{H}(k,q) = \mathbb{H}^{A_1}(k,q),$$

is defined to be the subalgebra of $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ generated by $X^{\pm 1} :=$ (the multiplication operator by $x^{\pm 1}$) and the image $\rho_{k,q}\big(H(k)\big)$.

As an abstract algebra, the DAHA $\mathbb{H}$ of type A_1 is presented with generators T_1, U, X and relations

$$(T_1 - k)(T_1 + k^{-1}) = 0, \quad U^2 = 1, \quad T_1XT_1 = X^{-1}, \quad UXU = q^{1/2}U^{-1}. \tag{4.1.20}$$

See [M03, §4.7] and [C05] for the detail. The map $\rho_{k,q}$ of (4.1.18) extends to the embedding $\rho_{k,q} \colon \mathbb{H} \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$.

We have the Poincaré-Birkhoff-Witt type decomposition of $\mathbb{H}$ as a $\mathbb{C}$-linear space:

$$\mathbb{H} \cong \mathbb{C}[X^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}]. \tag{4.1.21}$$

This decomposition is compatible with $H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}]$ in (4.1.13) under the identification of $H = H(k)$ with the faithful image $\rho_{k,q}(H) \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$. Below we often identify $X^{\pm 1}$ and $x^{\pm 1}$, and denote the decomposition (4.1.21) as $\mathbb{H} = \mathbb{C}[x^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}]$.

Let us also recall *the duality anti-involution* introduced by Cherednik ([C95c], [M03, (4.7.6)]). It is the unique $\mathbb{C}$-algebra anti-involution

$$*\colon \mathbb{H}(k, q) \longrightarrow \mathbb{H}(k^*, q), \quad h \longmapsto h^* \tag{4.1.22}$$

such that, denoting by $X^\lambda :=$ (the multiplication operator by x^l) for $\lambda = l\varpi \in \Lambda$, $l \in \mathbb{Z}$, we have

$$T_1^* = T_1, \quad (Y^\lambda)^* = X^{-\lambda}, \quad (X^\lambda)^* = Y^{-\lambda} \quad (\lambda \in \Lambda), \quad k^* = k.$$

Here and hereafter we use the redundant symbol k^* for the comparison with type $(C_1^\vee, C_1)$ (see (4.2.15)).

Finally, we denote by

$$H(k)^* \subset \mathbb{H}(k^*, q) = \mathbb{H}(k, q) \tag{4.1.23}$$

the image of $H(k) \subset \mathbb{H}(k, q)$ under the duality anti-involution $*$. Then $H(k)^*$ is equal to the subalgebra of $\mathbb{H}(k, q)$ generated by the finite Hecke algebra $H_0(k)$ (see (4.1.14)) and $X^{\pm 1} = x^{\pm 1}$.

4.1.2 Bispectral quantum Knizhnik-Zamolodchikov equation

Let us explain the bispectral qKZ equation of the affine root system $S(A_1)$, mainly following [vM11, §3.2]. Hereafter we fix the parameters $q^{1/2}, k \in \mathbb{C}^\times$, and consider the basic representation $\rho_{k,q}\colon H(k) \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ of the affine Hecke algebra $H(k)$ in (4.1.18) and the DAHA $\mathbb{H}(k, q)$ in Definition 4.1.1.4.

The affine intertwiners of type A_1

Following [C05, §1.3], [vMS09, §2.3] and [vM11, Proposition 3.3], we introduce the affine intertwines of type A_1. Corresponding to the generators s_1, s_0, u of the extended Weyl group W (and T_1, T_0, U of $H(k)$), we define $\widetilde{S}_1, \widetilde{S}_0, \widetilde{S}_u \in \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ by

$$\widetilde{S}_i = \widetilde{S}_i(k, q) := d_i(x; k, q) s_{i,q} \quad (i = 1, 0), \quad \widetilde{S}_u = \widetilde{S}_u(q) := u_q, \tag{4.1.24}$$

where $s_{i,q}$ and u_q are the operators in (4.1.16), and the function $d_i(x)$ is given by

$$d_i(x) = d(x_i; k, q) := k^{-1} - k x_i, \quad x_1 := x^2, \; x_0 := q x^{-2}. \tag{4.1.25}$$

The elements $\widetilde{S}_1$, $\widetilde{S}_0$ and $\widetilde{S}_u$ belong to the subalgebra $\mathbb{H} \subset \mathrm{End}(\mathbb{C}[x^{\pm 1}])$ since

$$\widetilde{S}_i = (1 - x_i)(\rho_{k,q}(T_i) - k) + k^{-1} - k x_i, \quad \widetilde{S}_u = \rho_{k,q}(U) \tag{4.1.26}$$

More generally, for each $w \in W$, taking a reduced expression $w = s_{j_1} \cdots s_{j_l} u^r$ with $j_1, \ldots, j_l, r \in \{0, 1\}$, we define the element $\widetilde{S}_w \in \mathbb{H}$ by

$$\widetilde{S}_w := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{l-1}} d_{j_l})(x) \cdot w_q. \tag{4.1.27}$$

Here we used the action of s_i's on functions in x and the operator w_q, both given in (4.1.16). Note that this definition includes (4.1.24) by setting $\widetilde{S}_0 = \widetilde{S}_{s_0}$ and $\widetilde{S}_1 = \widetilde{S}_{s_1}$. The element $\widetilde{S}_w \in \mathbb{H}$ is independent of the choice of reduced expression $w = s_{j_1} \cdots s_{j_l} u^r$, since

$$d_w(x) := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{l-1}} d_{j_l})(x) \tag{4.1.28}$$

depends only on w [M03, (2.2.9)]. Moreover, by [vM11, Proposition 3.3 (ii)], we have

$$\widetilde{S}_w = \widetilde{S}_{j_1} \cdots \widetilde{S}_{j_l} \widetilde{S}_u^r. \tag{4.1.29}$$

We call the elements $\widetilde{S}_w$ in (4.1.27) *the affine intertwiners of type A_1*.

Remark 4.1.2.1. Our affine intertwines are obtained from those in [vM11] by replacing k, x with k^{-1}, x^{-1}. We made this replacement to simplify the comparison with the type $(C_1^\vee, C_1)$ discussed in § 4.3.

The double extended Weyl group

Extending the representation space $\mathbb{C}[x^{\pm 1}]$ of the basic representation $\rho_{k,q}$ (see (4.1.15) and (4.1.18)), we introduce

$$\mathbb{L} := \mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[\xi^{\pm 1}] = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]. \tag{4.1.30}$$

We sometimes call x *the geometric variable* and ξ *the spectral variable.*

Remark 4.1.2.2. The papers [vMS09, vM11, St14] considered (for a root system of arbitrary type) the ring $\mathbb{L}' := \mathbb{C}[T \times T] \cong \mathbb{C}[T] \otimes \mathbb{C}[T]$ of regular functions on the product $T \times T$, where $T := \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^{\times})$ is the algebraic torus associated to the lattice Λ. In loc. cit., the value of $t \in T$ at $\lambda \in \Lambda$ is written as $t^{\lambda} \in \mathbb{C}^{\times}$, and a point of $T \times T$ is denoted by $(t, \gamma) \in T \times T$. For the type A_1 we are considering, the lattice is $\Lambda = \mathbb{Z}\varpi$, and there is a natural identification $\mathbb{L}' \cong \mathbb{L}$ given by $(t \mapsto t^{\varpi}) \mapsto x$ and $(\gamma \mapsto \gamma^{\varpi}) \mapsto \xi$. The geometric and spectral variables x, ξ are called the coordinate (functions) of $T \times T$ in loc. cit. The formulas and arguments given in the following text are obtained from those in loc. cit. by replacing $f(t, \gamma) \in \mathbb{L}'$ with $f(x, \xi) \in \mathbb{L}$.

Then the DADA $\mathbb{H} = \mathbb{H}(k, q)$ in Definition 4.1.1.4 has a structure of an $\mathbb{L}$-module by

$$(f \otimes g)h := f(X) \cdot h \cdot g(Y) \tag{4.1.31}$$

for $f = f(x) \in \mathbb{C}[x^{\pm 1}] \subset \mathbb{L}$, $g = g(\xi) \in \mathbb{C}[\xi^{\pm 1}] \subset \mathbb{L}$ and $h \in \mathbb{H}$. Here $X \in \mathbb{H}$ denotes the multiplication operator by x (see Definition 4.1.1.4), and $Y \in H = \rho_{k,q}(H) \subset \mathbb{H}$ denotes the Dunkl operator (4.1.11). The $\cdot$ in the right hand side means to take the multiplication of the ring $\mathbb{H}$. Note that the PBW type decomposition (4.1.21) yields the natural $\mathbb{L}$-module isomorphism

$$\mathbb{H} \cong H_0^{\mathbb{L}} := \mathbb{L} \otimes H_0, \tag{4.1.32}$$

where in the right hand side $\mathbb{L}$ acts on the first tensor component $\mathbb{L}$ by ring multiplication.

We turn to the introduction of *the double extended Weyl group* $\mathbb{W}$, following [vMS09, §3.1] and [vM11, §3.2]. Let ι denote the nontrivial element of the group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. We define the group $\mathbb{W}$ as the semi-direct product

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W), \tag{4.1.33}$$

where $\iota \in \mathbb{Z}_2$ acts on the product $W \times W$ of the extended affine Weyl group W by

$$\iota(w, w') = (w', w)\iota \quad (w, w' \in W).$$

The group $\mathbb{W}$ acts on $\mathbb{L}$ as follows. We define an involution $\diamond \colon W \to W$ by

$$w^{\diamond} := w, \quad \mathrm{t}(\lambda)^{\diamond} := \mathrm{t}(-\lambda) \tag{4.1.34}$$

for $w \in W_0$ and $\lambda \in \Lambda$. Then the $\mathbb{W}$-action on $\mathbb{L}$ is given by

$$(wf)(x) := (w_q f)(x), \quad (w'g)(\xi) := ((w'^{\diamond})_q g)(\xi), \quad (\iota F)(x, \xi) = F(\xi^{-1}, x^{-1}) \tag{4.1.35}$$

for $w \in W = W \times \{e\} \subset \mathbb{W}$, $w' \in W = \{e\} \times W \subset \mathbb{W}$ and $f = f(x), g = g(\xi), F = F(x, \xi) \in \mathbb{L}$. Here w_q denotes the W-action in (4.1.16).

Remark 4.1.2.3. The element $\iota \in \mathbb{W}$ is designed to be consistent with the duality anti-involution $*$ (4.1.22) and the actions of $\mathbb{W}$ and $\mathbb{H}$ on $\mathbb{L}$.

Now, following [vMS09, §3.1] and [vM11, §3.2], we define $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_{\iota} \in \mathrm{End}(\mathbb{H})$ by

$$\widetilde{\sigma}_{(w,w')}(h) := \widetilde{S}_w \cdot h \cdot (\widetilde{S}_{w'})^*, \quad \widetilde{\sigma}_{\iota}(h) := h^* \quad (h \in \mathbb{H}). \tag{4.1.36}$$

Here $*$ denotes the anti-involution (4.1.22), and $\cdot$ denotes the multiplication of the ring $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ (or the composition of operators on $\mathbb{C}[x^{\pm 1}]$). The action is well defined since $\widetilde{S}_w \in \mathbb{H}$.

Fact 4.1.2.4 ([vMS09, Lemma 3.2], [vM11, Lemma 3.5])**.** For $h \in \mathbb{H}$, $f \in \mathbb{L}$ and $w, w' \in W$, we have

$$\widetilde{\sigma}_{(w,w')}(fh) = ((w, w')f)\widetilde{\sigma}_{(w,w')}(h), \quad \widetilde{\sigma}_{\iota}(fh) = (\iota f)\widetilde{\sigma}_{\iota}(h). \tag{4.1.37}$$

The cocycles

Below we denote the field of meromorphic functions of variables x and ξ by

$$\mathbb{K} := \mathcal{M}(x, \xi),$$

and set

$$H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0. \tag{4.1.38}$$

An element $f \in H_0^{\mathbb{K}}$ is regarded as a meromorphic function of x, ξ valued in $H_0 \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x^{\pm 1}])$. Also, we have a $\mathbb{C}$-linear isomorphism $H_0^{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$ by (4.1.32), and $f \in H_0^{\mathbb{K}}$ can be expressed as

$$f = \sum_{w \in W_0} f_w T_w, \quad f_w \in \mathbb{K}. \tag{4.1.39}$$

The $\mathbb{W}$-action on $\mathbb{L}$ given by (4.1.35) naturally extends to that on $\mathbb{K}$. Now the group $\mathbb{W}$ acts on $H_0^{\mathbb{K}}$ by

$$\mathbf{w} f := \sum_{w \in W_0} (\mathbf{w} f_w) T_w \tag{4.1.40}$$

for $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

By Fact 4.1.2.4, we can extend the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ uniquely to $\mathbb{C}$-linear endomorphisms of $H_0^{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$ such that the formulas (4.1.37) are valid for $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. We denote them by the same symbols $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_\iota \in \mathrm{End}_{\mathbb{C}}(H_0^{\mathbb{K}})$.

Fact 4.1.2.5 ([vMS09, Theorem 3.3], [vM11, Theorem 3.6])**.** There is a unique group homomorphism

$$\tau \colon \mathbb{W} \longrightarrow \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$$

satisfying

$$\tau(w, w')(f) = d_w(x)^{-1} d_{w'}(\xi^{-1})^{-1} \cdot \widetilde{\sigma}_{(w,w')}(f), \quad \tau(\iota)(f) = \widetilde{\sigma}_\iota(f) \tag{4.1.41}$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. Here we used the function d_w given by (4.1.28), and $\cdot$ denotes the $\mathbb{K}$-action given by (4.1.31). Moreover, we have

$$\tau(\mathbf{w})(g f) = w g \tau(\mathbf{w})(f)$$

for $g \in \mathbb{K}$, $f \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

Remark 4.1.2.6. In [vM11, Theorem 3.6], the action of $\tau(w, w')$ is written using $d^{\diamond}_{w'}(Y)$, which is equal to $d_{w'}(Y^{-1})$ according to [vMS09, Proof of Lemma 3.2].

Now we recall a terminology of non-abelian group cohomology. Let G be a group, and M be a G-group. We denote by $m^g \in M$ the action of $g \in G$ on $m \in M$. Then, a (1-)cocycle means a map $z \colon G \to M$ such that $z(g_1 g_2) = z(g_1) z(g_2)^{g_1}$ for any $g_1, g_2 \in G$.

Recall that $\mathbb{W}$ acts on $H_0^{\mathbb{K}}$ by (4.1.40). This action makes the group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ into a $\mathbb{W}$-group by

$$(\mathbf{w}, A) \longmapsto \mathbf{w} A \mathbf{w}^{-1} \quad (\mathbf{w} \in \mathbb{W},\ A \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})).$$

Fact 4.1.2.7 ([vMS09, Corollary 3.4], [vM11, Corollary 3.8])**.** The map

$$\mathbf{w} \longmapsto C_{\mathbf{w}} := \tau(\mathbf{w}) \mathbf{w}^{-1} \tag{4.1.42}$$

is a cocycle of $\mathbb{W}$ with values in the $\mathbb{W}$-group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$. In other words, for any $\mathbf{w}, \mathbf{w}' \in \mathbb{W}$, we have $C_{\mathbf{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ and

$$C_{\mathbf{w}\mathbf{w}'} = C_{\mathbf{w}} \mathbf{w} C_{\mathbf{w}'} \mathbf{w}^{-1}. \tag{4.1.43}$$

Note that the cocycles $C_{\mathbf{w}}$ depend on the parameters (k, q). Also note that, by the natural isomorphism

$$\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K} \otimes \mathrm{GL}_{\mathbb{C}}(H_0), \tag{4.1.44}$$

we can regard an element $C_{\mathbf{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as a meromorphic function of x, ξ valued in $\mathrm{GL}_{\mathbb{C}}(H_0)$. To stress this point, we denote it as

$$C_{\mathbf{w}}(x, \xi). \tag{4.1.45}$$

The bispectral bispectral quantum KZ equations of type A_1

Let us focus on the cocycles associated to the translations in $\mathbb{W}$, i.e., the elements in the subgroup

$$\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda) \subset W \times W \subset \mathbb{W}.$$

Recalling $\Lambda = \mathbb{Z}\varpi$, we denote

$$C_{l,m} := C_{(\mathrm{t}(l\varpi),\mathrm{t}(m\varpi))} \quad (l, m \in \mathbb{Z}). \tag{4.1.46}$$

Definition 4.1.2.8 ([vMS09, Dfn. 3.7], [vM11, Dfn. 3.9], [St14, Dfn. 3.2])**.** Using (4.1.45) and (4.1.46), we call the system of q-difference equations

$$C_{l,m}(x,\xi)f(q^{-l}x, q^m\xi) = f(x,\xi) \quad (l, m \in \mathbb{Z})$$

for $f \in H_0^{\mathbb{K}}$ *the bispectral quantum KZ equations* (*the bqKZ equations* for short) *of type* A_1. The solution space is denote by

$$\mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k,q) := \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } A_1\}.$$

Remark 4.1.2.9. The solution space is denoted by SOL in [vMS09, vM11], and by $\mathcal{K}_{k,q}$ in [St14]. Our symbol is a modification of the notation Sol_{QAKZ} in [C05, Theorem 1.3.8].

The cocycle values

As before, let $H = H(k)$ be the affine Hecke algebra of type A_1, $H_0 = H_0(k)$ be the subalgebra of H generated by T_1, and $H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0$. We can write down the cocycles $C_{1,0}$ and $C_{0,1}$ by the following representations of the affine Hecke algebra H and its duality anti-involution image H^* (see (4.1.23)).

Definition 4.1.2.10. $H_0^{\mathbb{K}}$ has the following left H-module structure and the right H^*-module structure: We define an algebra homomorphism $\eta_L\colon H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_L(A)\Big(\sum_{w\in W_0} f_w T_w\Big) := \sum_{w\in W_0} f_w(AT_w) \quad (A \in H), \tag{4.1.47}$$

using the expression (4.1.39) of an element of $H_0^{\mathbb{K}}$. We also define an algebra anti-homomorphism $\eta_R\colon H^* \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_R(A)\Big(\sum_{w\in W_0} f_w T_w\Big) := \sum_{w\in W_0} f_w(T_w A) \quad (A \in H^*). \tag{4.1.48}$$

Remark 4.1.2.11. The map η_L was introduced in [vMS09, §4.1] and [vM11, §4.1], denoted by η, under the name of *the formal principal series representation of* H, since it is a formal version of the principal series representation used in [C92c, C94]. We borrowed the symbol η_R from [T10, §4.2].

Lemma 4.1.2.12 (c.f. [vM11, (5.3)])**.** Regarding the cocycles $C_{1,0}, C_{0,1}$ as $\mathrm{GL}(H_0)$-valued meromorphic functions of x, ξ (see (4.1.45)), we have

$$C_{1,0}(x,\xi) = R_0^L(x_0)\eta_L(U), \tag{4.1.49}$$

$$C_{0,1}(x,\xi) = R_0^R(\xi_0')\eta_R(U^*), \tag{4.1.50}$$

where we denoted $x_0 := qx^{-2}$, $\xi_0' := q\xi^2$ and

$$\begin{aligned} R_i^L(z) &:= c(z,k)^{-1}\big(\eta_L(T_i) - b(z;k)\big) \quad\ = c(z;k)^{-1}\big(\eta_L(T_i) - k\big) + 1, \\ R_i^R(z) &:= c(z,k^*)^{-1}\big(\eta_R(T_i^*) - b(z;k^*)\big) = c(z;k^*)^{-1}\big(\eta_R(T_i^*) - (k^*)^{-1}\big) + 1, \end{aligned}$$

using $c(z;k), b(z;k)$ in (4.1.17) and the duality anti-involution $*$ in (4.1.22). We also used the redundant notation $k^* = k$.

Proof. We first calculate $C_{1,0} = C_{(\mathrm{t}(\varpi),e)} = \tau(\mathrm{t}(\varpi), e)\,(\mathrm{t}(\varpi), e)^{-1}$. We have $\mathrm{t}(\varpi) = us_1 = s_0 u$ by (4.1.8). Then, using (4.1.42) and (4.1.41), for any element $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$ ($f_w \in \mathbb{K}$), we have

$$\begin{aligned}C_{1,0} f &= \tau(s_0u, e)\,(s_0u, e)^{-1}\Big(\sum_{w\in W_0} f_w T_w\Big) = \tau(s_0u, e)\Big(\sum_{w\in W_0} ((s_0u, e)^{-1} f_w) T_w\Big) \\ &= d_{s_0u}(x)^{-1}\widetilde{\sigma}_{(s_0u,e)}\Big(\sum_{w\in W_0} (s_0u, e)^{-1} f_w T_w\Big) \\ &= d_{s_0u}(x)^{-1}\Big(\sum_{w\in W_0} ((s_0u, e)(s_0u, e)^{-1} f_w)\widetilde{S}_{s_0u} T_w\Big) = d_{s_0u}(x)^{-1}\Big(\sum_{w\in W_0} f_w \widetilde{S}_{s_0u} T_w\Big).\end{aligned}$$

Now, by (4.1.26), we have

$$\widetilde{S}_{s_0u} = \widetilde{S}_0\widetilde{S}_u = \big((1 - x_0)(\rho_{k,q}(T_0) - k) + k^{-1} - kx_0\big)\rho_{k,q}(U).$$

On the other hand, (4.1.28) and (4.1.25) yield $d_{s_0u}(x) = k^{-1} - kx_0$, and by (4.1.17), we have

$$d_{s_0u}(x)^{-1}(1 - x_0) = c(x_0; k)^{-1}. \tag{4.1.51}$$

Then, using Definition 4.1.2.10, we have

$$C_{1,0} f = \big(c(x_0; k)^{-1}(\eta_L(T_0) - k^{-1}) + 1\big)\eta_L(U)(f),$$

which yields (4.1.49).

Similarly, the action of $C_{0,1}$ on $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$ is computed as

$$C_{0,1} f = \tau(e, s_0u)(e, s_0u)^{-1}\Big(\sum_{w\in W_0} f_w T_w\Big) = d_{s_0u}(\xi^{-1})^{-1}\cdot\Big(\sum_{w\in W_0} f_w T_w \widetilde{S}^*_{s_0u}\Big),$$

where $\cdot$ denotes the $\mathbb{K}$-action (see (4.1.31)). By (4.1.24) and (4.1.26), we have

$$\widetilde{S}^*_{s_0u} = \widetilde{S}^*_u\widetilde{S}^*_0 = \rho_{k,q}(U)^*\big((\rho_{k,q}(T_0)^* - k)(1 - q^{-1}Y^{-2}) + k^{-1} - kq^{-1}Y^{-2}\big).$$

Now recall that a function $g(\xi)$ acts on H_0 by the right multiplication of $g(Y)$ (see (4.1.31)). Then, by (4.1.51) and Definition 4.1.2.10, we have

$$C_{0,1} f = \big((\eta_R(T_0^*) - k)c(qY^2; k)^{-1} + 1\big)\eta_R(U^*)(f),$$

which yields (4.1.50). □

Remark 4.1.2.13. A few comments on Lemma 4.1.2.12 are in order.

(1) By [vM11, Remark 4.4], we have

$$C_{(e,w)}(x, \xi) = C_\iota C_{(w,e)}(\xi^{-1}, x^{-1})C_\iota \tag{4.1.52}$$

for any $w \in W$, where we used the notation (4.1.45). The result of Lemma 4.1.2.12 is consistent with this equality.

(2) As shown in [vMS09, Lemma 4.3], the rational function

$$R_i(z) := c(z, k)^{-1}\big(\eta_L(T_i) - b(z; k)\big)$$

valued in $\mathrm{End}(H_0)$ satisfies the Yang-Baxter equation $R_0(z)R_1(zz')R_0(z') = R_1(z')R_0(zz')R_1(z)$. In the terminology [C05, §1.3.6], $R_i(z)$ is called the baxterization of T_i.

For later use, let us cite the following two facts.

Fact 4.1.2.14 ([vM11, Lemma 5.1]). Let $\mathcal{A} := \mathbb{C}[x^{-1}] \subset \mathbb{L} = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]$, and $\mathcal{Q}_0(\mathcal{A})$ be the subring of the quotient field $\mathcal{Q}(\mathcal{A}) = \mathbb{C}(x)$ consisting of rational functions which are regular at $x^{-1} = 0$. Considering $\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm 1}]$ as a subring of $\mathbb{C}(x, \xi)$, we have

$$C_{1,0} \in (\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm 1}]) \otimes \mathrm{End}(H_0). \tag{4.1.53}$$

Moreover, setting $C_{1,0}^{(0)} := C_{1,0}|_{x^{-1}=0} \in \mathbb{C}[\xi^{\pm 1}] \otimes \mathrm{End}(H_0)$, we have

$$C_{1,0}^{(0)} = k\eta_L(T_1 Y^{-1} T_1^{-1}). \tag{4.1.54}$$

Similarly, defining $\mathcal{B} := \mathbb{C}[\xi] \subset \mathbb{L}$, and $\mathcal{Q}_0(\mathcal{B}) \subset \mathcal{Q}(\mathcal{B})$ to be the subring consisting of rational functions which are regular at $\xi = 0$, we have

$$C_{0,1} \in (\mathbb{C}[x^{\pm 1}] \otimes \mathcal{Q}_0(\mathcal{B})) \otimes \mathrm{End}(H_0).$$

Moreover, setting $C_{0,1}^{(0)} := C_{0,1}|_{\xi=0} \in \mathbb{C}[x^{\pm 1}] \otimes \mathrm{End}(H_0)$, we have

$$C_{0,1}^{(0)} = k^* \eta_R(T_1 Y^{-1} T_1^{-1}).$$

Proof. We only show the statements for $C_{1,0}$ using Lemma 4.1.2.12. Let us denote $A(x) \approx A_0$ if $A(x) = A_0 + O(x^{-1})$ by expansion in terms of x^{-1}. Then we have $c(x_0; k) = c(qx^{-2}; k) \approx k$, and the expression (4.1.49) yields

$$C_{1,0} \approx C_{1,0}^{(0)} := \big(k(\eta_L(T_0) - k) + 1\big)\eta_L(U) = k\eta_L(T_1 Y^{-1} T_1^{-1}),$$

where we used $T_0 U = U T_1$ and $T_1^{-1} = T_1 - k + k^{-1}$ in $H = H(k)$ from (4.1.10), and $Y^{-1} = T_1^{-1} U$ from (4.1.11). Thus we have (4.1.53) and (4.1.54). $\square$

For the next fact, note that we have $\widetilde{S}_w^* \in H \subset \mathbb{H}$ for all $w \in W_0$.

Fact 4.1.2.15 ([vMS09, Lemma 4.2])**.** For $w \in W_0$, we set

$$\tau_w := \eta_L(\widetilde{S}_{w^{-1}}^*) T_e \in \mathbb{C}[\{e\} \times T] \otimes H_0 \subset H_0^{\mathbb{K}}.$$

Then the following statements hold.

(1) $\{\tau_w \mid w \in W_0\}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of simultaneous eigenfunctions for the η_L-action of $\mathbb{C}[Y^{\pm 1}] \subset \mathbb{H}$ on $H_0^{\mathbb{K}}$.

(2) For $p \in \mathbb{C}[T]$ and $w \in W_0$, we have

$$\eta_L(p(Y))(\gamma)\, \tau_w(\gamma) = (w^{-1}p)(\gamma)\, \tau_w(\gamma)$$

as H_0-valued regular functions in $\gamma \in T$.

We close this subsection with:

Lemma 4.1.2.16. The cocycles $C_{2,0}$ and $C_{0,2}$ are given by

$$C_{2,0} = R_0^L(x_0) R_1^L(x_1'), \quad C_{0,2} = R_0^R(\xi_0') R_1^R(\xi_1')$$

Here we used the notation of Lemma 4.1.2.12: $x_0 := qx^{-2}$, $\xi_0' := q\xi^2$ and

$$\begin{aligned} R_i^L(z) &:= c(x_i, k)^{-1}\big(\eta_L(T_i) - b(x_i; k)\big) &&= c(x_i; k)^{-1}\big(\eta_L(T_i) - k\big) + 1, \\ R_i^R(z) &:= c(\xi_i, k^*)^{-1}\big(\eta_R(T_i^*) - b(\xi_i; k^*)\big) &&= c(\xi_i; k^*)^{-1}\big(\eta_R(T_i^*) - (k^*)^{-1}\big) + 1. \end{aligned}$$

We further used $x_1' := q^2 x^{-2}$ and $\xi_1' := q^2 \xi^2$.

Proof. It is a consequence of the cocycle relation (4.1.43) and a similar calculation of Lemma 4.1.2.12. We omit the detail. $\square$

4.1.3 Bispectral Macdonald-Ruijsenaars equations

As in the previous § 4.1.2, we fix generic complex numbers $q^{1/2}$ and k.

We consider the crossed product algebra (the smash product algebra)

$$\mathbb{D}_q^{\mathbb{W}} := \mathbb{W} \ltimes \mathbb{C}(x, \xi),$$

where $\mathbb{W}$ acts as field automorphisms on $\mathbb{C}(x,\xi)$ by (4.1.35), and also the subalgebra $\mathbb{D}_q$ of $\mathbb{D}_q^{\mathbb{W}}$ defined by

$$\mathbb{D}_q := \big(\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)\big) \ltimes \mathbb{C}(x,\xi) \subset \mathbb{D}_q^{\mathbb{W}},$$

where $\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)$ is regarded as a subgroup of $W \times W \subset \mathbb{W}$. The subalgebra $\mathbb{D}_q$ is identified with the algebra of q-difference operators on $\mathbb{C}(x,\xi)$. We can expand each $D \in \mathbb{D}_q^{\mathbb{W}}$ as

$$D = \sum_{\mathbf{w}\in\mathbb{W}} f_{\mathbf{w}}\mathbf{w} = \sum_{\mathbf{s}\in W_0\times W_0} D_{\mathbf{s}}\mathbf{s} \tag{4.1.55}$$

with $f_{\mathbf{w}} \in \mathbb{C}(x,\xi)$ and $D_{\mathbf{s}} = \sum_{\mathbf{t}\in\mathrm{t}(\Lambda)\times\mathrm{t}(\Lambda)} g_{\mathbf{ts}}\mathbf{t} \in \mathbb{D}_q$. Then we define the restriction map $\mathrm{Res}\colon \mathbb{D}_q^{\mathbb{W}} \to \mathbb{D}_q$ to be the $\mathbb{C}(x,\xi)$-linear map

$$\mathrm{Res}(D) := \sum_{\mathbf{s}\in W_0\times W_0} D_{\mathbf{s}}. \tag{4.1.56}$$

Next, we introduce two realizations of the basic representation ρ of H. One is given by

$$\rho^x_{1/k,q}\colon H(1/k) \longrightarrow \mathbb{D}_q^{\mathbb{W}} \tag{4.1.57}$$

which is the map $\rho_{1/k,q}$ from (4.1.18), regarded as an algebra homomorphism from $H(1/k)$ to the subalgebra $\mathbb{C}(x)[W \times \{e\}]$ of $\mathbb{D}_q^{\mathbb{W}}$. The other is given by

$$\rho^\xi_{k,1/q}\colon H(k) \longrightarrow \mathbb{D}_q^{\mathbb{W}} \tag{4.1.58}$$

defined as the map $\rho_{k,1/q}$ from (4.1.18), regarded as an algebra homomorphism from $H(1/k)$ to the subalgebra $\mathbb{C}(\xi)[\{e\} \times W]$ of $\mathbb{D}_q^{\mathbb{W}}$.

Definition 4.1.3.1. For $h \in H(1/k)$, we define

$$D^x_h := \rho^x_{1/k,q}(h) \in \mathbb{D}_q^{\mathbb{W}}.$$

Also, for $h' \in H(k)$, we define

$$D^\xi_{h'} := \rho^\xi_{k,1/q}(h') \in \mathbb{D}_q^{\mathbb{W}}.$$

Remark 4.1.3.2. Our choice (4.1.57) and (4.1.58) of the basic representations affects the parameters in the bispectral correspondence (4.1.66) of quantum Knizhnik-Zamolodchikov and Macdonald-Ruijsenaars equations. Our argument is equivalent to [vMS09, §6.2] and [vM11, §6.1], and opposite to [St14, Definition 2.17]. See Definition 4.2.3.1 for the $(C_1^\vee, C_1)$ case.

Let $\mathbb{C}[z^{\pm1}]^{W_0}$ denote the ring of Laurent polynomials of variable z which are invariant under the W_0-action $s_1(z) := z^{-1}$. Using the restriction map Res in (4.1.56), we introduce:

Definition 4.1.3.3. For $p \in \mathbb{C}[z^{\pm1}]^{W_0}$, we define $L^x_p, L^\xi_p \in \mathbb{D}_q$ by

$$L^x_p = L^x_p(k,q) := \mathrm{Res}(D^x_{p(Y)}), \quad L^\xi_p = L^\xi_p(k,q) := \mathrm{Res}(D^\xi_{p(Y)}), \tag{4.1.59}$$

where we regard $p(Y) \in H(1/k)$ for L^x_p, and $p(Y) \in H(k)$ for L^ξ_p.

Since we have $\mathbb{C}[z^{\pm1}]^{W_0} \cong \mathbb{C}[z+z^{-1}]$, it is natural to introduce:

Definition 4.1.3.4. We denote $p_1 := z + z^{-1}$, the generator of the invariant ring $\mathbb{C}[z^{\pm1}]^{W_0}$.

Using the function $c(\cdot;k)$ in (4.1.17), we can write down

$$L^x_{p_1}, L^\xi_{p_1} \in \mathbb{D}_q \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big).$$

Let us denote the action of $w \in W$ on functions of x given in (4.1.16) as

$$w^x \in \mathrm{End}(\mathbb{C}(x)) \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big).$$

Explicitly, for $f = f(x) \in \mathbb{C}(x)$, we have

$$(s_0^x f)(x) := f(qx^{-1}), \quad (s_1^x f)(x) = f(x^{-1}), \quad (u^x f)(x) = f(q^{1/2}x^{-1}), \quad (\mathrm{t}(\varpi)^x f)(x) = f(q^{1/2}x). \tag{4.1.60}$$

Recall that it is compatible with $\rho^x_{1/k,q}$ in (4.1.57). We also denote by

$$w^\xi \in \mathrm{End}(\mathbb{C}(\xi)) \subset \mathrm{End}\big(\mathbb{C}(x,\xi)\big)$$

the action on functions $g = g(\xi) \in \mathbb{C}(\xi)$. It is given by

$$(s_0^\xi g)(\xi) := g(q^{-1}\xi^{-1}), \quad (s_1^\xi g)(\xi) = g(\xi^{-1}), \quad (u^\xi g)(\xi) = g(q^{-1/2}\xi^{-1}), \quad (\mathrm{t}(\varpi)^\xi g)(\xi) = g(q^{-1/2}\xi), \tag{4.1.61}$$

and is compatible with $\rho^\xi_{k,1/q}$ in (4.1.58).

Proposition 4.1.3.5. We have

$$L^x_{p_1}(k,q) = A(x)T_{q^{1/2},x} + A(x^{-1})T_{q^{-1/2},x}, \quad L^\xi_{p_1}(k,q) = A^*(\xi^{-1})T_{q^{1/2},\xi} + A^*(\xi)T_{q^{-1/2},\xi} \tag{4.1.62}$$

with

$$A(z) := c(z^2;k) = \frac{k^{-1} - kz^2}{1-z^2}, \quad A^*(z) := c(z^2;k^*) = A(z).$$

Here we used the redundant notation $k^* = k$ for the comparison with $(C_1^\vee, C_1)$ case (Proposition 4.2.3.2).

Proof. Let us compute $L^x_{p_1} = \mathrm{Res}(D^x_{Y+Y^{-1}})$. Since $Y = UT_1$ and $u = \mathrm{t}(\varpi)s_1$, using (4.1.10) and (4.1.19), we have

$$\begin{aligned} D^x_{Y+Y^{-1}} &= \rho^x_{1/k,q}(UT_1 + T_1^{-1}U) \\ &= \big(\mathrm{t}(\varpi)^x s_1^x\big)\big(k^{-1} + c(x^2;k^{-1})(s_1^x - 1)\big) + \big(k + c(x^2;k^{-1})(s_1^x-1)\big)\big(\mathrm{t}(\varpi)^x s_1^x\big). \end{aligned}$$

Then, using

$$\begin{aligned} &\mathrm{Res}\big(\mathrm{t}(\varpi)^x s_1^x\big) = \mathrm{t}(\varpi)^x, \quad \mathrm{Res}\big(\mathrm{t}(\varpi)^x s_1^x (s_1^x - 1)\big) = 0, \\ &\mathrm{Res}\big((s_1^x - 1)\,\mathrm{t}(\varpi)^x s_1^x\big) = \mathrm{t}(-\varpi)^x - \mathrm{t}(\varpi)^x, \end{aligned}$$

$k + k^{-1} - c(x^2;k^{-1}) = c(x^2;k)$ and $c(x^2;k^{-1}) = c(x^{-2};k)$, we have

$$\begin{aligned} \mathrm{Res}(D^x_{Y+Y^{-1}}) &= k^{-1}\,\mathrm{t}(\varpi)^x + k\,\mathrm{t}(\varpi)^x + c(x^2;k^{-1})(\mathrm{t}(-\varpi)^x - \mathrm{t}(\varpi)^x) \\ &= \big(k + k^{-1} - c(x^2;k^{-1})\big)\,\mathrm{t}(\varpi)^x + c(x^2;k^{-1})\,\mathrm{t}(-\varpi)^x \\ &= c(x^2;k)\,\mathrm{t}(\varpi)^x + c(x^{-2};k)\,\mathrm{t}(-\varpi)^x. \end{aligned}$$

By (4.1.60), we obtain the first half of (4.1.62).

For $L^\xi_{p_1}$, we replace (x,k,q) in $L^x_{p_1}$ with (ξ, k^{-1}, q^{-1}) and calculate

$$L^\xi_{p_1}(k,q) = c(\xi^2;k^{-1})\,\mathrm{t}(-\varpi)^\xi + c(\xi^{-2};k^{-1})\,\mathrm{t}(\varpi)^\xi = c(\xi^{-2};k)\,\mathrm{t}(-\varpi)^\xi + c(\xi^2;k)\,\mathrm{t}(\varpi)^\xi.$$

Then, by (4.1.61), we obtain the second half of (4.1.62). □

Remark 4.1.3.6. By the expression (4.1.17) of $c(\cdot;k)$ and (4.1.60), the formula of $L^x_{p_1} \in \mathbb{D}_q$ in (4.1.62) can be rewritten by

$$L^x_{p_1}(k,q) = \frac{kx - k^{-1}x^{-1}}{x - x^{-1}}T_{q^{1/2},x} + \frac{k^{-1}x - kx^{-1}}{x - x^{-1}}T_{q^{-1/2},x},$$

where $T_{q,x}$ denotes the q-shift operator acting on a function f in x as $(T_{q,x}f)(x) = f(qx)$. Similarly, for $L^\xi_{p_1}$, recalling $\mathrm{t}(\varpi)^\xi = T^{-1}_{q^{1/2},\xi}$ from (4.1.61), we have

$$L^\xi_{p_1}(k,q) = \frac{k^{-1}\xi - k\xi^{-1}}{\xi - \xi^{-1}}T_{q^{1/2},\xi} + \frac{k\xi - k^{-1}\xi^{-1}}{\xi - \xi^{-1}}T_{q^{-1/2},\xi}.$$

Now let us recall the Macdonald q-difference operator of type GL_2 [Ma95, Chap. VI], or the two-variable trigonometric Ruijsenaars operator [R87]:

$$D_{\mathrm{MR}}(x_1, x_2; q, t) := \frac{tx_1 - x_2}{x_1 - x_2} T_{q,x_1} + \frac{tx_2 - x_1}{x_2 - x_1} T_{q,x_2}$$

The specialization $D_{\mathrm{MR}}(x, x^{-1}; q, t)$ is essentially equal to the Macdonald q-difference operator of type A_1 (see [M87, (9.13)] and [M03, §6.3]). Comparing these operators, we have

$$\begin{aligned} L^x_{p_1}(k, q) &= k^{-1} D_{\mathrm{MR}}(x, x^{-1}; q^{1/2}, k^2), \\ L^\xi_{p_1}(k, q) &= k\, D_{\mathrm{MR}}(\xi, \xi^{-1}; q^{1/2}, k^{-2}) = k^{-1} D_{\mathrm{MR}}(\xi^{-1}, \xi; q^{1/2}, k^2). \end{aligned}$$

Lem42 In particular, using the action (4.1.35) of ι and noting $\iota T_{q,x} \iota = T_{q,\xi}$, we have

$$L^\xi_{p_1} = \iota L^x_{p_1} \iota.$$

See [vM11, Lemma 6.2] for a generalization of this relation.

Now we reach the main object in this § 4.1.3.

Definition 4.1.3.7. The following system of eigen-equations for $f = f(x, \xi) \in \mathbb{K} = \mathcal{M}(x, \xi)$ is called *the bispectral Macdonald-Ruijsenaars equation of type* A_1, and *the bMR equation* for short.

$$\begin{cases} (L^x_{p_1}(k, q)f)(x, \xi) &= p_1(\xi^{-1}) f(x, \xi) \\ (L^\xi_{p_1}(k, q)f)(x, \xi) &= p_1(x) f(x, \xi) \end{cases}. \tag{4.1.63}$$

The solution space is denoted as

$$\mathrm{SOL}_{\mathrm{bMR}}(k, q) := \{f \in \mathbb{K} \mid f \text{ satisfies } (4.1.63)\}.$$

Remark 4.1.3.8. Continuing Remark 4.1.2.9, the solution space is denoted as BiSP in [vMS09, vM11]. Our symbol is a modification of Sol_{Mac} in [C05, Theorem 1.3.8].

4.1.4 Bispectral qKZ/MR correspondence

The works [vMS09, vM11] established the following correspondence between the two solution spaces $\mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k, q)$ (Definition 4.1.2.8) and $\mathrm{SOL}_{\mathrm{bMR}}(k, q)$ (Definition 4.1.3.7).

Definition 4.1.4.1. We define a $\mathbb{K}$-linear function $\chi_+ \colon H_0 \to \mathbb{C}$ by

$$\chi_+(T_w) := k^{\ell(w)} \tag{4.1.64}$$

for the basis element $T_w \in H_0$ ($w \in W_0$). It is extended to $H^{\mathbb{K}}_0$ as

$$\chi_+ \colon H^{\mathbb{K}}_0 \longrightarrow \mathbb{K}, \quad \sum_{w \in W_0} f_w T_w \longmapsto \sum_{w \in W_0} f_w \chi_+(T_w), \tag{4.1.65}$$

where we used the expression (4.1.39).

Remark 4.1.4.2. This is a bispectral analogue of the map tr in [C05, §1.3.4, Theorem 1.3.8].

Fact 4.1.4.3 ([vMS09, Theorem 6.16, Corollary 6.21], [vM11, Theorem 6.6])**.** Assume $0 < q < 1$. Then the map χ_+ restricts to an injective $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map

$$\chi_+ \colon \mathrm{SOL}^{A_1}_{\mathrm{bqKZ}}(k, q) \longrightarrow \mathrm{SOL}_{\mathrm{bMR}}(k, q), \tag{4.1.66}$$

where $\mathbb{F}$ is the subspace of $\mathbb{K} = \mathcal{M}(x, \xi)$ defined by

$$\mathbb{F} := \left\{ f(x, \xi) \in \mathbb{K} \mid \big((\mathrm{t}(\lambda), \mathrm{t}(\mu)) f\big)(x, \xi) = f(x, \xi),\ \forall\, (\lambda, \mu) \in \Lambda \times \Lambda \right\},$$

and $\mathbb{W}_0$ is the subgroup of $\mathbb{W}$ defined by

$$\mathbb{W}_0 := \mathbb{Z}_2 \ltimes (W_0 \times W_0) \subset \mathbb{W}.$$

Remark 4.1.4.4. As mentioned in Remark 4.1.3.2, we follow the arguments in [vMS09, vM11] giving the bispectral correspondence $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(k,q) \to \mathrm{SOL}_{\mathrm{bMR}}(k,q)$. The claim in [St14, Theorem 3.1] is based on the correspondence $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(1/k,q) \to \mathrm{SOL}_{\mathrm{bMR}}(k,q)$, $\chi_+(T_w) = k^{-\ell(w)}$.

Let us explain the outline of the proof. We abbreviate $\mathrm{SOL}_{\mathrm{bqKZ}} \coloneqq \mathrm{SOL}_{\mathrm{bqKZ}}(k,q)$ and $\mathrm{SOL}_{\mathrm{bMR}} \coloneqq \mathrm{SOL}_{\mathrm{bMR}}(k,q)$. The proof is divided into three parts.

(i) χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$.

(ii) The image $\chi_+(\mathrm{SOL}_{\mathrm{bqKZ}})$ is contained in $\mathrm{SOL}_{\mathrm{bMR}}$.

(iii) $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathrm{SOL}_{\mathrm{bMR}}$ is injective

We omit the part (iii), and refer to [vMS09, Corollary 6.21] for the detail. For the part (i), we give a preliminary lemma.

Lemma 4.1.4.5 ([vMS09, Lemma 6.6])**.** For each $\mathbf{w} \in \mathbb{W}_0$ and $F \in H_0^{\mathbb{K}}$, we have

$$\chi_+(C_{\mathbf{w}} F) = \chi_+(F).$$

Proof. First, we have $\chi_+ \circ C_\iota = \chi_+$ since, for any $w \in W_0$, the element $T_w \in H_0 \subset H_0^{\mathbb{K}}$ satisfies $C_\iota(T_w) = T_{w^{-1}}$. Second, since $C_{(e,s_1)} = C_\iota C_{(s_1,e)} C_\iota$ by Remark 4.1.2.13, (4.1.52), it is sufficient to show $\chi_+ \circ C_{(s_1,e)} = \chi_+$. But it is a consequence of

$$C_{(s_1,e)} h = c(x_1;k,q)^{-1}(\eta_L(T_1) - k)h + h, \quad \chi_+(T_1) = k, \quad \chi_+ \circ \eta_L = \eta_L \circ \chi_+ \tag{4.1.67}$$

for any $h \in H_0$. $\square$

Part (i) *of the proof of Fact 4.1.4.3.* We first show that χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$. By (4.1.42), Lemma 4.1.4.5 and (4.1.35), for any $f \in H_0^{\mathbb{K}}$ and $w \in \mathbb{W}_0$, we have

$$\chi_+(\tau(w)f) = \chi_+(C_w w f) = \chi_+(wf) = w\big(\chi_+(f)\big).$$

Hence χ_+ is $\mathbb{W}_0$-equivariant. Then, by Definition 4.1.2.8, (4.1.64) and (4.1.65), we obtain the $\mathbb{W}_0$-equivariant and $\mathbb{F}$-linear map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$ by restriction. $\square$

The part (ii) of the proof consists of several arguments, and we may say that this part is one of the main body of [vMS09]. It is further divided into the following steps.

- Describe of $\mathrm{SOL}_{\mathrm{bqKZ}}$ in terms of the basic asymptotically free solution Φ.
- Analyze the map χ_+ using Φ.

The first step requires the following Fact 4.1.4.6 and Fact 4.1.4.8.

Fact 4.1.4.6 ([vMS09, §§5.1–5.2], [vM11, §5.2], [St14, §3.2])**.** Denote $w_0 \coloneqq s_1 \in W_0$. Let

$$\mathcal{W}(x,\xi) = \mathcal{W}(x,\xi;k,q) \in \mathbb{K} = \mathcal{M}(x,\xi) \tag{4.1.68}$$

be a meromorphic function satisfying the q-difference equations (quasi-periodicity)

$$\mathcal{W}(q^{l/2}x,\xi) = (k/\xi)^l \mathcal{W}(x,\xi) \quad (l \in \mathbb{Z}) \tag{4.1.69}$$

and the self-duality

$$\mathcal{W}(\xi^{-1},x^{-1};k^*,q) = \mathcal{W}(x,\xi;k,q). \tag{4.1.70}$$

Here we used the redundant notation $k^* = k$ for the comparison with the $(C_1^\vee, C_1)$ case (4.2.51). Then, there is a unique element $\Psi \in H_0^{\mathbb{K}}$ satisfying the following conditions (i)–(iii).

(i) We have the self-dual solution

$$\Phi \coloneqq \mathcal{W}\Psi \in \mathrm{SOL}_{\mathrm{bqKZ}}(k,q), \quad \iota(\Phi) = \Phi.$$

(ii) We have a series expansion

$$\Psi(t,\gamma) = \sum_{m,n \in \mathbb{N}} K_{m,n} x^{-2m} \xi^{2n} \quad (K_{\alpha,\beta} \in H_0)$$

for $(x,\xi) \in B_\varepsilon^{-1} \times B$ with B_ε being some open ball of radius $\varepsilon > 0$, which is normally convergent on compact subsets of $B_\varepsilon^{-1} \times B_\varepsilon$.

(iii) $K_{0,0} = T_{w_0}$.
The solution Φ is called *the basic asymptotically free solution of the bqKZ equation* in [vMS09, Definition 5.5], [vM11, Definition 5.5] and *the self-dual basic Harish-Chandra series* in [St14, Definition 3.8].

Remark 4.1.4.7. The function $\mathcal{W}$ is designed so that the element $\mathcal{W}(x,\xi)T_{w_0} = \mathcal{W}(x,\xi)T_1$ is a solution of the formal asymptotic form of the quantum KZ equation $C_{(l\varpi,e)}(x,\xi)f(q^{-l/2}x,\xi) = f(x,\xi)$ in the region $|x| \gg 0$. Indeed, noting that we are working in $H(1/k)$, recall from (4.1.54) the asymptotic form of $C_{(\varpi,e)} = C_{1,0}$ in this region:

$$C_{1,0} \approx C_{1,0}^{(0)} = k\eta_L(T_1 Y^{-1} T_1^{-1}).$$

The definition (4.1.47) of the map η_L and the $\mathbb{K}$-module structure (4.1.31) yield $\eta_L(T_1Y^{-1}T_1^{-1})T_1 = Y^{-1}T_1 = \xi^{-1}T_1$. Thus we have

$$\begin{aligned} C_{1,0}^{(0)}(x,\xi)\big(\mathcal{W}(q^{-1/2}x,\xi)T_1\big) = \mathcal{W}(x,\xi)T_1 &\iff k\xi^{-1}\mathcal{W}(q^{-1/2}x,\xi)T_1 = \mathcal{W}(x,\xi)T_1 \\ &\iff \mathcal{W}(q^{-1/2}x,\xi) = k^{-1}\xi\mathcal{W}(t,\gamma), \end{aligned}$$

which holds by (4.1.69). See also the argument in [vMS09, §5.1]. We give an example of such $\mathcal{W}$ in Example 4.1.4.12.

Fact 4.1.4.8 ([vMS09, (5.18), Lem. 5.12, Prop. 5.13], [vM11, Prop. 5.12])**.** Denoting $w_0 := s_1 \in W_0$, we define $U \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big) = \mathbb{K} \otimes \mathrm{End}\big(H_0\big)$ by

$$U(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) := \tau(e,w)\Phi \quad (w \in W_0).$$

Then the following statements hold.
1. U is an invertible $\mathrm{End}(H_0)$-valued solution of the bqKZ equation. In particular, under the natural isomorphism $\mathbb{K} \otimes \mathrm{End}(H_0) \cong \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$, we have $U \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$.
2. $U' \in \mathbb{K} \otimes \mathrm{End}\big(H_0\big)$ is an $\mathrm{End}\big(H_0\big)$-valued meromorphic solution of the bqKZ equation if and only if $U' = UF$ for some $F \in \mathbb{F} \otimes \mathrm{End}\big(H_0\big)$.
3. $U \in \mathrm{GL}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$ restricts to an $\mathbb{F}$-linear isomorphism $U\colon H_0^{\mathbb{F}} \to \mathrm{SOL}_{\mathrm{bqKZ}}$.
4. $\{\tau(e,w)\Phi \mid w \in W_0\}$ is an $\mathbb{F}$-basis of $\mathrm{SOL}_{\mathrm{bqKZ}}$.

We turn to the second step, which requires the following Fact 4.1.4.9–Fact 4.1.4.11.

Fact 4.1.4.9 ([vMS09, Lemma 6.5 (ii), (6.3)])**.** For $F \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$, we denote by

$$\phi_{\chi,v}^F := \chi(Fv) \in \mathbb{K} \tag{4.1.71}$$

the matrix coefficient of F with respect to $\chi \in H_0^*$ and $v \in H_0$. Also, using U in Fact 4.1.4.8, we define a twisted algebra homomorphism $\vartheta'\colon D_q \to \mathrm{End}\big(\mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})\big)$ by

$$\vartheta'(f)F = fF, \quad \vartheta'(\mathbf{w})F = \mathbf{w}(F)U^{-1}(\tau(\mathbf{w})U)$$

for $f \in \mathbb{C}(x,\xi)$, $\mathbf{w} \in \mathbb{W}$ and $F \in \mathrm{End}_{\mathbb{K}}\big(H_0^{\mathbb{K}}\big)$. Then we have the following.
1. ϑ' is an algebra homomorphism.
2. For $D = \sum_{\mathbf{s}\in\mathbb{W}_0} D_{\mathbf{s}}\mathbf{s} \in D_q^{\mathbb{W}}$ (see (4.1.55)), we have
$$\phi_{\chi,v}^{\vartheta'(D)U} = \sum_{\mathbf{s}\in\mathbb{W}_0} D_{\mathbf{s}}(\phi_{\chi,v}^{C_{\mathbf{s}}^{-1}U}). \tag{4.1.72}$$
3. If $\chi \in H_0^*$ satisfies $\chi(C_{\mathbf{s}}U) = \chi(U)$ for all $\mathbf{s} \in \mathbb{W}_0$, then we have
$$\mathrm{Res}(D)(\phi_{\chi,v}^U) = \phi_{\chi,v}^{\vartheta'(D)U}$$
for any $D \in D_q^{\mathbb{W}}$ and $v \in H_0$.

Fact 4.1.4.10 ([vMS09, Proposition 6.9])**.** For $h \in H(1/k)$, we have

$$\vartheta'(D^x_h)U = \eta_L(h^\dagger)U, \tag{4.1.73}$$

where $\dagger\colon H(1/k) \to H(k)$ is the unique algebra anti-isomorphism satisfying

$$T_1^\dagger = T_1^{-1}, \quad \pi^\dagger = \pi^{-1}.$$

Similarly, for $h' \in H(k)$, we have

$$\vartheta'(D^\xi_{h'})U = C_\iota \iota(\eta_L(h'^\ddagger))C_\iota U, \tag{4.1.74}$$

where $\ddagger\colon H(k) \to H(k)$ is the unique algebra anti-involution satisfying

$$T_1^\ddagger = T_1, \quad \pi^\ddagger = \pi^{-1}.$$

Fact 4.1.4.11 ([vMS09, Lemma 6.10])**.** For $p \in \mathbb{C}[z^{\pm 1}]^{W_0}$, we have

$$p(Y)^\dagger = p(Y)^\ddagger = p(Y^{-1}).$$

Now we can explain:

Part (ii) *of the proof of Fact 4.1.4.3.* We want to show $\chi_+(f) \in \mathrm{SOL}_{\mathrm{bMR}}(k,q)$ for $f \in \mathrm{SOL}_{\mathrm{bqKZ}}(1/k,q)$. By Fact 4.1.4.8 (2) and the $\mathbb{F}$-linearity of χ_+, it is enough to consider the case $f = Uv$ with $v \in H_0(1/k)$. Then $\chi_+(f) = \phi^U_{\chi_+,v}$ by (4.1.71).

Let us check the first equality of (4.1.63), extending it to general $p \in \mathbb{C}[T]^{W_0}$. By (4.1.59), we have

$$(L^x_p \phi^U_{\chi_+,v})(t,\gamma) = \bigl(\mathrm{Res}(D^x_{p(Y)})(\phi^U_{\chi_+,v})\bigr)(t,\gamma).$$

Now, by Lemma 4.1.4.5, χ_+ satisfies the condition of Fact 4.1.4.9 (3). Then we have

$$\bigl(\mathrm{Res}(D^x_{p(Y)})(\phi^U_{\chi_+,v})\bigr)(t,\gamma) = \phi^{\vartheta'(D^x_{p(Y)})U}_{\chi_+,v}(t,\gamma),$$

Then, by (4.1.73) in Fact 4.1.4.10 and by Fact 4.1.4.11, we have

$$\phi^{\vartheta'(D^x_{p(Y)})U}_{\chi_+,v}(t,\gamma) = \phi^{\eta_L(p(Y)^\dagger)U}_{\chi_+,v}(t,\gamma) = \phi^{\eta_L(p(Y^{-1}))U}_{\chi_+,v}(t,\gamma).$$

Finally, by Fact 4.1.2.15 and that p is W_0-invariant, we have

$$\phi^{\eta_L(p(Y^{-1}))U}_{\chi_+,v}(t,\gamma) = p(\gamma^{-1})\phi^U_{\chi_+,v}(t,\gamma).$$

Hence we have the desired equality $(L^x_p\chi_+(f))(t,\gamma) = p(\gamma^{-1})\chi_+(f)(t,\gamma)$.

Similarly, we can prove the second equality of (4.1.63), using (4.1.74) instead of (4.1.73). □

Example 4.1.4.12. We cite from [vMS09, vM11, St14] two examples of the function $\mathcal{W}$ in (4.1.68).

(1) We denote the Jacobi theta function with elliptic nome q by

$$\theta(z;q) := (q, z, q/z; q)_\infty = \prod_{n\in\mathbb{N}} (1-q^{n+1})(1-q^n z)(1-q^{n+1}/z),$$

using the q-shifted factorial (1.1.1). It enjoys the properties

$$\theta(qx;q) = \theta(x^{-1};q) = -x^{-1}\theta(x;q), \quad \theta(qx^{-1};q) = \theta(x;q), \tag{4.1.75}$$

Then, denoting

$$\theta(z,z';q) := \theta(z;q)\theta(z';q), \tag{4.1.76}$$

we define the meromorphic function $\mathcal{W}^{A_1}$ of x,ξ by

$$\mathcal{W}^{A_1}(x,\xi) = \mathcal{W}^{A_1}(x,\xi;k,q) := \frac{\theta(-q^{1/4}x\xi;q^{1/2})}{\theta(-q^{1/4}kx, -q^{1/4}k^{-1}\xi;q^{1/2})}. \tag{4.1.77}$$

By the above identities, it satisfies the properties (4.1.69) and (4.1.70). Let us write them again:

$$\mathcal{W}^{A_1}(q^{\pm 1/2}x,\xi;k,q) = (k/\xi)^{\pm 1}\mathcal{W}^{A_1}(x,\xi;k,q), \tag{4.1.78}$$

$$\mathcal{W}^{A_1}(\xi^{-1},x^{-1};k,q) = \mathcal{W}^{A_1}(x,\xi;k^*,q). \tag{4.1.79}$$

We used the redundant notation $k^* = k$ again for the comparison with the $(C_1^\vee, C_1)$ case (4.2.56).

(2) For later use, let us cite another function $\widehat{\mathcal{W}} \in \mathbb{K} = \mathcal{M}(x, \xi)$ from [St14, p.279]:

$$\widehat{\mathcal{W}}^{A_1}(x, \xi) = \widehat{\mathcal{W}}^{A_1}(x, \xi; k, q) := \frac{\theta(-q^{1/4}k^{-1}x\xi; q^{1/2})}{\theta(-q^{1/4}x; q^{1/2})}. \tag{4.1.80}$$

This function satisfies the q-difference equation

$$\widehat{\mathcal{W}}^{A_1}(q^{\pm 1/2}x, \xi; k, q) = (k/\xi)^{\pm 1}\widehat{\mathcal{W}}^{A_1}(x, \xi; k, q), \tag{4.1.81}$$

but does not satisfy the self-duality.

Remark 4.1.4.13. We give a few comments on the function $\mathcal{W}^{A_1}$ in Example 4.1.4.12 (1).

(1) The function $\mathcal{W}^{A_1}$ is equivalent to $G(t, \gamma)$ in [vM11, (5.8)], and equivalent to the function $\mathcal{W}$ [St14, §3.2] with k replaced by k^{-1}. This parameter difference comes from the choice of the basic representation $\rho^x_{k^{-1},q}$ in [vMS09, vM11] and $\rho^x_{k,q}$ in [St14] (see Remark 4.1.3.2).

(2) Let us explain the function $G(t, \gamma)$ in [vM11], and how to obtain the function $\mathcal{W}^{A_1}(x, \xi)$ from it. We use the torus $T = \mathrm{Hom}_{\mathrm{Group}}(\Lambda, \mathbb{C}^\times)$, the notation t^λ of the value of $t \in T$ at $\lambda \in \Lambda$, the notation of a point $(t, \gamma) \in T \times T$, the ring $\mathbb{L}' = \mathbb{C}[T \times T]$ and the isomorphism $\mathbb{L}' \cong \mathbb{L} = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]$ explained in Remark 4.1.2.2. The outline is that $G(t, \gamma)$ is defined to be an element of $\mathcal{M}(T \times T)$, i.e., a meromorphic function on $T \times T$, and the function $\mathcal{W}^{A_1}(x, \xi)$ is obtained from $G(t, \gamma)$ under the isomorphism $\mathcal{M}(T \times T) \cong \mathcal{M}(x, \xi)$ induced by $\mathbb{L}' \cong \mathbb{L}$.

Let $\vartheta = \vartheta^{A_1}$ be the theta function associated to the weight lattice $\Lambda = \mathbb{Z}\varpi$ of type A_1 in the sense of Looijenga [L76]. It is a meromorphic function on the torus $T := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}^\times)$, and the value at a point $t \in T$ is given by

$$\vartheta(t) = \vartheta^{A_1}(t) := \sum_{\lambda \in \Lambda} q^{1/2\langle \lambda, \lambda \rangle} t^\lambda \tag{4.1.82}$$

Let us also denote $w_0 := s_1 \in W_0$ and

$$\gamma_0 = \gamma_0^* := k^\alpha \in T,$$

which are borrowed from [St14, (2.3),(2.4)]. There the general types are treated in a uniform way under the notation $\gamma_{0,d}$ for our γ_0^*. The symbol $*$ indicates the duality anti-involution (4.1.22). Then, the meromorphic function G on $T \times T$ is defined to be

$$G(t, \gamma) := \frac{\vartheta(\mathrm{t}(w_0\gamma)^{-1})}{\vartheta(\gamma_0 t)\, \vartheta((\gamma_0^*)^{-1}\gamma)}. \tag{4.1.83}$$

Next we explain how to obtain $\mathcal{W}^{A_1}(x, \xi)$ from $G(t, \gamma)$. Using the coordinate $x = (t \mapsto t^\varpi)$, we can rewrite the lattice theta function as

$$\vartheta(t) = \sum_{l \in \mathbb{Z}} q^{l^2/4} x^l = \theta(-q^{1/4}x; q^{1/2}).$$

Using the other coordinate $\xi = (\gamma \mapsto \gamma^\varpi)$, we can also rewrite $tw_0(\gamma)^{-1}$ as $(tw_0(\gamma)^{-1})^\varpi = (t\gamma)^\varpi = x\xi$, $\gamma_0 t$ as $(\gamma_0 t)^\varpi = k^{\langle \alpha, \varpi \rangle} t^\varpi = kx$, and $(\gamma_0^*)^{-1}\gamma$ as $((\gamma_0^*)^{-1}\gamma)^\varpi = k^{-\langle \alpha, \varpi \rangle}\gamma^\varpi = k^{-1}\xi$. Hence, we obtain the function $\mathcal{W}^{A_1}(x, \xi)$.

4.1.5 Bispectral Macdonald-Ruijsenaars function of type A_1

In this subsection, we give an explicit solution of the bispectral Macdonald-Ruijsenaars q-difference equation of type A_1, following [NSh] and [St14, §5.3]. One caution is that we work on

$$\mathrm{SOL}_{\mathrm{bMR}}(1/k, q),$$

so that the reciprocal parameter k^{-1} is used in this subsection. As in the previous Fact 4.1.4.3, we assume $0 < q < 1$. Let us denote $\nu := q^{1/2}$.

Let us write again the bispectral Macdonald-Ruijsenaars equation (4.1.63):

$$\begin{cases} (L^x_{p_1} f)(x,\xi) & = (\xi+\xi^{-1}) f(x,\xi) \\ (L^\xi_{p_1} f)(x,\xi) & = (x+x^{-1}) f(x,\xi) \end{cases}. \tag{4.1.84}$$

By Proposition 4.1.3.5 and Remark 4.1.3.6, the operators can be written as

$$L^x_{p_1} = L(x;k,q), \quad L^\xi_{p_1} = L(\xi;k^{-1},q^{-1}), \tag{4.1.85}$$

$$L(x;k,q) := \frac{k-k^{-1}x^{-2}}{1-x^{-2}} T_{\nu,x} + \frac{k^{-1}-kx^{-2}}{1-x^{-2}} T^{-1}_{\nu,x}. \tag{4.1.86}$$

First, we consider the asymptotic form of the x-side q-difference equation

$$\big(L^x_{p_1} - (\xi+\xi^{-1})\big) f(x) = 0$$

in the region $|x| \gg 1$. From (4.1.86) (also recall Remark 4.1.3.6), the asymptotic form is

$$L^x_{p_1} \approx L^x_{(\infty)} := kT_{\nu,x} + k^{-1} T^{-1}_{\nu,x}.$$

Similarly, in the region $|\xi| \ll 1$, we have

$$L^\xi_{p_1} \approx L^\xi_{(0)} := k^{-1} T_{\nu,\xi} + k T^{-1}_{\nu,\xi},$$

Now recall the functions $\mathcal{W}^{A_1}(x,\xi;1/k,q)$ and $\widehat{\mathcal{W}}^{A_1}(x,\xi;1/k,q)$:

$$\mathcal{W}^{A_1}(x,\xi;1/k,q) = \frac{\theta(-\nu^{1/2}x\xi;\nu)}{\theta(-\nu^{1/2}k^{-1}x, -\nu^{1/2}k\xi;\nu)}, \quad \widehat{\mathcal{W}}^{A_1}(x,\xi;1/k,q) := \frac{\theta(-\nu^{1/2}kx\xi;\nu)}{\theta(-\nu^{1/2}x;\nu)}. \tag{4.1.87}$$

Lemma 4.1.5.1. The sets $\{\mathcal{W}^{A_1}(x,\xi^{\pm 1};1/k,q)\}$ and $\{\widehat{\mathcal{W}}^{A_1}(x,\xi^{\pm 1};1/k,q)\}$ are bases of solutions of the asymptotic q-difference equation

$$\big(L^x_{(\infty)} - (\xi+\xi^{-1})\big) f(x) = 0.$$

Similarly, the sets $\{\mathcal{W}^{A_1}(x^{\pm 1},\xi;1/k,q)\}$ and $\{\widehat{\mathcal{W}}^{A_1}(x^{\pm 1},\xi;k1/,q)\}$ are bases of solutions of

$$\big(L^\xi_{(0)} - (x+x^{-1})\big) g(\xi) = 0.$$

Proof. As seen before, we have $T^{\pm 1}_{\nu,x} f(x) = (k\xi)^{\mp 1} f(x)$ for $f(x) := \mathcal{W}^{A_1}(x^{\pm 1},\xi;1/k,q)$, so that these functions are solutions of the x-side equation. Since the equation is second-order and these functions are linear independent by the property of the Jacobi theta function $\theta(x;q)$, we have the x-side statement. The ξ-side is shown similarly using $T^{\pm 1}_{\nu,\xi} \mathcal{W}^{A_1}(x,\xi;1/k,q) = (x/k)^{\mp 1} \mathcal{W}^{A_1}(x,\xi;1/k,q)$. The same argument works for $\widehat{\mathcal{W}}^{A_1}$. □

Next, let us recall Heine's basic hypergeometric q-difference equation [GR04, Chap. 1, Exercise 1.13]:

$$\big(D^z_H(a,b,c;q) u\big)(z) = 0, \tag{4.1.88}$$

where the operator D^z_H is given by

$$D^z_H(a,b,c;q) := z(c-abqz)\partial_q^2 + \left(\frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q} z\right)\partial_q + \frac{(1-a)(1-b)}{(1-q)^2} \tag{4.1.89}$$

with $(\partial_q u)(z) := \big(u(z)-u(qz)\big)/\big((1-q)z\big)$. A solution of (4.1.88) is given by Heine's basic hypergeometric function

$$u(z) = {}_2\phi_1\left[\begin{matrix} a, \ b \\ c \end{matrix}; q, \ z\right], \tag{4.1.90}$$

where we used the notation (4.0.2).

The following relation between the Macdonald q-difference operator of type A_1 and Heine's basic hypergeometric q-difference equation is well known.

Lemma 4.1.5.2 (c.f. [St14, Lemma 5.4]). Let $\mathcal{W}(x)$ be a meromorphic function in x satisfying

$$\mathcal{W}(q^{\pm 1/2}x) = (k\xi)^{\mp 1}\mathcal{W}(x). \tag{4.1.91}$$

Then, the function $f(x) = \mathcal{W}(x)u(k^{-2}qx^{-2})$ is a meromorphic solution of the q-difference equation

$$(L^x_{p_1} f)(x) = (\xi + \xi^{-1})f(x)$$

if and only if $u(z)$ is a meromorphic solution of the q-difference equation

$$(D^z_H(k^2, k^2\xi^2, q\xi^2)u)(z) = 0, \quad z = k^{-2}qx^{-2}.$$

Proof. A direct computation yields that the operator $D^z_H(a,b,c;q)$ in (4.1.89) is proportional to

$$D'(a,b,c;q) := (c/q - abz)T^2_{q,z} - (1 + c/q - (a+b)z)T_{q,z} + (1-z).$$

If $a/b = q/c$, then $D'(a, ac/q, c; q) = (c/q)(1-a^2z)T^2_{q,z} - (1+c/q)(1-az)T_{q,z} + (1-z)$. Hence, defining

$$D''(a,c;q) := T^{-1}_{q,z}\frac{1}{1-az}D'_z(a, ac/q, c; q) = cq^{-1}\frac{1-a^2z/q}{1-az/q}T_{q,z} + \frac{1-z/q}{1-az/q}T^{-1}_{q,z} - (1+c/q),$$

we have $(D^z_H(a, ac/q, c; q)u)(z) = 0 \iff (D''(a,c;q)u)(z) = 0$. If moreover $z = k^{-2}qx^{-2}$, $a = k^2$ and $c = q\xi^2$, then we have

$$\begin{aligned}
&\big(D^z_H(k^2, k^2\xi^2, q\xi^2; q)u\big)(z) = 0 \iff \big(\xi^{-1}D''(k^2, q\xi^2; q)u\big)(z) = 0\\
\iff &\Big(\frac{1-k^2x^{-2}}{1-x^{-2}}\xi T_{q,z} + \frac{1-k^{-2}x^{-2}}{1-x^{-2}}\xi^{-1}T^{-1}_{q,z} - (\xi+\xi^{-1})\Big)u(z) = 0.
\end{aligned}$$

On the other hand, by the expression (4.1.85) and the condition (4.1.91), we have

$$\begin{aligned}
&\big((L^x_{p_1} - (\xi+\xi^{-1}))f\big)(x) = 0\\
\iff &\Big(\frac{k-k^{-1}x^{-2}}{1-x^{-2}}k^{-1}\xi^{-1}T^{-1}_{q,z} + \frac{k^{-1}-kx^{-2}}{1-x^{-2}}k\xi T_{q,z} - (\xi+\xi^{-1})\Big)u(z) = 0\\
\iff &\Big(\frac{1-k^2x^{-2}}{1-x^{-2}}\xi T_{q,z} + \frac{1-k^{-2}x^{-2}}{1-x^{-2}}\xi^{-1}T^{-1}_{q,z} - (\xi+\xi^{-1})\Big)u(z) = 0.
\end{aligned}$$

Thus we have the desired equivalence. □

Now we give an explicit bispectral solution of (4.1.84).

Proposition 4.1.5.3 (c.f. [NSh, Theorems 2.1, 2.2, (3.13)], [St14, Cor. 5.5]). We denote $\nu := q^{1/2}$.

(1) Define the function $f^{A_1}(x,\xi)$ by

$$\begin{aligned}
f^{A_1}(x,\xi) &= f^{A_1}(x,\xi;k,q) := \mathcal{W}^{A_1}(x,\xi;1/k,q)\,\varphi^{A_1}(x,\xi;k,q),\\
\varphi^{A_1}(x,\xi) &= \varphi^{A_1}(x,\xi;k,q) := \frac{(q\xi^2;q)_\infty}{(k^{-2}q\xi^2;q)_\infty}\,{}_2\phi_1\left[\begin{matrix}k^2,\ k^2\xi^2\\ q\xi^2\end{matrix};q,\ \frac{q}{k^2x^2}\right].
\end{aligned} \tag{4.1.92}$$

Here we used the function $\mathcal{W}^{A_1}(x,\xi;1/k,q)$ in (4.1.87), and assumed $\left|k^{-2}qx^{-2}\right| < 1$. Then f^{A_1} satisfies the following properties.

(i) It is a solution of the bispectral problem (4.1.84).

(ii) It has the symmetry (the inversion invariance in [St14])

$$f^{A_1}(x,\xi) = f^{A_1}(x^{-1};\xi) = f^{A_1}(x,\xi^{-1}).$$

(iii) It has the self-duality

$$f^{A_1}(x,\xi;k,q) = f^{A_1}(\xi^{-1};x^{-1};k^*,q),$$

using the redundant notation $k^* = k$ for the comparison with the $(C^\vee_1, C_1)$ case.

Recalling the $\mathbb{W}$-action on $\mathbb{K} = \mathcal{M}(T \times T)$ in (4.1.35), we express the subset of $\mathrm{SOL}_{\mathrm{bMR}}(1/k,q)$ satisfying these properties as

$$\mathrm{SOL}^{\mathbb{W}^*}_{\mathrm{bMR}}(1/k,q) := \{f \in \mathrm{SOL}_{\mathrm{bMR}}(1/k,q) \mid \text{(ii), (iii)}\}.$$

Thus, we can restate the claim as

$$f^{A_1} \in \mathrm{SOL}^{\mathbb{W}^*}_{\mathrm{bMR}}(1/k,q).$$

(2) Defining $\xi_l := k^{-1}\nu^{-l}$ for $l \in \mathbb{N}$, we have

$$\begin{aligned} &f^{A_1}(x,\xi_l) = c_l P^{A_1}_l(x), \\ &c_l := \frac{(-k)^{-l}\nu^{-\binom{l+1}{2}}}{\theta(-k^2\nu^{l+\frac{1}{2}};\nu)} \frac{(k^{-2}q^{1-l};q)_\infty}{(k^{-4}q^{1-l};q)_\infty}, \quad P^{A_1}_l(x) := x^l {}_2\phi_1\left[\begin{matrix} k^2,\ q^{-l} \\ k^{-2}q^{1-l} \end{matrix}; q,\ \frac{q}{k^2x^2}\right]. \end{aligned} \tag{4.1.93}$$

The function $P^{A_1}_l(x)$ satisfies the following three conditions.

(i) It is an eigenfunction of the Macdonald-Ruijsenaars q-difference operator $L^x_{p_1}$ of type A_1.

(ii) It is a Laurent polynomial in x belonging to $x^l\mathbb{C}[x^{-1}]$, and is invariant under the replacement $x \mapsto x^{-1}$.

Moreover, these conditions uniquely determine the function $P^{A_1}_l(x)$ up to constant multiplication, and the eigenvalue in (i) is $p_1(\xi_l^{-1}) = \xi_l^{-1} + \xi_l$.

We will give an almost self-consistent proof, except the following equality (4.1.94).

Fact 4.1.5.4 ([NSh, (4.11)]). The function $\varphi^{A_1}(x,\xi)$ satisfies

$$\varphi^{A_1}(x,\xi) = \frac{(k^2, qx^{-2}\xi^2;q)_\infty}{(k^{-2}qx^{-2}, k^{-2}q\xi^2;q)_\infty} {}_2\phi_1\left[\begin{matrix} k^{-2}qx^{-2},\ k^{-2}q\xi^2 \\ qx^{-2}\xi^2 \end{matrix}; q,\ k^2\right] \tag{4.1.94}$$

under the condition $|k| < 1$. In particular, we have

$$\varphi^{A_1}(x,\xi) = \varphi^{A_1}(\xi^{-1};x^{-1}). \tag{4.1.95}$$

The equality (4.1.94) can be shown using Heine's transformation formula for ${}_2\phi_1$ series [GR04, (1.4.1)]. See also [NSh, (4.10)] for the calculation.

Proof of Proposition 4.1.5.3. For (1), we follow the argument of [St14, Lemma 2.18]. Let us denote $\mathcal{W}^{A_1}(x,\xi) := \mathcal{W}^{A_1}(x,\xi;1/k,q)$ for simplicity, and recall the quasi-periodicity and the self-duality:

$$\mathcal{W}^{A_1}(\xi^{-1},x^{-1}) = \mathcal{W}^{A_1}(x,\xi), \quad \mathcal{W}^{A_1}(\nu x,\xi) = (k\xi)^{-1}\mathcal{W}(x,\xi). \tag{4.1.96}$$

The first equality of (4.1.96) and (4.1.95) yield the self-duality (iii). The second equality of (4.1.96) is nothing but the condition (4.1.91), so that Lemma 4.1.5.2 and (4.1.90) yield

$$L^x_{p_1} f^{A_1}(x,\xi) = (\xi+\xi^{-1})f^{A_1}(x,\xi) = p_1(\xi^{-1})f^{A_1}(x,\xi). \tag{4.1.97}$$

On the other hand, (4.1.85) shows $L^\xi_{p_1} = L(\xi;k^{-1},q^{-1}) = IL(\xi;k,q)I$, where I is the operator $g(\xi) \mapsto (Ig)(\xi) := g(\xi^{-1})$ for a function $g(\xi)$. Then, the self-duality (iii) and the eigen-property (4.1.97) imply

$$\begin{aligned} L^\xi_{p_1} f^{A_1}(x,\xi) &= \big(IL(\xi;k,q)I\big)f^{A_1}(\xi^{-1};x^{-1}) = IL(\xi;k,q)f^{A_1}(\xi;x^{-1}) = I\big(p_1(x)f^{A_1}(\xi;x^{-1})\big) \\ &= p_1(x)f^{A_1}(\xi^{-1};x^{-1}) = (x+x^{-1})f^{A_1}(x,\xi). \end{aligned}$$

Hence (iii) holds.

Before showing (1) (ii), we show (2). The equality in the statement is a consequence of

$$\mathcal{W}^{A_1}(x,\xi_l;1/k,q) = (-\nu^{-1/2}k^{-1}x)^l\nu^{-\binom{l}{2}} = x^l c_l,$$

which can be checked using $\theta(x;q) = (q,x,q/x;q)_\infty$. The condition (2) (i) is a consequence of (4.1.97). The condition (2) (ii) can be checked by the formula 4.1.93 (see also Remark 4.1.5.5 (1)). The uniqueness is well-known in the theory of Macdonald polynomials (see also Remark 4.1.5.5 (1)).

Now we show the remaining (1) (ii). By (2) (ii), we have $f^{A_1}(x,\xi_l) = f^{A_1}(x^{-1};\xi_l)$ for any $l \in \mathbb{N}$. Then, applying the identity theorem in complex analysis to the analytic function $g(\xi) := f^{A_1}(x,\xi) - f^{A_1}(x^{-1};\xi)$, we have $f^{A_1}(x,\xi) = f^{A_1}(x^{-1};\xi)$ for any ξ in the domain of definition. Combining it with the self-duality (1) (iii), we have $f^{A_1}(x,\xi) = f^{A_1}(x,\xi^{-1})$. Hence we have (1) (ii). □

Remark 4.1.5.5. Some comments on Proposition 4.1.5.3 are in order.

(1) Defining $\beta \in \mathbb{C}$ by $k = \nu^{\beta}$, the Laurent polynomial $P_l^{A_1}$ is equal to

$$P_l^{A_1}(x) = \begin{bmatrix} \beta+l-1 \\ l \end{bmatrix}_q^{-1} \sum_{i+j=l} \begin{bmatrix} \beta+i-1 \\ i \end{bmatrix}_q \begin{bmatrix} \beta+j-1 \\ j \end{bmatrix}_q x^{i-j}, \tag{4.1.98}$$

where we used the q-binomial coefficient (4.0.3). It is nothing but the Macdonald symmetric polynomial of type A_1 [M03, (6.3.7)], and is proportional to the continuous q-ultraspherical polynomial, or the Rogers polynomial. See [M03, §6.3, pp.156–157] for the detail.

(2) In [NSh], Noumi and Shiraishi gave an explicit bispectral solution $f(x_1,\dots,x_n;s_1,\dots,s_n)$ of type GL_n. The above solution $f^{A_1}(x,\xi)$ is obtained by specializing $(x_1,x_2)=(x,x^{-1})$ and $(s_1,s_2)=(\xi,\xi^{-1})$ in the solution $f(x_1,x_2;s_1,s_2)$ of type GL_2. See also Stokman [St14, Corollary 5.5] for the uniqueness of $f(x_1,x_2;s_1,s_2)$.

Let us cite another bispectral solution.

Fact 4.1.5.6 ([St14, Theorem 4.6, (5.18)])**.** Define a meromorphic function $\mathcal{E}_+^{A_1}(x,\xi) = \mathcal{E}_+^{A_1}(x,\xi;k,q) \in \mathbb{K} = \mathcal{M}(x,\xi)$ by

$$\begin{aligned} \mathcal{E}_+^{A_1}(x,\xi;k,q) &:= \frac{\theta(-\nu^{1/2}k;\nu)}{\theta(-\nu^{1/2}\xi;\nu)} \frac{(k^2\xi^{-2},k^2;q)_\infty}{(\xi^{-2},k^4;q)_\infty} \widehat{\mathcal{W}}^{A_1}(x,\xi;1/k,q)\,{}_2\phi_1\left[\begin{matrix} k^2,\ k^2\xi^2 \\ q\xi^2 \end{matrix};q,\ \frac{q}{k^2x^2}\right] + (\xi \mapsto \xi^{-1}) \\ &= \frac{\theta(-\nu^{1/2}k,-\nu^{1/2}kx\xi;\nu)}{\theta(-\nu^{1/2}\xi,-\nu^{1/2}x;\nu)} \frac{(k^2\xi^{-2},k^2;q)_\infty}{(\xi^{-2},k^4;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} k^2,\ k^2\xi^2 \\ q\xi^2 \end{matrix};q,\ \frac{q}{k^2x^2}\right] + (\xi \mapsto \xi^{-1}), \end{aligned} \tag{4.1.99}$$

where the second term is obtained by replacing ξ in the first term with ξ^{-1}. Then the function $\mathcal{E}_+^{A_1}$ enjoys the following properties (i)–(iii).

(i) It is a solution of the bispectral problem (4.1.84).

(ii) It has the symmetry (the inversion invariance in [St14])

$$\mathcal{E}_+^{A_1}(x,\xi) = \mathcal{E}_+^{A_1}(x^{-1};\xi) = \mathcal{E}_+^{A_1}(x,\xi^{-1}).$$

(iii) It has the self-duality

$$\mathcal{E}_+^{A_1}(x,\xi;k,q) = \mathcal{E}_+^{A_1}(\xi^{-1};x^{-1};k^*,q),$$

using the redundant notation $k^*=k$ for the comparison with the $(C_1^\vee,C_1)$ case.

Recalling the $\mathbb{W}$-action on $\mathbb{K}=\mathcal{M}(x,\xi)$ in (4.1.35), we express the subset of $\mathrm{SOL}_{\mathrm{bMR}}(1/k,q)$ satisfying these properties as

$$\mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q) := \{f \in \mathrm{SOL}_{\mathrm{bMR}}(1/k,q) \mid \text{(ii), (iii)}\}.$$

Thus, we can restate the claim as

$$\mathcal{E}_+^{A_1} \in \mathrm{SOL}_{\mathrm{bMR}}^{\mathbb{W}^*}(1/k,q).$$

Following [St14], we call it *the basic hypergeometric function of type* A_1.

Remark 4.1.5.7. Some comments on the function $\mathcal{E}_+^{A_1}$ are in order.

(1) As explained right after [St14, Definition 2.19], we have the basic hypergeometric function of arbitrary type. The reduced case, including the above $\mathcal{E}_+^{A_1}(x,\xi;k,q)$, was introduced by Cherednik [C97b, C09] under the name of global spherical function. The non-reduced case (type $(C_1^\vee,C_1)$) was introduced by Stokman [St03], and the uniform approach was discussed in [St14]. The GL_2 type is written down in [St14, (5.18)], from which we can recover the A_1 case.

(2) Although we take (4.1.99) as the definition of the basic hypergeometric function $\mathcal{E}_+^{A_1}$, the actual statement of [St14, Theorem 4.6] is that $\mathcal{E}_+$ (of arbitrary type) has *the c-function expansion* with respect to the self-dual basic Harish-Chandra series Φ (see Fact 4.1.4.6 for type A_1), and defined for generic $\eta \in T$. The c-function expansion is given in the form

$$\mathcal{E}_+(t,\gamma;k,q) = \sum_{w\in W_0} \mathfrak{c}(t,w\gamma;k,q)\Phi(t,w\gamma;k,q).$$

4.2 Type $(C_1^\vee, C_1)$

We discuss the type $(C_1^\vee, C_1)$, or the non-reduced type. See also [St14, §3, §5.2].

4.2.1 Extended affine Hecke algebra

First, we recall the affine root system of type $(C_1^\vee, C_1)$ and the extended affine Weyl group, following [M03, §1, §2, §6.4].

We consider the one-dimensional real Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with

$$V = \mathbb{R}\epsilon, \quad \langle \epsilon, \epsilon \rangle = 1.$$

Similarly as in §4.1.1, we denote by F the space of affine real functions on V, and identify it with $V \oplus \mathbb{R}c$. Using the gradient map $D\colon F \to V$, we extend $\langle \cdot, \cdot \rangle$ to F.

Let $S(C_1^\vee, C_1) := \{m(\pm\epsilon + \frac{1}{2}n) \mid m \in \{1, 2\},\ n \in \mathbb{Z}\}$ be the affine root system $S(C_1^\vee, C_1)$ in the sense of Macdonald [M03]. A basis is given by $\{a_0 := \frac{1}{2}c - \epsilon, a_1 := \epsilon\}$, and the corresponding simple reflections $s_i\colon V \to V$ for $i = 0, 1$ are given by the formula (4.1.2) with $a_i^\vee := 2a_i/\langle a_i, a_i \rangle = 2a_i \in F$. Explicitly, we have

$$s_1(r\epsilon) = -r\epsilon, \quad s_0(r\epsilon) = (1 - r)\epsilon \quad (r \in \mathbb{R}). \tag{4.2.1}$$

We denote $W_0 := \langle s_1 \rangle \subset O(V, \langle \cdot, \cdot \rangle)$, which is isomorphic to $\mathfrak{S}_2$. The W_0-action (4.2.1) on V preserves

$$\Lambda := \mathbb{Z}\epsilon \subset V,$$

the coroot lattice of the root system $R(C_1) = \{\pm 2\epsilon\}$ generated by $(2\epsilon)^\vee = \epsilon$. We also denote by $\mathrm{t}(\Lambda) = \{\mathrm{t}(\lambda) \mid \lambda \in \Lambda\}$ is the abelian group with relations $\mathrm{t}(\lambda)\,\mathrm{t}(\mu) = \mathrm{t}(\lambda + \mu)$ for $\lambda, \mu \in \Lambda$. The group $\mathrm{t}(\Lambda)$ acts on V by translation (4.1.4). Then, the extended affine Weyl group W of $S(C_1^\vee, C_1)$ is defined to be the subgroup of the isometries on $(V, \langle \cdot, \cdot \rangle)$ generated by W_0 and $\mathrm{t}(\Lambda)$.

$$W := W_0 \ltimes \mathrm{t}(\Lambda). \tag{4.2.2}$$

In particular, we have the relation

$$s_1\, \mathrm{t}(\lambda) s_1 = \mathrm{t}(s_1(\lambda)) \quad (\lambda \in \Lambda) \tag{4.2.3}$$

with $s_1(\lambda)$ given by (4.2.1).

As an abstract group, W is generated by s_0 and s_1 with fundamental relations

$$s_0^2 = s_1^2 = e. \tag{4.2.4}$$

The following relations hold in W.

$$\mathrm{t}(\epsilon) = s_0 s_1, \quad \mathrm{t}(-\epsilon) = s_1 s_0. \tag{4.2.5}$$

Compare the first relation with (4.1.9): denoting $s_i^{A_1}$ $(i = 0, 1)$ for the generators of the extended Weyl group W^{A_1} of $S(A_1)$, we have $\mathrm{t}(\alpha) = s_0^{A_1} s_1^{A_1}$.

Next, we recall the extended affine Hecke algebra H associated to the affine root system $S(C_1^\vee, C_1)$. For the detail, see [M03, §4, §6.4]. Hereafter we fix nonzero complex numbers k_1, k_0, l_1, l_0 and denote

$$\underline{k} := (k_1, k_0), \quad \underline{l} := (l_1, l_0). \tag{4.2.6}$$

The symbols k_1 and k_0 are borrowed from [NS04].

Remark 4.2.1.1. Our parameters (k_1, k_0, l_1, l_0) correspond to $(t_1^{1/2}, t_0^{1/2}, l_1^{1/2}, l_0^{1/2})$ in [N95] and [T10].

Definition 4.2.1.2. The extended affine Hecke algebra $H(\underline{k})$ is the $\mathbb{C}$-algebra generated by T_1 and T_0 with fundamental relations

$$(T_i - k_i)(T_i + k_i^{-1}) = 0 \quad (i = 1, 0). \tag{4.2.7}$$

In this § 4.2, we denote $H := H(\underline{k})$ for simplicity.

As in § 4.1.1, we denote by $\ell(w)$ the length of $w \in W$. If we have a reduced expression $w = s_{i_1} \cdots s_{i_l}$, $i_j \in \{0,1\}$, then $\ell(w) = l$. For such $w \in W$, we set

$$T_w := T_{i_1} \cdots T_{i_l} \in H.$$

Then T_w is independent of the choice of reduced expression. We also define $Y^{\pm 1} \in H$ by

$$Y := T_0 T_1, \quad Y^{-1} := T_1^{-1} T_0^{-1}, \tag{4.2.8}$$

which can be regarded as deformations of $\mathrm{t}(\epsilon) \in W$ given in (4.2.5). As in the case of type A_1 (§ 4.1.1), the monomials in $\mathbb{C}[Y^{\pm 1}] \subset H$ are denoted as $Y^\lambda := Y^l$ for $\lambda = l\epsilon \in \Lambda$, $l \in \mathbb{Z}$. We also have a $\mathbb{C}$-linear isomorphism $H \cong H_0 \otimes \mathbb{C}[Y^{\pm 1}]$, where

$$H_0 := \mathbb{C} + \mathbb{C}T_1$$

is the subalgebra of H generated by T_1.

Remark 4.2.1.3. Our choice (4.2.8) of the Dunkl operator Y follows [M03, §6.4], which is the opposite of [N95, T10, St14]. The choice (4.2.8) is compatible with the choice for type A_1 (see (4.1.12)).

Next, we review Noumi's [N95] basic representation $\rho_{\underline{k},\underline{l},q}$ of $H = H(\underline{k})$. Choose and fix a parameter $q^{1/2} \in \mathbb{C}^\times$. The extended affine Weyl group W acts on the Laurent polynomial ring $\mathbb{C}[x^{\pm 1}]$ by

$$(s_{1,q} f)(x) = f(x^{-1}), \quad (s_{0,q} f)(x) = f(qx^{-1}), \quad (\mathrm{t}(\epsilon)_q f)(x) = f(qx) = (T_{q,x} f)(x), \tag{4.2.9}$$

where $T_{q,x}$ denotes the q-shift operator on the variable x. Then, we have an algebra embedding

$$\rho_{\underline{k},\underline{l},q} \colon H(\underline{k}) \longrightarrow \mathrm{End}(\mathbb{C}[x^{\pm}]), \quad \rho(T_i) := c(x_i; k_i, l_i) s_{i,q} + b(x_i; k_i, l_i) \quad (i = 1, 0) \tag{4.2.10}$$

with $x_1 := x^2$, $x_0 := qx^{-2}$ and

$$\begin{aligned} c(z; k, l) &:= k^{-1} \frac{(1 - klz^{1/2})(1 + kl^{-1} z^{1/2})}{1 - z}, \\ b(z; k, l) &:= k - c(z; k, l) = \frac{(k - k^{-1}) + (l - l^{-1}) z^{1/2}}{1 - z}. \end{aligned} \tag{4.2.11}$$

Here we understand $x_1^{1/2} = x$ and $x_0^{1/2} = q^{1/2} x^{-1}$. We call $\rho_{\underline{k},\underline{l},q}$ *the basic representation of* $H(\underline{k})$.

Definition 4.2.1.4. *The double affine Hecke algebra (DAHA) of type* $(C_1^\vee, C_1)$, denoted as

$$\mathbb{H} = \mathbb{H}(\underline{k}, \underline{l}, q) = \mathbb{H}^{(C_1^\vee, C_1)}(\underline{k}, \underline{l}, q),$$

is defined to be the $\mathbb{C}$-subalgebra of $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ generated by the multiplication operators by $x^{\pm 1}$ and the image $\rho_{\underline{k},\underline{l},q}(H(\underline{k}))$.

As an abstract algebra, the DAHA $\mathbb{H}$ of type $(C_1^\vee, C_1)$ is presented with generators $T_1, T_0, T_1^\vee, T_0^\vee$ and relations

$$\begin{aligned} &(T_i - k_i)(T_i + k_i^{-1}) = 0 \quad (T_i^\vee - l_i)(T_i^\vee + l_i^{-1}) = 0 \quad (i = 1, 0), \\ &T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}. \end{aligned} \tag{4.2.12}$$

See [Sa99], [NS04], [M03, §4.7] and [C05] for the detail. The symbols $T_i^\vee$ are borrowed from [NS04]. To recover Definition 4.2.1.4, we put

$$T_1^\vee = X^{-1} T_1^{-1}, \quad T_0^\vee = q^{-1/2} T_0^{-1} X, \tag{4.2.13}$$

by which we can extend the map $\rho_{\underline{k},\underline{l},q}$ of (4.2.12) to the embedding $\rho_{\underline{k},\underline{l},q} \colon \mathbb{H} \hookrightarrow \mathrm{End}(\mathbb{C}[x^{\pm 1}])$.

Similarly as the type A_1, we have the Poincaré-Birkhoff-Witt decomposition of $\mathbb{H}$:

$$\mathbb{H} \cong \mathbb{C}[X^{\pm 1}] \otimes H_0 \otimes \mathbb{C}[Y^{\pm 1}], \tag{4.2.14}$$

and *the duality anti-involution*

$$*\colon \mathbb{H}(\underline{k},\underline{l},q) \longrightarrow \mathbb{H}(\underline{k}^*,\underline{l}^*,q), \quad h \longmapsto h^*, \tag{4.2.15}$$

which is a unique $\mathbb{C}$-algebra anti-involution determined by

$$T_1^* := T_1, \quad (Y^\lambda)^* := x^{-\lambda}, \quad (x^\lambda)^* := Y^{-\lambda}$$

for $\lambda \in \Lambda$ and

$$(\underline{k}^*,\underline{l}^*) = (k_1^*,k_0^*,l_1^*,l_0^*) := (k_1,l_1,k_0,l_0). \tag{4.2.16}$$

We also denote by

$$H(\underline{k},\underline{l})^* \subset \operatorname{End}(\mathbb{C}[x^{\pm 1}]) \tag{4.2.17}$$

the image of $H(\underline{k},\underline{l}) \subset \mathbb{H}(\underline{k},\underline{l},q)$ under the duality anti-involution $*$.

4.2.2 Bispectral quantum Knizhnik-Zamolodchikov equation

Let us explain the bispectral qKZ equation of the affine root system $S(C_1^\vee, C_1)$, mainly following [T10, §4.1, §4.2]. Hereafter we choose and fix $k_1,k_0,l_1,l_0,q^{1/2} \in \mathbb{C}^\times$, and consider the affine Hecke algebra $H = H(\underline{k})$, the basic representation $\rho_{\underline{k},\underline{l},q}\colon H(\underline{k}) \hookrightarrow \operatorname{End}(\mathbb{C}[x^{\pm 1}])$ and the DAHA $\mathbb{H} = \mathbb{H}(\underline{k},\underline{l},q)$.

The affine intertwiners

Following [C05, §1.3] and [T10, §4.2], we introduce the affine intertwines of type $(C_1^\vee, C_1)$. We set $x_1 := x^2$, $x_0 := qx^{-2}$, and define $\widetilde{S}_1, \widetilde{S}_0 \in \operatorname{End}(\mathbb{C}[x^{\pm 1}])$ by

$$\widetilde{S}_i := d_i(x)s_i, \quad d_i(x) = d_i(x;\underline{k},\underline{l},q) := k_i^{-1}(1 - k_i l_i x_i^{1/2})(1 + k_i l_i^{-1} x_i^{1/2}) \quad (i = 0,1). \tag{4.2.18}$$

The elements $\widetilde{S}_1$ and $\widetilde{S}_0$ belong to the subalgebra $\mathbb{H} \subset \operatorname{End}(\mathbb{C}[x^{\pm 1}])$ since

$$\widetilde{S}_i = (1 - x_i)\rho_{\underline{k},\underline{l},q}(T_i) - (k_i - k_i^{-1}) - (l_i - l_i^{-1})x_i^{1/2}. \tag{4.2.19}$$

More generally, for each $w \in W$, taking a reduced expression $w = s_{j_1} \cdots s_{j_r}$ with $j_1,\dots,j_r \in \{0,1\}$, we define the element $\widetilde{S}_w \in \mathbb{H}$ by

$$\widetilde{S}_w := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{r-1}} d_{j_r})(x) \cdot w, \tag{4.2.20}$$

The element $\widetilde{S}_w \in \mathbb{H}$ is independent of the choice of reduced expression $w = s_{j_1} \cdots s_{j_r}$ by the same argument as the type A_1 case, using

$$d_w(x) := d_{j_1}(x) \cdot (s_{j_1} d_{j_2})(x) \cdot \cdots \cdot (s_{j_1} \cdots s_{j_{r-1}} d_{j_r})(x) \tag{4.2.21}$$

Also, by [T10, §4.1], we have

$$\widetilde{S}_w = \widetilde{S}_{j_1} \cdots \widetilde{S}_{j_r}. \tag{4.2.22}$$

We call the elements $\widetilde{S}_w$ in (4.2.20) *the affine intertwiners of type* $(C_1^\vee, C_1)$.

The double extended affine Weyl group

As in the case of type A_1 (§ 4.1.2), let us consider the ring

$$\mathbb{L} := \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}] \cong \mathbb{C}[x^{\pm 1}] \otimes \mathbb{C}[\xi^{\pm 1}].$$

We can regard $\mathbb{H}$ as an $\mathbb{L}$-module by

$$(f \otimes g)h := f(x)\, h\, g(Y) \tag{4.2.23}$$

for $f = f(x) \in \mathbb{C}[x^{\pm 1}]$, $g = g(\xi) \in \mathbb{C}[\xi^{\pm 1}]$ and $h \in \mathbb{H}$, where x is understood as the multiplication operator by x itself, and Y is the Dunkl operator. By the PBW type decomposition (4.2.14), we have an $\mathbb{L}$-module isomorphism

$$\mathbb{H} \cong H_0^{\mathbb{L}} := \mathbb{L} \otimes H_0. \tag{4.2.24}$$

As in the case of type A_1, we regard $f(x, \xi) \in H_0^{\mathbb{L}}$ as a function of x, ξ valued in H_0.

The double extended Weyl group $\mathbb{W}$ is introduced in the same way (4.1.33) as the type A_1 case. Let ι denote the nontrivial element of the group $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, and define $\mathbb{W}$ to be the semi-direct product group

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W),$$

with $\iota \in \mathbb{Z}_2$ acting on $W \times W$ by $\iota(w, w') = (w', w)\iota$ for $(w, w') \in W \times W$.

The group W acts on $\mathbb{L}$ in the same way as the type A_1 (see §4.1.2). Define the involution $\diamond \colon W \to W$ by (4.1.34), i.e., $w^{\diamond} := w$ for $w \in W_0$ and $\mathrm{t}(\lambda)^{\diamond} := \mathrm{t}(-\lambda)$ for $\lambda \in \Lambda$. Then the $\mathbb{W}$-action on $\mathbb{L}$ is given by

$$(wf)(x) := (w_q f)(x), \quad (w'g)(\xi) := ((w'^{\diamond})_q g)(\xi), \quad (\iota F)(x, \xi) = F(\xi^{-1}, x^{-1}) \tag{4.2.25}$$

for $w \in W = W \times \{e\} \subset \mathbb{W}$, $w' \in W = \{e\} \times W \subset \mathbb{W}$ and $f = f(x), g = g(\xi), F = F(x, \xi) \in \mathbb{L}$. Here w_q denotes the W-action in (4.2.9).

We also define $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_{\iota} \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})$ by

$$\widetilde{\sigma}_{(w,w')}(h) := \widetilde{S}_w h \widetilde{S}_{w'}^*, \quad \widetilde{\sigma}_{\iota}(h) := h^*$$

for $h \in \mathbb{H}$, where $*$ is the duality anti-involution (4.2.15). Then, as in Fact 4.1.2.4, we have

$$\widetilde{\sigma}_{(w,w')}(fh) = ((w, w')f)\widetilde{\sigma}_{(w,w')}(h), \quad \widetilde{\sigma}_{\iota}(fh) = (\iota f)\widetilde{\sigma}_{\iota}(h) \tag{4.2.26}$$

for $h \in \mathbb{H}$, $f \in \mathbb{L}$ and $w, w' \in W$. The proof is essentially the same as Fact 4.1.2.4 ([vM11, Lemma 3.5]).

The cocycle

As in the case of type A_1 (see (4.1.38)), we denote by

$$\mathbb{K} := \mathcal{M}(x, \xi)$$

the meromorphic functions of variables x, ξ, and define

$$H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0 \cong \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H},$$

We can express an element $f \in H_0^{\mathbb{K}}$ as (4.1.39): $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$, $f_w \in \mathbb{K}$. The $\mathbb{W}$-action (4.2.25) on $\mathbb{L}$ naturally extends to that on $\mathbb{K}$, and we have a $\mathbb{W}$-action on $H_0^{\mathbb{K}}$ by the formula (4.1.40):

$$\mathbf{w} f := \sum_{w \in W_0} (\mathbf{w} f_w) T_w \tag{4.2.27}$$

for $f = \sum_{w \in W_0} f_w T_w \in H_0^{\mathbb{K}}$ and $\mathbf{w} \in \mathbb{W}$.

By the argument right before Fact 4.1.2.5, we have $\widetilde{\sigma}_{(w,w')}, \widetilde{\sigma}_{\iota} \in \mathrm{End}_{\mathbb{C}}(H_0^{\mathbb{K}})$ such that the formulas (4.2.26) are valid for $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. Then, similarly as Fact 4.1.2.5, we have:

Fact 4.2.2.1 ([T10, §4.2])**.** There is a unique group homomorphism $\tau \colon \mathbb{W} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$ satisfying

$$\tau(w, w')(f) = d_w(x)^{-1} d_{w'}^*(\xi^{-1})^{-1} \cdot \widetilde{\sigma}_{(w,w')}(f), \quad \tau(\iota)(f) = \widetilde{\sigma}_{\iota}(f)$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. Here we denoted by $d_{w'}^*$ the image of $d_{w'}$ under the duality anti-involution $*$ in (4.2.17), and $\cdot$ denotes the $\mathbb{L}$-action (4.2.23).

By the $\mathbb{W}$-action (4.2.27) on $H_0^{\mathbb{K}}$, we can regard $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as a $\mathbb{W}$-group via the corresponding conjugation action:

$$(\mathbf{w}, A) \longmapsto \mathbf{w} A \mathbf{w}^{-1} \quad (\mathbf{w} \in \mathbb{W},\ A \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})).$$

Then, we have the following analogue of Fact 4.1.2.7.

Fact 4.2.2.2 ([T10, §4.2])**.** The map

$$\mathbf{w} \longmapsto C_{\mathbf{w}} := \tau(\mathbf{w})\mathbf{w}^{-1} \tag{4.2.28}$$

is a cocycle of $\mathbb{W}$ with values in the $\mathbb{W}$-group $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K} \otimes \mathrm{GL}_{\mathbb{C}}(H_0)$.

We denote $C_{\mathbf{w}}(x,\xi)$ to stress that the cocycle can be regarded as a meromorphic function of x,ξ valued in $\mathrm{GL}_{\mathbb{C}}(H_0)$

Definition 4.2.2.3. Denote $C_{l,m} := C_{(\mathrm{t}(l\epsilon),\mathrm{t}(m\epsilon))}$ for $l,m \in \mathbb{Z}$. The system of q-difference equations

$$C_{l,m}(x,\xi)f(q^{-l}x, q^m\xi) = f(x,\xi) \quad (l,m \in \mathbb{Z})$$

for $f = f(x,\xi) \in H_0^{\mathbb{K}}$ is called *the bispectral quantum KZ equations* (*the bqKZ equations* for short) *of type* $(C_1^\vee, C_1)$. We also denote

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)} = \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)}(\underline{k},\underline{l},q) := \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } (C_1^\vee, C_1)\}.$$

In this § 4.2, we abbreviate $\mathrm{SOL}_{\mathrm{bqKZ}} := \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee,C_1)}$.

Similarly as Lemma 4.1.2.12, we can compute the action of $C_{1,0}$ and $C_{0,1}$ on $H_0^{\mathbb{K}}$. We define an algebra homomorphisms $\eta_L \colon H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_L(A)\Big(\sum_{w\in W_0} f_w T_w\Big) := \sum_{w\in W_0} f_w(AT_w), \tag{4.2.29}$$

for $A \in H$ and $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$. Similarly, using the subspace $H^* \subset \mathbb{H}$ in (4.2.17), we define an algebra anti-homomorphism $\eta_R \colon H^* \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ by

$$\eta_R(A)\Big(\sum_{w\in W_0} f_w T_w\Big) := \sum_{w\in W_0} f_w(T_w A) \tag{4.2.30}$$

for $A \in H^*$ and $f = \sum_{w\in W_0} f_w T_w \in H_0^{\mathbb{K}}$.

Lemma 4.2.2.4. The cocycles $C_{1,0}, C_{0,1} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \cong \mathbb{K} \otimes \mathrm{GL}(H_0)$, regarded as functions of x and ξ are expressed as

$$C_{1,0} = R_0^L(x_0)R_1^L(x_1'), \quad C_{0,1} = R_0^R(\xi_0')R_1^R(\xi_1'), \tag{4.2.31}$$

where we denoted $x_0 := qx^{-2}$, $x_1' := q^2x^{-2}$, $\xi_0' := q\xi^2$, $\xi_1' := q^2\xi^2$ and

$$\begin{aligned}
R_i^L(z) &:= c_i(z)^{-1}\big(\eta_L(T_i) - b_i(z)\big) \\
&= \frac{k_i}{(1-k_il_iz^{1/2})(1+k_il_i^{-1}z^{1/2})}\big((1-z)\eta_L(T_i) - (k_i - k_i^{-1}) - (l_i - l_i^{-1})z^{1/2}\big), \\
R_i^R(z) &:= c_i^*(z)^{-1}\big(\eta_R(T_i^*) - b_i^*(z)\big) \\
&= \frac{k_i^*}{(1-k_i^*l_i^*z^{1/2})(1+k_i^*(l_i^*)^{-1}z^{1/2})}\big((1-z)\eta_R(T_i^*) - (k_i^* - (k_i^*)^{-1}) - (l_i^* - (l_i^*)^{-1})z^{1/2}\big)
\end{aligned}$$

for $i = 0,1$, using the duality anti-involution $*$ in (4.2.15).

Proof. We denote by s_i^x and s_i^ξ for $i = 0,1$ the action (4.2.25) of s_i in terms of variables x and ξ of $\mathbb{K} = \mathcal{M}(x,\xi)$. Explicitly, for $f(x,\xi) \in \mathbb{K}$, we have

$$\begin{aligned}
(s_1^x f)(x,\xi) &= f(x^{-1},\xi), \quad (s_0^x f)(x,\xi) = f(qx^{-1},\xi), \\
(s_1^\xi f)(x,\xi) &= f(x,\xi^{-1}), \quad (s_0^\xi f)(x,\xi) = f(x,q^{-1}\xi^{-1}).
\end{aligned}$$

By a similar calculation as Lemma 4.1.2.12, the cocycle values for (s_1,e) and (s_0,e) are given by $C_{(s_1,e)} = R_1^L(x_1)$ with $x_1 := x^2$ and $C_{(s_0,e)} = R_0^L(x_0)$, respectively. Then the cocycle condition gives

$$C_{1,0} = C_{(s_0s_1,e)} = C_{(s_0,e)}(C_{(s_1,e)})^{(s_0,e)} = R_0^L(x)\big(s_0^x R_1^L(x_1)\big) = R_0^L(x_0)R_1^L(x_1'),$$

where s_0^x means the (s_1, e)-action given in (4.2.25).

Next, using the duality anti-involution $*$ and the $\mathbb{K}$-action (4.2.23), the cocycle values for (e, s_1) and (e, s_0) are given by $C_{(e,s_1)} = R_1^L(x_1)^* = R_1^R(\xi^{-2})$ and $C_{(e,s_0)} = R_0^L(x_0)^* = R_0^R(\xi_0')$ with $\xi_0' = (x_0)^* = q\xi^2$. Thus, we have

$$C_{0,1} = C_{(e,s_0s_1)} = C_{(e,s_0)}(C_{(e,s_1)})^{(e,s_0)} = R_0^R(\xi_0')\big(s_0^\xi R_1^R(\xi^{-2})\big) = R_0^R(\xi_0')R_1^R(\xi_1').$$

□

Remark 4.2.2.5. Some comments on Lemma 4.2.2.4 are in order.

(1) Explicitly, we have

$$C_{1,0} = J_0(x)J_1(x), \quad C_{1,0} = K_0(\xi)K_1(\xi) \tag{4.2.32}$$

with

$$\begin{aligned}
J_0(x) &\coloneqq \frac{k_0}{(1 - k_0l_0q^{1/2}x^{-1})(1 + k_0l_0^{-1}q^{1/2}x^{-1})} \cdot \\
&\quad \cdot \big((1 - qx^{-2})\eta_L(T_0) - (k_0 - k_0^{-1}) - (l_0 - l_0^{-1})q^{1/2}x^{-1}\big), \\
J_1(x) &\coloneqq \frac{k_1}{(1 - k_1l_1qx^{-1})(1 + k_1l_1^{-1}qx^{-1})}\big((1 - q^2x^{-2})\eta_L(T_1) - (k_1 - k_1^{-1}) - (l_1 - l_1^{-1})qx^{-1}\big), \\
K_0(\xi) &\coloneqq \frac{l_1}{(1 - l_1l_0q^{1/2}\xi)(1 + l_1l_0^{-1}q^{1/2}\xi)}\big((1 - q\xi^2)\eta_R(T_0^*) - (l_1 - l_1^{-1}) - (l_0 - l_0^{-1})q^{1/2}\xi\big), \\
K_1(\xi) &\coloneqq \frac{k_1}{(1 - k_1k_0q\xi)(1 + k_1k_0^{-1}q\xi)}\big((1 - q^2\xi^2)\eta_R(T_1) - (k_1 - k_1^{-1}) - (k_0 - k_0^{-1})q\xi\big).
\end{aligned}$$

(2) As in Remark 4.1.2.13, we have

$$C_{(e,w)}(x, \xi) = C_\iota C_{(w,e)}(\xi^{-1}, x^{-1})C_\iota \tag{4.2.33}$$

for any $w \in W$. The formulas (4.2.31) are compatible with 4.2.33.

(3) The formulas (4.2.31) are also consistent with the computation of $C_{0,1}$ in the final paragraph of [T10, §4.2]. Note that we are working on the different choice (4.2.8) of Y from loc. cit.

For later use, we give a $(C_1^\vee, C_1)$-analogue of Fact 4.1.2.14.

Lemma 4.2.2.6. Let $\mathcal{A} \coloneqq \mathbb{C}[x^{-1}] \subset \mathbb{L} = \mathbb{C}[x^{\pm1}, \xi^{\pm1}]$, and $\mathcal{Q}_0(\mathcal{A})$ be the subring of the quotient field $\mathcal{Q}(\mathcal{A}) = \mathbb{C}(x)$ consisting of rational functions which are regular at $x^{-1} = 0$. Considering $\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm1}]$ as subring of $\mathbb{C}(x, \xi)$, we have

$$C_{1,0} \in (\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm1}]) \otimes \operatorname{End} H_0. \tag{4.2.34}$$

Moreover, setting $C_{1,0}^{(0)} \coloneqq C_{1,0}|_{x^{-1}=0} \in \mathbb{C}[\xi^{\pm1}] \otimes \operatorname{End} H_0$, we have

$$C_{1,0}^{(0)} = k_1k_0\eta_L(T_1Y^{-1}T_1^{-1}). \tag{4.2.35}$$

Similarly, defining $\mathcal{B} \coloneqq \mathbb{C}[\xi] \subset \mathbb{L}$ and $\mathcal{Q}_0(\mathcal{B}) \subset \mathcal{Q}(\mathcal{B}) = \mathbb{C}(\xi)$ to be the subring consisting of rational functions which are regular at the point $\xi = 0$, we have

$$C_{0,1} \in (\mathbb{C}[x^{\pm1}] \otimes \mathcal{Q}_0(\mathcal{B})) \otimes \operatorname{End} H_0.$$

Moreover, setting $C_{0,1}^{(0)} \coloneqq C_{0,1}|_{\xi=0} \in \mathbb{C}[x^{\pm1}] \otimes \operatorname{End} H_0$, we have

$$C_{0,1}^{(0)} = k_1l_1\eta_R(T_1Y^{-1}T_1^{-1}). \tag{4.2.36}$$

Proof. We only show the statements for $C_{1,0}$. By the expression (4.2.31) of $C_{1,0}$, we have $C_{1,0} \in (\mathcal{Q}_0(\mathcal{A}) \otimes \mathbb{C}[\xi^{\pm1}]) \otimes \operatorname{End} H_0$. To get (4.2.35), we compute

$$\begin{aligned}
\lim_{x\to\infty} C_{1,0} &= \big(\lim_{x\to\infty} J_1(x)\big)\big(\lim_{x\to\infty} J_0(x)\big) = k_0(\eta_L(T_0) - k_0 + k_0^{-1})k_1(\eta_L(T_1) - k_1 + k_1^{-1}) \\
&= k_1k_0\eta_L(T_0^{-1})\eta_L(T_1^{-1}) = k_1k_0\eta_L(T_1YT_1^{-1}).
\end{aligned}$$

Here we used $T_i^{-1} = T_i - k_i + k_i^{-1}$ from (4.2.7) and $Y = T_1T_0$ from (4.2.8). □

Let us also record the $(C_1^\vee, C_1)$-version of Fact 4.1.2.15.

Fact 4.2.2.7 (c.f. [vM11, Lemma 4.2])**.** For $w \in W_0$, we set

$$\tau_w := \eta_L(\widetilde{S}^*_{w^{-1}})T_e \in \mathbb{C}[\xi^{\pm 1}] \otimes H_0 \subset H_0^{\mathbb{K}}.$$

Then the following statements hold.

(1) $\{\tau_w \mid w \in W_0\}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of eigenfunctions for the η_L-action of $\mathbb{C}[Y^{\pm 1}] \subset \mathbb{H}$ on $H_0^{\mathbb{K}}$.

(2) For $p \in \mathbb{C}[\xi^{\pm 1}]$ and $w \in W_0$, we have $\eta_L(p(Y))\tau_w(\xi) = (w^{-1}p)(\xi)\tau_w(\xi)$ as H_0-valued regular functions in ξ.

The proof for the reduced type in [vM11] also works for the non-reduced type $(C_1^\vee, C_1)$, so we omit it.

4.2.3 Bispectral Askey-Wilson q-difference equation

As in § 4.1.3, we consider the crossed product algebra

$$\mathbb{D}_q^{\mathbb{W}} := \mathbb{W} \ltimes \mathbb{C}(x, \xi)$$

where $\mathbb{W}$ acts on $\mathbb{C}(x, \xi)$ by (4.1.35), and also consider the subalgebra

$$\mathbb{D}_q := \big(\mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)\big) \ltimes \mathbb{C}(x, \xi) \subset \mathbb{D}_q^{\mathbb{W}},$$

which is identified with the algebra of q-difference operators on $\mathbb{C}(x, \xi)$. We can expand $D \in \mathbb{D}_q^{\mathbb{W}}$ as

$$D = \sum_{\mathbf{w} \in \mathbb{W}} f_{\mathbf{w}} \mathbf{w} = \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}} \mathbf{s}, \tag{4.2.37}$$

where $f_{\mathbf{w}} \in \mathbb{C}(T \times T)$ and $D_{\mathbf{s}} = \sum_{\mathbf{t} \in \mathrm{t}(\Lambda) \times \mathrm{t}(\Lambda)} g_{\mathbf{ts}} \mathbf{t} \in \mathbb{D}_q$. We also use $\mathrm{Res}\colon \mathbb{D}_q^{\mathbb{W}} \to \mathbb{D}_q$ given by

$$\mathrm{Res}(D) := \sum_{\mathbf{s} \in W_0 \times W_0} D_{\mathbf{s}}. \tag{4.2.38}$$

Next, following (4.1.57) and (4.1.58), we introduce two realizations of the basic representation of type $(C_1^\vee, C_1)$. Let us denote

$$(1/\underline{k}, 1/\underline{l}) := (1/k_1, 1/k_0, 1/l_1, 1/l_0).$$

Then, the first is given by the algebra homomorphism

$$\rho^x_{1/\underline{k}, 1/\underline{l}, q}\colon H(1/\underline{k}) \longrightarrow \mathbb{C}(x)[W \times \{e\}] \subset \mathbb{D}_q^{\mathbb{W}} \tag{4.2.39}$$

given by the map $\rho_{1/\underline{k}, 1/\underline{l}, q}$ in (4.2.10). The second is

$$\rho^\xi_{\underline{k}^*, \underline{l}^*, 1/q}\colon H(\underline{k}^*) \longrightarrow \mathbb{C}(\xi)[\{e\} \times W] \subset \mathbb{D}_q^{\mathbb{W}}. \tag{4.2.40}$$

Then, recalling Definitions 4.1.3.1 and 4.1.3.3, let us introduce:

Definition 4.2.3.1. For $h \in H(1/\underline{k})$ and $h' \in H(\underline{k}^*)$, we define $D^x_h, D^\xi_{h'} \in \mathbb{D}_q^{\mathbb{W}}$ by

$$D^x_h := \rho^x_{1/\underline{k}, 1/\underline{l}, q}(h), \quad D^\xi_{h'} := \rho^\xi_{\underline{k}^*, \underline{l}^*, 1/q}(h').$$

Also, for an invariant polynomial $p = p(z) \in \mathbb{C}[z^{\pm 1}]^{W_0} = \mathbb{C}[z + z^{-1}]$, we define $L^x_p, L^\xi_p \in \mathbb{D}_q$ by

$$L^x_p = L^x_p(\underline{k}, \underline{l}, q) := \mathrm{Res}(D^x_{p(Y)}), \quad L^\xi_p = L^\xi_p(\underline{k}, \underline{l}, q) := \mathrm{Res}(D^\xi_{p(Y)}), \tag{4.2.41}$$

where we regarded $p(Y) \in H(1/\underline{k})$ for L^x_p, and $p(Y) \in H(\underline{k}^*)$ for L^ξ_p, and used the map Res in (4.2.38).

As in Definition 4.1.3.4, we denote by $p_1(z) \coloneqq z + z^{-1}$, which is the generator of the invariant polynomial ring $\mathbb{C}[z^{\pm 1}]^{W_0}$. Then, similarly as in Proposition 4.1.3.5, we can compute $L^x_{p_1}$ and $L^\xi_{p_1}$ using the function $c(z;t,l)$ in (4.2.11). Let us denote the action of $w \in W$ on functions of x given in (4.2.9) as w^x. It is compatible with $\rho^x_{1/k,q}$ in (4.2.39), and explicitly,

$$s_0^x(x) \coloneqq qx^{-1}, \quad s_1^x(x) = x^{-1}, \quad \mathrm{t}(\varpi)^x(x) = q^{1/2}x. \tag{4.2.42}$$

We also denote by w^ξ the action on functions of ξ. It is compatible with $\rho^\xi_{k,1/q}$ in (4.2.40), and explicitly,

$$s_0^\xi(\xi) \coloneqq q^{-1}\xi^{-1}, \quad s_1^\xi(\xi) = \xi^{-1}, \quad \mathrm{t}(\varpi)^\xi(\xi) = q^{-1/2}\xi. \tag{4.2.43}$$

Proposition 4.2.3.2. We have

$$L^x_{p_1} = k_1k_0 + (k_1k_0)^{-1} + (k_1k_0)^{-2}D^x_{\mathrm{AW}}, \qquad D^x_{\mathrm{AW}} \coloneqq A(x)(T_{q,x} - 1) + A(x^{-1})(T^{-1}_{q,x} - 1), \tag{4.2.44}$$

$$L^\xi_{p_1} = k_1l_1 + (k_1l_1)^{-1} + (k_1l_1)^{2}D^\xi_{\mathrm{AW}}, \qquad D^\xi_{\mathrm{AW}} \coloneqq A^*(\xi^{-1})(T_{q,\xi} - 1) + A^*(\xi)(T^{-1}_{q,\xi} - 1) \tag{4.2.45}$$

with

$$\begin{aligned} A(z) &\coloneqq \frac{(1-k_1l_1z)(1+k_1l_1^{-1}z)(1-k_0l_0q^{-1/2}z)(1+k_0l_0^{-1}q^{-1/2}z)}{(1-z^2)(1-q^{-1}z^2)}, \\ A^*(z) &\coloneqq \frac{(1-k_1k_0z)(1+k_1l_1^{-1}z)(1-l_1l_0q^{-1/2}z)(1+l_1l_0^{-1}q^{-1/2}z)}{(1-z^2)(1-q^{-1}z^2)}. \end{aligned}$$

Proof. Let us compute $L^x_{p_1} = \mathrm{Res}(D^x_{Y+Y^{-1}})$. Since $Y = T_0T_1$ and $s_0 = \mathrm{t}(\epsilon)s_1$, using (4.2.7), (4.2.42) and (4.2.10), we have

$$\begin{aligned} D^x_{Y+Y^{-1}} &= \rho^x_{1/\underline{k},1/\underline{l},q}(T_0T_1 + T_1^{-1}T_0^{-1}) \\ &= \big(k_0^{-1} + c_0(\mathrm{t}(\epsilon)^x s_1^x - 1)\big)\big(k_1^{-1} + c_1(s_1^x - 1)\big) + \big(k_1 + c_1(s_1^x - 1)\big)\big(k_0 + c_0(\mathrm{t}(\epsilon)^x s_1^x - 1)\big) \\ &= k_1^{-1}k_0^{-1} + k_1^{-1}c_0(\mathrm{t}(\epsilon)^x s_1^x - 1) + k_0^{-1}c_1(s_1^x - 1) + c_0(c_1'\,\mathrm{t}(\epsilon)^x s_1^x - c_1)(s_1^x - 1) \\ &\quad + k_1k_0 + k_1c_0(\mathrm{t}(\epsilon)^x s_1^x - 1) + k_0c_1(s_1^x - 1) + c_1(c_0's_1^x - c_0)(\mathrm{t}(\epsilon)^x s_1^x - 1), \end{aligned}$$

where w^x is given by (4.2.42) and, using the function c in (4.2.11), we denoted

$$\begin{aligned} c_1 &\coloneqq c(x^2; k_1^{-1}, l_1^{-1}), & c_1' &\coloneqq \mathrm{t}(\epsilon)^x s_1^x(c_1), \\ c_0 &\coloneqq c(qx^{-2}; k_0^{-1}, l_0^{-1}), & c_0' &\coloneqq s_1^x(c_0) = c(qx^2; k_0^{-1}, l_0^{-1}). \end{aligned}$$

Then, using $(c_0's_1^x - c_0)(\mathrm{t}(\epsilon)^x s_1^x - 1) = c_0'\,\mathrm{t}(-\epsilon)^x - c_0's_1^x - c_0\,\mathrm{t}(\epsilon)^x s_1^x + c_0$ and

$$\mathrm{Res}(\mathrm{t}(\epsilon)^x s_1^x - 1) = \mathrm{t}(\epsilon)^x - 1, \quad \mathrm{Res}(s_1^x - 1) = 0,$$

we have

$$\begin{aligned} \mathrm{Res}(D^x_{Y+Y^{-1}}) &= k_1^{-1}k_0^{-1} + k_1^{-1}c_0(\mathrm{t}(\epsilon)^x - 1) \\ &\quad + k_1k_0 + k_1c_0(\mathrm{t}(\epsilon)^x - 1) + c_1(c_0'\,\mathrm{t}(-\epsilon)^x - c_0' - c_0\,\mathrm{t}(\epsilon)^x + c_0) \\ &= k_1k_0 + k_1^{-1}k_0^{-1} + c_0(k_1 + k_1^{-1} - c_1)(\mathrm{t}(\epsilon)^x - 1) + c_1c_0'(\mathrm{t}(-\epsilon)^x - 1). \end{aligned}$$

Now, using the identity

$$k_1 + k_1^{-1} - c_1 = k_1^{-1}\frac{(1-k_1l_1x)(1+k_1l_1^{-1}x)}{1-x^2} = c(x^2; k_1, l_1) = c(x^{-2}; k_1^{-1}, l_1^{-1}) = s_1^x(c_1),$$

we have $c_0(k_1 + k_1^{-1} - c_1) = c_0 \cdot s_1^x(c_1) = s_1^x(c_0'c_1)$. Then, by $\mathrm{t}(\epsilon)^x = T_{q,x}$, we have

$$L^x_{p_1} = \mathrm{Res}(D^x_{Y+Y^{-1}}) = k_1k_0 + k_1^{-1}k_0^{-1} + \big(s_1^x(c_0'c_1)\big)(T_{q,x} - 1) + c_0'c_1(T^{-1}_{q,x} - 1).$$

Denoting $A(x) \coloneqq s_1^x(c_0'c_1)$, we obtain (4.2.44). The formula (4.2.45) of $L^\xi_{p_1}$ is obtained from $L^x_{p_1}$ by replacing $(x, k_0, k_1, l_0, l_1, q)$ with $(\xi, l_1^{-1}, k_1^{-1}, l_0^{-1}, k_0^{-1}, q^{-1})$. $\square$

Remark 4.2.3.3 (c.f. [N95, pp.54–55]). The operators D^x_{AW} and D^ξ_{AW} are equivalent to *the Askey-Wilson second order q-difference operator* [AW85, (5.7)]:

$$D_{\mathrm{AW}}(z;a,b,c,d,q) := A^+(z;a,b,c,d,q)(T_{q,z}-1) + A^+(z^{-1};a,b,c,d,q)(T_{q,z}^{-1}-1),$$
$$A^+(z;a,b,c,d,q) := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

The precise relation with $A(x), A^*(\xi)$ in (4.2.44), (4.2.45) is given by

$$A(x) = A^+(x;a,b,c',d',q), \quad A^*(\xi) = A^+(\xi;a^*,b^*,c'^*,d'^*,q)$$

with the parameters

$$\{a,b,c',d'\} := \{k_1l_1, -k_1l_1^{-1}, q^{-1/2}k_0l_0, -q^{-1/2}k_0l_0^{-1}\},$$
$$\{a^*,b^*,c'^*,d'^*\} := \{k_1^{-1}k_0^{-1}, -k_1^{-1}k_0, q^{-1/2}l_1^{-1}l_0^{-1}, -q^{-1/2}l_1^{-1}l_0\}.$$

The reciprocal parameter q^{-1} appearing above originates from our choice (4.2.8) of the Dunkl operator Y. As mentioned in Remark 4.2.1.3, the choice in [N95, T10, St14] is the opposite, and for that choice, the above construction of the q-difference operator on x which is equal to the original Askey-Wilson operator $D^x_{\mathrm{AW}}(x;a,b,c,d,q)$.

The ordinary parameters and *the dual parameters* of Askey-Wilson polynomials are given as

$$\{a,b,c,d\} := \{k_1l_1, -k_1l_1^{-1}, q^{1/2}k_0l_0, -q^{1/2}k_0l_0^{-1}\},$$
$$\{a^*,b^*,c^*,d^*\} := \{k_1k_0, -k_1k_0^{-1}, q^{1/2}l_1l_0, -q^{1/2}l_1l_0^{-1}\}.$$

There are related by the duality anti-involution $*$ (see (4.2.15)) as

$$a^* = \sqrt{abcd/q}, \quad b^* = ab/a^*, \quad c^* = ac/a^*, \quad d^* = ad/a^*.$$

By Remark 4.2.3.3, it is natural to name the bispectral problem as:

Definition 4.2.3.4. The following system of eigen-equations for $f = f(x,\xi) \in \mathbb{K}$ is called *the bispectral Askey-Wilson q-difference equation of type* $(C_1^\vee, C_1)$, and *the bAW equation* for short.

$$\begin{cases} (L^x_{p_1}f)(x,\xi) & = p_1(\xi^{-1})f(x,\xi), \\ (L^\xi_{p_1}f)(x,\xi) & = p_1(x)f(x,\xi). \end{cases} \tag{4.2.46}$$

The solution space is denoted as

$$\mathrm{SOL}_{\mathrm{bAW}}(\underline{k},\underline{l},q) := \{f \in \mathbb{K} \mid f \text{ satisfies (4.2.3.4)}\}.$$

4.2.4 Bispectral qKZ/AW correspondence

Here we give a $(C_1^\vee, C_1)$-analogue of § 4.1.4, using the reciprocal parameters

$$(1/\underline{k}, 1/\underline{l}) := (1/k_1, 1/k_0.1/l_1, 1/l_0).$$

Similarly as in Definition 4.1.4.1, we define a $\mathbb{K}$-linear function $\chi_+ \colon H_0(1/\underline{k}) \to \mathbb{C}$ by

$$\chi_+(T_w) := k_1^{-\ell(w)} \tag{4.2.47}$$

for the basis element $T_w \in H_0(1/\underline{k})$ $(w \in W_0)$. It is extended to $H_0(1/\underline{k})^{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{L}} H_0(1/\underline{k})$ as

$$\chi_+ \colon H_0(1/\underline{k})^{\mathbb{K}} \longrightarrow \mathbb{K}, \quad \sum_{w\in W_0} f_w T_w \longmapsto \sum_{w\in W_0} f_w \chi_+(T_w). \tag{4.2.48}$$

Below is a $(C_1^\vee, C_1)$-analogue of Fact 4.1.4.3.

Theorem 4.2.4.1 (c.f. [St14, §3]). Assume $0 < q < 1$. Then the map χ_+ restricts to an injective $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map

$$\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}}(1/\underline{k}, 1/\underline{l}, q) \longrightarrow \mathrm{SOL}_{\mathrm{bAW}}(\underline{k}, \underline{l}, q),$$

where $\mathbb{W}_0$ is the subgroup of $\mathbb{W}$ defined by

$$\mathbb{W}_0 := \mathbb{Z}_2 \ltimes (W_0 \times W_0) \subset \mathbb{W},$$

and $\mathbb{F}$ is the subspace of $\mathbb{K} = \mathcal{M}(T \times T)$ defined by

$$\mathbb{F} := \left\{ f(t,\gamma) \in \mathbb{K} \mid \big((\mathrm{t}(\lambda), \mathrm{t}(\mu)) f\big)(t,\gamma) = f(t,\gamma),\ \forall\, (\lambda,\mu) \in \Lambda \times \Lambda \right\}.$$

The strategy of proof is the same as the type A_1 (§4.1.4). Denoting $\mathrm{SOL}_{\mathrm{bqKZ}} := \mathrm{SOL}_{\mathrm{bqKZ}}(1/\underline{k}, 1/\underline{l}, q)$ and $\mathrm{SOL}_{\mathrm{bAW}} := \mathrm{SOL}_{\mathrm{bAW}}(1/\underline{k}, 1/\underline{l}, q)$, we can divide the proof into three parts.

(i) χ_+ restricts to an $\mathbb{F}$-linear $\mathbb{W}_0$-equivariant map $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$.
(ii) The image $\chi_+(\mathrm{SOL}_{\mathrm{bqKZ}})$ is contained in $\mathrm{SOL}_{\mathrm{bAW}}$.
(iii) $\chi_+ \colon \mathrm{SOL}_{\mathrm{bqKZ}} \to \mathrm{SOL}_{\mathrm{bAW}}$ is injective

We write down the arguments of part (i) and the first half of part (ii). The rest arguments are similar as the type A_1, and we omit them.

Part (i) *of the proof of Theorem 4.2.4.1.* Similarly as Lemma 4.1.4.5, we have

$$\chi_+(C_{\mathbf{w}} F) = \chi_+(F) \tag{4.2.49}$$

for each $\mathbf{w} \in \mathbb{W}_0$ and $F \in H_0(1/\underline{k})^{\mathbb{K}}$. The proof is quite similar as Lemma 4.1.4.5, once we use $C_{(e,s_1)} = C_\iota C_{(s_1,e)} C_\iota$ and replace the expression (4.1.67) of $C_{(s_1,e)}h$ for $h \in H_0$ by

$$C_{(s_1,e)} h = d(x^2; 1/k_1, 1/l_1)^{-1} \big((1-x^2)\eta_L(T_1) - (k_1^{-1} - k_1) - (l_1^{-1} - l_1)x \big) h.$$

Then, in the same way as §4.1.4, we can show that χ_+ is $\mathbb{W}_0$-equivariant using (4.2.28), (4.2.49) and (4.2.25), and that χ_+ restricts to an $\mathbb{F}$-linear map $\mathrm{SOL}_{\mathrm{bqKZ}} \to \mathbb{K}$ using Definition 4.2.2.3, (4.2.47) and (4.2.48). □

Similarly as the type A_1, the part (ii) of the proof consists of two steps.

- Describe of $\mathrm{SOL}_{\mathrm{bqKZ}}$ in terms of the basic asymptotically free solution Φ.
- Analyze the map χ_+ using Φ.

The second step is quite the same as the type A_1, and we omit the detail. The first step requires the following Proposition 4.2.4.2, which is a $(C_1^\vee, C_1)$-analogue of Fact 4.1.4.6, and a simple modification of Fact 4.1.4.8.

Proposition 4.2.4.2. Denote $w_0 := s_1 \in W_0$. Let

$$\mathcal{W}(x,\xi) = \mathcal{W}(x,\xi;\underline{k},\underline{l},q) \in \mathbb{K} = \mathcal{M}(x,\xi)$$

be a meromorphic function satisfying the q-difference equations

$$\mathcal{W}(q^l x, \xi) = (k_1 k_0 \xi)^{-l} \mathcal{W}(x,\xi) \qquad (l \in \mathbb{Z}) \tag{4.2.50}$$

and the self-duality

$$\mathcal{W}(\xi^{-1}, x^{-1}; \underline{k}^*, \underline{l}^*, q) = \mathcal{W}(x,\xi;\underline{k},\underline{l},q). \tag{4.2.51}$$

Then, there is a unique element $\Psi \in H_0(1/\underline{k})^{\mathbb{K}}$ satisfying the following conditions.

(i) We have

$$\Phi := \mathcal{W}\Psi \in \mathrm{SOL}_{\mathrm{bqKZ}}.$$

(ii) We have a series expansion

$$\Psi(x,\xi) = \sum_{m,n \in \mathbb{N}} K_{m,n} x^{-m} \xi^{n\alpha} \quad (K_{\alpha,\beta} \in H_0)$$

for $(x,\xi) \in B_\varepsilon^{-1} \times B_\varepsilon$ with B_ε being some open ball of radius $\varepsilon > 0$, which is normally convergent on compact subsets of $B_\varepsilon^{-1} \times B_\varepsilon$.

(iii) $K_{0,0} = T_{w_0}$.
We call the solution Φ *the basic asymptotically free solution of the bqKZ equation of type* $(C_1^\vee, C_1)$.

Let us give some preliminaries for the proof of Proposition 4.2.4.2. Given a function $\mathcal{W} \in \mathbb{K}$ satisfying (4.2.50) and (4.2.51), we write

$$D_{1,0}(x,\xi) := \mathcal{W}(x,\xi)^{-1} C_{1,0}(x,\xi) \mathcal{W}(q^{-\epsilon}x,\xi),$$
$$D_{0,1}(x,\xi) := \mathcal{W}(x,\xi)^{-1} C_{0,1}(x,\xi) \mathcal{W}(x,q^{\epsilon}\xi),$$

which are regarded as $\mathrm{End}\big(H_0(1/\underline{k})\big)$-valued meromorphic functions in x,ξ. We have $f \in H_0(1/\underline{k})^{\mathbb{K}}$ if and only if $g := \mathcal{W}(x,\xi)^{-1} f$ satisfies the holonomic system of q-difference equations

$$\begin{cases} D_{1,0}(x,\xi) g(q^{-\epsilon}x,\xi) = g(x,\xi) \\ D_{0,1}(x,\xi) g(x,q^{\epsilon}\xi) = g(x,\xi) \end{cases}$$

as $\mathrm{End}\big(H_0(1/\underline{k})\big)$-valued rational functions in x,ξ. Now recall from Lemma 4.2.2.6

$$\mathcal{A} := \mathbb{C}[x^{-1}] \subset \mathbb{C}[x^{\pm 1}], \quad \mathcal{B} := \mathbb{C}[\xi] \subset \mathbb{C}[\xi^{\pm 1}]$$

and

$$Q_0(\mathcal{A}) := \big\{ f(x^{-1})/g(x^{-1}) \in Q(\mathcal{A}) \mid g(0) \neq 0 \big\} \subset Q(\mathcal{A}) = \mathbb{C}(x),$$
$$Q_0(\mathcal{B}) := \{ f(\xi)/g(\xi) \in Q(\mathcal{B}) \mid g(0) \neq 0 \} \subset Q(\mathcal{B}) = \mathbb{C}(\xi).$$

Lemma 4.2.4.3 (c.f. [vMS09, Lemma 5.2])**.** The operators $D_{1,0}$ and $D_{0,1}$ satisfy the following properties.

(1) $D_{1,0} \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}\big(H_0(1/\underline{k})\big)$ and $D_{0,1} \in (\mathcal{A} \otimes Q_0(\mathcal{B})) \otimes \mathrm{End}\big(H_0(1/\underline{k})\big)$

(2) Define $D_{1,0}^{(0)}, D_{0,1}^{(0)} \in \mathrm{End}\big(H_0(1/\underline{k})\big)$ by

$$D_{1,0}^{(0)} := D_{1,0}|_{x^{-1}=0}, \quad D_{0,1}^{(0)} := D_{0,1}|_{\xi=0}.$$

Then, denoting $w_0 := s_1$, we have

$$D_{1,0}^{(0)}(T_{w_0}T_w) = \begin{cases} T_1 & (w=e) \\ 0 & (w=s_1) \end{cases}, \quad D_{0,1}^{(0)}(T_{w_0}T_w) = \begin{cases} T_1 & (w=e) \\ 0 & (w=s_1) \end{cases}. \tag{4.2.52}$$

Proof. For the first half of (1), note that the q-difference equation (4.2.50) with $\lambda = -\epsilon$ yields

$$D_{1,0}(x,\xi) = \mathcal{W}(x,\xi)^{-1} C_{1,0}(x,\xi) \mathcal{W}(q^{-1}x,\xi) = k_1 k_0 \xi C_{1,0}(x,\xi), \tag{4.2.53}$$

By the explicit expression of $C_{1,0}$ (Lemma 4.2.2.4), we have $D_{1,0} \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}(H_0)$.

For the second half, using (4.2.50) and (4.2.51), we have

$$D_{0,1}(x,\xi) = \mathcal{W}^{(C_1^\vee,C_1)}(x,\xi)^{-1} C_{0,1}(x,\xi) \mathcal{W}^{(C_1^\vee,C_1)}(x,q\xi) = (k_1 u_1 x)^{-1} C_{0,1}(x,\xi).$$

By the explicit expression of $C_{0,1}$ (Lemma 4.2.2.4), we have $D_{0,1} \in (\mathcal{A} \otimes Q_0(\mathcal{B})) \otimes \mathrm{End}(H_0)$.

Next, we will show the first half of (2). By the above computation (4.2.53) and Lemma 4.2.2.6, we have

$$D_{1,0}^{(0)} = D_{1,0}|_{x^{-1}=0} = k_1 k_0 \xi C_{1,0}^{(0)}. \tag{4.2.54}$$

Let us compute $D_{1,0}^{(0)}(T_1)$. Since $\eta_L(T_1 Y^{-1} T_1^{-1})(T_1) = \xi^{-1} T_1$, we have

$$D_{1,0}^{(0)}(T_1) = k_1 k_0 \xi C_{1,0}^{(0)}(T_1) = \xi \eta_L(T_1 Y^{-1} T_1^{-1})(T_1) = T_1,$$

using (4.2.35) with reciprocal parameters $1/\underline{k}$ in the second equality. Hence we obtain $D_{1,0}^{(0)}(T_1) = T_1$. For $D_{1,0}^{(0)}(T_e)$, note that $\tau_w := \eta_L(\widetilde{S}^*_{w^{-1}}) T_e$ $(w \in W_0)$ form a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ (Fact 4.2.2.7) and $\eta(T_{w_0})\tau_w \in \mathcal{B} \otimes \mathrm{End}(H_0)$. By Fact 4.2.2.7 and (4.2.35), we obtain

$$D_{1,0}^{(0)}(\eta(T_1)\tau_{s_1}) = k_1 k_0 \xi C_{1,0}^{(0)}(\eta(T_1)\tau_{s_1}) = \xi \eta_L(T_1 Y^{-1} T_1^{-1})(\eta(T_1)\tau_{s_1}) = \xi^2 \eta(T_1)\tau_{s_1}.$$

as identities in $\mathcal{B} \otimes \mathrm{End}(H_0)$. Specializing at $\xi = 0$, we obtain $D_{1,0}^{(0)}(T_e) = 0$.

The second half of (2) can be shown similarly using (4.2.36). We omit the detail. □

Proof of Proposition 4.2.4.2. Lemma 4.2.4.3 implies that the operators $D^{(0)}_{1,0}$ and $D^{(0)}_{0,1}$ on $H_0(1/\underline{k}, 1/\underline{l})$ commute with each other. We denote the simultaneous eigenspace decomposition of $H_0(1/\underline{k}, 1/\underline{l})$ as

$$H_0(1/\underline{k}, 1/\underline{l}) = \bigoplus_{(a,b)\in\mathbb{C}^2} H_0[a,b], \quad H_0[a,b] := \left\{ v \in H_0 \mid D^{(0)}_{1,0}(v) = av,\ D^{(0)}_{0,1}(v) = bv \right\}$$

Since $H_0(1/\underline{k}, 1/\underline{l})$ is finite dimensional, the subset $S \subset \mathbb{C}^2$ for which $H_0[a,b] \neq 0$ is finite. We also have $(1,1) \in S$ and $H_0[1,1] = \mathbb{C}T_1$ by Lemma 4.2.4.3. Furthermore, $a, b \in q^{\mathbb{N}}$ for all $(a,b) \in S$. Under these conditions, the holonomic system of q-difference equations 4.2.52 admits a unique solution Ψ satisfying the desired properties by the general theory developed in [vMS09, Theorem A.6]. $\square$

Example 4.2.4.4. We give an example of the function $\mathcal{W}$ in Proposition 4.2.4.2. As in the case of type A_1 (Example 4.1.4.12 (1)), using the Jacobi theta function $\theta(z;q) := (q, z, q/z; q)_\infty$, we define

$$\mathcal{W}^{(C_1^\vee, C_1)}(x,\xi) = \mathcal{W}^{(C_1^\vee, C_1)}(x,\xi;\underline{k},\underline{l}) := \frac{\theta(-q^{1/2}x\xi;q)}{\theta(-q^{1/2}(k_1k_0)^{-1}x, -q^{1/2}k_1l_1\xi;q)}. \tag{4.2.55}$$

It satisfies the q-difference equation (4.2.50) in the form

$$\mathcal{W}^{(C_1^\vee, C_1)}(q^{\pm 1}x,\xi) = (k_1k_0\xi)^{\mp 1}\mathcal{W}^{(C_1^\vee, C_1)}(x,\xi),$$

and the self-duality (4.2.51) in the form

$$\mathcal{W}^{(C_1^\vee, C_1)}(\gamma^{-1}, t^{-1}; \underline{k}^*, \underline{l}^*) = \mathcal{W}^{(C_1^\vee, C_1)}(t,\gamma;\underline{k},\underline{l}). \tag{4.2.56}$$

Here we used the duality anti-involution $*$ in (4.2.15).

Remark 4.2.4.5. As in the case of type A_1 case (Remark 4.1.4.13), the function $\mathcal{W}^{(C_1^\vee, C_1)}$ is nothing but the function G of Remark 4.1.4.13 (2) introduced by [vM11]:

$$G(t,\gamma) := \frac{\vartheta(\mathrm{t}(w_0\gamma)^{-1})}{\vartheta(\gamma_0 t)\,\vartheta((\gamma_0^*)^{-1}\gamma)}$$

whose lattice theta function $\vartheta(t) = \vartheta^{A_1}(t)$ is replaced by

$$\vartheta(t) := \sum_{\lambda\in\Lambda} q^{\langle\lambda,\lambda\rangle/2} t^\lambda, \quad \Lambda = \mathbb{Z}\epsilon,$$

and the parameters γ_0, γ_0^* are replaced by

$$\gamma_0 := (k_1k_0)^{-\epsilon},\ \gamma_0^* := (k_1l_1)^{-\epsilon} \in T. \tag{4.2.57}$$

4.2.5 Bispectral Askey-Wilson function

In this subsection, we cite from [St02, St14] an example of explicit solution of the bispectral Askey-Wilson q-difference equation. As in the previous Theorem 4.2.4.1, we assume $0 < q < 1$.

Let us write again the bispectral Askey-Wilson q-difference equation (4.2.46) for $f(x,\xi) \in \mathbb{L} = \mathbb{C}[x^{\pm 1}, \xi^{\pm 1}]$ for the reciprocal parameters $\mathrm{SOL}_{\mathrm{bAW}}(1/\underline{k}, 1/\underline{l})$:

$$\begin{cases} (L^x_{p_1} f)(x,\xi) & = (\xi + \xi^{-1}) f(x,\xi) \\ (L^\xi_{p_1} f)(x,\xi) & = (x + x^{-1}) f(x,\xi) \end{cases}. \tag{4.2.58}$$

By Proposition 4.2.3.2 and Remark 4.2.3.3, the operators are given by

$$L^x_{p_1} = k_1k_0 + (k_1k_0)^{-1} + (k_1k_0)^{-1} D^x_{\mathrm{AW}}, \quad L^\xi_{p_1} = k_1l_1 + (k_1l_1)^{-1} + (k_1l_1) D^\xi_{\mathrm{AW}}, \tag{4.2.59}$$

$$D^x_{\mathrm{AW}} := D_{\mathrm{AW}}(x; a,b,c,d,q), \quad D^\xi_{\mathrm{AW}} := D_{\mathrm{AW}}(\xi; (a^*)^{-1}, (b^*)^{-1}, (c^*)^{-1}, (d^*)^{-1}, q^{-1}),$$

$$\{a,b,c,d\} := \{k_1l_1, -k_1l_1^{-1}, q^{1/2}k_0l_0, -q^{1/2}k_0l_0^{-1}\}, \tag{4.2.60}$$

$$\{a^*,b^*,c^*,d^*\} := \{k_1k_0, -k_1k_0^{-1}, q^{1/2}l_1l_0, -q^{1/2}l_1l_0^{-1}\} \tag{4.2.61}$$

with

$$D_{\mathrm{AW}}(x;q,a,b,c,d) := A(x)(T_{q,x}-1)+A(x^{-1})(T_{q,x}^{-1}-1),$$
$$A(x) := \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}. \tag{4.2.62}$$

As mentioned in Remark 4.2.3.3, the q-difference operator D_{AW}^x was introduced by Askey and Wilson [AW85]. Using the symbol $(x_1,\ldots,x_r;q)_l$ in (1.1.2), they showed that the basic hypergeometric polynomial

$$P_l(x;a,b,c,d;q) := \frac{(ab,ac,ad;q)_l}{a^l}\,{}_4\phi_3\left[\begin{matrix} q^{-l},\ abcdq^{l-1},\ ax,\ a/x \\ ab,\ ac,\ ad \end{matrix};q,\ q\right] \quad (l\in\mathbb{N}) \tag{4.2.63}$$

is an eigenfunction of D_{AW}^x, and the eigenvalue is $-(1-q^{-l})(1-q^{l-1}abcd)$. This claim is restated as

$$L_{p_1}^x P_l(x;a,b,c,d;q) = (q^l a^* + q^{-l}(a^*)^{-1})P_l(x;a,b,c,d;q)$$

under the parameter correspondence (4.2.60) and (4.2.61) (c.f. [N95, p.55]). The Laurent polynomial $P_l(x;a,b,c,d;q)$ is called *the Askey-Wilson polynomial*.

In order to treat the bispectral problem (4.2.58), we need to consider non-polynomial eigenfunctions of the Askey-Wilson second order q-difference operator D_{AW}. In literature, such an eigenfunction is given in terms of a very-well-poised ${}_8\phi_7$ series under the name of *the Askey-Wilson function*. Here we give a brief review, and refer to [St02, §3] for more information.

Following Gasper and Rahman [GR04, (2.1.11)], we denote

$${}_8W_7(a_1;a_4,a_5,a_6,a_7,a_8;q,z) := {}_8\phi_7\left[\begin{matrix} a_1,\ qa_1^{1/2},\ -qa_1^{1/2},\ a_4,\ a_5,\ a_6,\ a_7,\ a_8 \\ a_1^{1/2},\ -a_1^{1/2},\ \frac{qa_1}{a_4},\ \frac{qa_1}{a_5},\ \frac{qa_1}{a_6},\ \frac{qa_1}{a_7},\ \frac{qa_1}{a_8} \end{matrix};q,\ z\right],$$

which is a very-well-poised basic hypergeometric series in the sense of [GR04, the line after (2.1.9)]. Then, the Askey-Wilson function $\phi_\xi(x)=\phi_\xi(x;a,b,c,d;q)$ is defined by [St02, (3.1)]

$$\phi_\xi(x) := \frac{(qax\xi/d^*,qa\xi/d^*x,qabc/d;q)_\infty}{(a^*b^*c^*\xi,q\xi/d^*,qx/d,q/dx,bc,qb/d,qc/d;q)_\infty}\,{}_8W_7(a^*b^*c^*\xi/q;ax,a/x,a^*\xi,b^*\xi,c^*\xi;q,q/d^*\xi).$$

It satisfies the eigen-equation

$$(L_{p_1}^x\phi_\xi)(x) = (\xi+\xi^{-1})\phi_\xi(x), \tag{4.2.64}$$

the self-duality

$$\phi_\xi(x;a,b,c,d;q) = \phi_x(\xi;a^*,b^*,c^*,d^*;q), \tag{4.2.65}$$

and the symmetry (the inversion invariance in [St14])

$$\phi_\xi(x)=\phi_\xi(x^{-1})=\phi_{\xi^{-1}}(x). \tag{4.2.66}$$

The properties (4.2.65) and (4.2.66) are the consequences of the equality [St14, (3.2)]:

$$\begin{aligned}\phi_\xi(x) = {} & \frac{(qabc/d;q)_\infty}{(bc,qa/d,qb/d,qc/d,q/ad;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} ax,\ a/x,\ a^*\xi,\ a^*/\xi \\ ab,\ ac,\ ad \end{matrix};q,\ q\right] \\ & + \frac{(ax,a/x,a^*\xi,a^*/\xi,qabc/d;q)_\infty}{(qx/d,q/dx,q\xi/d^*,q/d^*\xi,ab,ac,bc,qa/d,ad/q;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} qx/d,\ q/dx,\ q\xi/d^*,\ q/d^*\xi \\ qb/d,\ qc/d,\ q^2/ad \end{matrix};q,\ q\right],\end{aligned}$$

which can be shown by a form [GR04, (2.10.10)] of Bailey's transformation formulas. The above equality also yields

$$\phi_{\xi_l}(x) = \frac{(qabc/d;q)_\infty}{(bc,qa/d,qb/d,qc/d,q/ad;q)_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-l},\ abcdq^{l-1},\ ax,\ a/x \\ ab,\ ac,\ ad \end{matrix};q,\ q\right],\quad \xi_l := (a^*)^{-1}q^{-l},$$

which is proportional to the Askey-Wilson polynomial $P_l(x)$ (4.2.63).

Let us consider the asymptotic form of the Askey-Wilson q-difference equation $\big(L^x_{p_1} - (\xi+\xi^{-1})\big)f(x) = 0$ in the region $|x| \gg 1$. Since the functions $A(x)$ and $A(x^{-1})$ in (4.2.62) behave as $A(x) \approx (a^*)^2$ and $A(x^{-1}) \approx 1$, we have the asymptotic form

$$L^x_{p_1} \approx a^* T_{q,x} + (a^*)^{-1} T^{-1}_{q,x}.$$

Now, recall the function $\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi)$ given in (4.2.55):

$$\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi) = \frac{\theta(-\nu x\xi; q)}{\theta(-\nu x/a^*, -\nu\xi a; q)},$$

where $\nu \coloneqq q^{1/2}$. By $\theta(qx; q) = -x^{-1}\theta(x; q)$, we have $T^{\pm 1}_{q,x}\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi) = (a^*\xi)^{\mp 1}\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi)$, which implies that the set $\{\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi^{\pm 1})\}$ is a basis of solutions of the asymptotic q-difference equation

$$\big(a^* T_{q,x} + (a^*)^{-1} T^{-1}_{q,x} - (\xi + \xi^{-1})\big) f(x) = 0.$$

Similarly, the ξ-side asymptotic q-difference equation in the region $|\xi| \ll 1$ is given by

$$L^\xi_{p_1} \approx a T^{-1}_{q,\xi} + a^{-1} T_{q,\xi},$$

and since $T^{\pm 1}_{q,\xi}\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi) = (a/x)^{\pm 1}\mathcal{W}^{(C_1^\vee, C_1)}(x, \xi)$, the set $\{\mathcal{W}^{(C_1^\vee, C_1)}(x^{\pm 1}, \xi)\}$ is a basis of solutions of the asymptotic equation

$$\big(a^{-1} T_{q,\xi} + a T^{-1}_{q,\xi} - (x + x^{-1})\big) g(\xi) = 0,$$

By the argument in § 4.2.4, we have a unique element $\widehat{\Phi} \coloneqq \chi_+(\Phi) \in \mathrm{SOL}_{\mathrm{bAW}}$ of the form $\widehat{\Phi} = \mathcal{W}^{(C_1^\vee, C_1)} g$, where $g = g(x)$ has a convergent series expansion around $|x| = \infty$ with constant coefficient being 1. By [St14, Proposition 5.2, (5.8)], $\widehat{\Phi}$ is written down as

$$\begin{aligned}\widehat{\Phi}(x, \xi) = \mathcal{W}^{(C_1^\vee, C_1)}(x, \xi) \cdot \frac{(qa\xi/a^*x, qb\xi/a^*x, qc\xi/a^*x, qa^*\xi/dx, d/x; q)_\infty}{(q/ax, q/bx, q/dx, q^2\gamma^2/dx; q)_\infty} \\ \cdot {}_8W_7(q\xi^2/dx; q\xi/a^*, q\xi/d^*, b^*\xi, c^*\xi, q/dx; q, d/x).\end{aligned}$$

Remark 4.2.5.1. Our solution $\widehat{\Phi}(x, \xi)$ is equivalent to the solution $\widehat{\Phi}_\eta(t, \gamma)$ in [St14, (5.8)] up to quasi-constant multiplication.

Now we cite a $(C_1^\vee, C_1)$-analogue of Fact 4.1.5.6.

Fact 4.2.5.2 (c.f. [St14, Proposition 5.2])**.** The function $\mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi) = \mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi; \underline{k}, \underline{l}, q)$ given by

$$\mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi) \coloneqq \frac{(qax\xi/d^*, qa\xi/d^*x, qa/d, q/ad; q)_\infty}{(a^*b^*c^*\xi, q\xi/d^*, qx/d, q/dx; q)_\infty} {}_8W_7(a^*b^*c^*\xi/q; ax, a/x, a^*\xi, b^*\xi, c^*\xi; q, q/d^*x).$$

enjoys the following properties.

(i) It is a solution of the bispectral problem (4.2.58).

(ii) It has the symmetry

$$\mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi) = \mathcal{E}_+^{(C_1^\vee, C_1)}(x^{-1}; \xi) = \mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi^{-1}).$$

(iii) It has the self-duality

$$\mathcal{E}_+^{(C_1^\vee, C_1)}(x, \xi; \underline{k}, \underline{l}, q) = \mathcal{E}_+^{(C_1^\vee, C_1)}(\xi^{-1}; x^{-1}, \underline{k}^*, \underline{l}^*, q). \tag{4.2.67}$$

Thus, defining $\mathrm{SOL}^{\mathbb{W}^*}_{\mathrm{bAW}} \coloneqq \{f \in \mathrm{SOL}_{\mathrm{bAW}} \mid \text{(ii), (iii)}\}$, we have

$$\mathcal{E}_+^{(C_1^\vee, C_1)} \in \mathrm{SOL}^{\mathbb{W}^*}_{\mathrm{bAW}}.$$

The function $\mathcal{E}_+^{(C_1^\vee, C_1)}$ is called *the basic hypergeometric series of type* $(C_1^\vee, C_1)$.

4.3 Specialization

In [YY22, §2.6], we introduced four embeddings of affine root systems of type A_1 into type $(C_1^\vee, C_1)$. They are given by particular specializations of the parameters $(\underline{k}, \underline{l})$, and characterized to preserve the Macdonald inner product under which the Macdonald-Koornwinder polynomials are orthogonal. Among the four specializations, the one given by

$$(\underline{k}, \underline{l}) = (k, 1, 1, 1) \tag{4.3.1}$$

has a special feature that it is also compatible with the duality anti-involution (4.2.15). In this section, we show that this specialization yields the commutative diagram mentioned in Preface:

$$\begin{tikzcd} \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)} \arrow[r, hook, "\chi_+^{(C_1^\vee, C_1)}"] \arrow[d, hook, "\mathrm{sp}"'] & \mathrm{SOL}_{\mathrm{bAW}} \arrow[d, hook, "\mathrm{sp}"] \\ \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1} \arrow[r, hook, "\chi_+^{A_1}"'] & \mathrm{SOL}_{\mathrm{bMR}} \end{tikzcd}$$

4.3.1 The bispectral qKZ equations

Recall the subalgebras $H_0^{A_1}(k) \subset \mathbb{H}^{A_1}(k, q)$ and $H_0^{(C_1^\vee, C_1)}(\underline{k}) \subset \mathbb{H}^{(C_1^\vee, C_1)}(\underline{k}, \underline{l}, q)$, both of which have the basis $\{T_e = 1, T_{s_1} = T_1\}$. Let us identify these linear spaces, and denote it by H_0. As in the previous sections, let us use the notation $\mathbb{K} = \mathcal{M}(x, \xi)$ and $H_0^{\mathbb{K}} = \mathbb{K} \otimes H_0$.

Then, the solution spaces of bispectral qKZ equations of type A_1 and of type $(C_1^\vee, C_1)$ (Definition 4.1.2.8 and Definition 4.2.2.3) can be expressed as

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{A_1}(k, q) = \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } A_1\},$$
$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)}(\underline{k}, \underline{l}, q) = \{f \in H_0^{\mathbb{K}} \mid f \text{ satisfies the bqKZ equations of type } (C_1^\vee, C_1)\}.$$

Then we can show:

Proposition 4.3.1.1. For the specialized parameters $(\underline{k}, \underline{l}) = (k, 1, 1, 1)$, we have the relation

$$\mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q) \subset \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1}(k, q).$$

Proof. Denoting by $c^{A_1}(z; k, q) := c(z : k, q)$ the function in (4.1.17), and by $c^{(C_1^\vee, C_1)}(z; k, l, q) := c(z : k, l, q)$ the function in (4.2.11), we have

$$c^{(C_1^\vee, C_1)}(z; k, 1, q) = c^{A_1}(z; k, q).$$

Then, comparing Lemma 4.1.2.16 and Lemma 4.2.2.4, we have

$$C_{1,0}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q) = C_{2,0}^{A_1}(k, q), \quad C_{0,1}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q) = C_{0,2}^{A_1}(k, q), \tag{4.3.2}$$

from which we have the claim. □

Theorem 4.3.1.2. The specialization (4.3.1) yields the commutative diagram

$$\begin{tikzcd} \mathrm{SOL}_{\mathrm{bqKZ}}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q) \arrow[r, hook, "\chi_+^{(C_1^\vee, C_1)}"] \arrow[d, hook, "\mathrm{sp}"'] & \mathrm{SOL}_{\mathrm{bAW}}(k, 1, 1, 1, q) \arrow[d, hook, "\mathrm{sp}"] \\ \mathrm{SOL}_{\mathrm{bqKZ}}^{A_1}(k, q) \arrow[r, hook, "\chi_+^{A_1}"'] & \mathrm{SOL}_{\mathrm{bMR}}(k, q) \end{tikzcd} \tag{4.3.3}$$

Proof. We saw the left vertical embedding in Proposition 4.3.1.1. Thus, it is enough to check that the specialization maps the bispectral Askey-Wilson equation (4.2.46) to the bispectral Macdonald-Ruijsenaars equation (4.1.63). Since $(k_1, k_0, l_1, l_0) = (k, 1, 1, 1)$ yields the Askey-Wilson parameters $\{a, b, c, d\} = \{k, -k, q^{1/2}, -q^{1/2}\}$, the specialization of the x-side equation is computed as

$$\begin{aligned} L^x_{(C_1^\vee, C_1)}(k, 1, 1, 1, q) &= k + k^{-1} + \frac{k - k^{-1}x^{-2}}{1 - x^{-2}}(T_{q,x} - 1) + \frac{k^{-1} - kx^{-2}}{1 - x^{-2}}(T_{q,x}^{-1} - 1) \\ &= \frac{k - k^{-1}x^{-2}}{1 - x^{-2}}T_{q,x} + \frac{k^{-1} - kx^{-2}}{1 - x^{-2}}T_{q,x}^{-1} = L^x_{A_1}(k, q^2). \end{aligned}$$

Note that the parameter q^2 in type A_1 is compatible with the relation (4.3.2). The ξ-side is similarly checked directly, or by the compatibility of the duality anti-involution and the specialization. □

So far we give a computational argument to show the commutative diagram (4.3.3). Let us give another, more conceptual argument.

Lemma 4.3.1.3. There is an isomorphism of algebras

$$\mathbb{H}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q) \xrightarrow{\sim} \mathbb{H}^{A_1}(k, q).$$

Proof. Recall the presentations (4.1.20) of $\mathbb{H}^{A_1}$ and (4.2.12) of $\mathbb{H}^{(C_1^\vee, C_1)}$. The former gives $\mathbb{H}^{A_1}(k, q)$ as the quotient of the free algebra $\mathbb{C}\langle T, U, X\rangle$ by the relations

$$(T - k)(T + k^{-1}) = 0, \quad U^2 = 1, \quad TXT = X^{-1}, \quad UXU = q^{1/2}X^{-1}.$$

Under the specialization $(\underline{k}, \underline{l}) = (k, 1, 1, 1)$, the latter gives $\mathbb{H}^{(C_1^\vee, C_1)}(k, 1, 1, 1, q)$ as the quotient of $\mathbb{C}\langle T_1, T_0, T_1^\vee, T_0^\vee\rangle$ by the relations

$$(T_1 - k)(T_1 + k^{-1}) = 0, \quad (T_0)^2 = (T_1^\vee)^2 = (T_0^\vee)^2 = 1, \quad T_1^\vee T_1 T_0 T_0^\vee = q^{-1/2}. \tag{4.3.4}$$

Now, recalling (4.2.13), we find that the correspondence $T_1 = T$, $T_0 = U$ and $T_0^\vee = q^{-1/2}UX$ gives the desired isomorphism □

Since the bispectral correspondence $\chi_+^{A_1}$ is defined in terms of the DAHA $\mathbb{H}^{A_1}(k, q)$, the restriction to the subalgebra $\mathbb{H}^{(C_1^\vee, C_1)}(k, 1, 1, 1, 1, q)$ will give the correspondence $\chi_+^{(C_1^\vee, C_1)}$. Thus we have the commutative diagram (4.3.3).

Remark 4.3.1.4. We leave it for a future study to give an explicit element in $\mathrm{SOL}_{\mathrm{bAW}}(k, 1, 1, 1, q)$ which is mapped to $\mathrm{SOL}_{\mathrm{bMR}}(k, q)$ under the right vertical embedding sp in (4.3.3). Here we only give a clue to find such an element. If the spectral variable ξ is specialized to $\xi_l = k^{-1}q^{-1/2}$ (see Proposition 4.1.5.3 (2)), we have

$$P_l^{A_1}(x; k^2, q) := x^l {}_2\phi_1\left[\begin{matrix} k^2,\ q^{-l} \\ q^{1-l}/k^2 \end{matrix}; q,\ \frac{q}{k^2x^2}\right] = \frac{1}{(q^l k^2; q)_l} P_l(x; k, 1, 1, 1; q) = P_l^{(C_1^\vee, C_1)}(x; k, 1, 1, 1; q).$$

We expect that there is an element $f(x, \xi) \in \mathrm{SOL}_{\mathrm{bAW}}(k, 1, 1, 1, q)$ such that the specialized $f(x, \xi_l)$ is equal to $P_l^{(C_1^\vee, C_1)}(x; k, 1, 1, 1; q)$ and the image $\mathrm{sp}(f(x, \xi_l))$ is equal to $P_l^{A_1}(x; t, q)$.

Bibliography

[AW85] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials*, Mem. Amer. Math. Soc., **54** (1985), no. 319.

[B68b] N. Bourbaki, *Groupes et algèbres de Lie : Chapitres 4, 5 et 6*, Hermann (1968).

[C92a] I. Cherednik, *Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, Int. Math. Res. Not., **9** (1992), 171–179.

[C92b] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Commun. Math. Phys., **150** (1992), 109–136.

[C92c] I. Cherednik, *Double Affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators*, Int. Math. Res. Not., **9** (1992), 171–179.

[C94] I. Cherednik, *Induced representations of double affine Hecke algebras and applications*, Math. Res. Lett., **1** (1994), 319–337.

[C95a] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. Math., **141** (1995), 191–216.

[C95b] I. Cherednik, *Non-symmetric Macdonald polynomials*, Int. Math. Res. Not., **10** (1995), 483–515.

[C95c] I. Cherednik, *Macdonald's evaluation conjectures and difference Fourier transform*, Inv. Math., **122** (1995), 119–145

[C97a] I. Cherednik, *Intertwining operators of double affine Hecke algebras*, Sel. Math. new series, **3** (1997), 459–495.

[C97b] I. Cherednik, *Difference Macdonald-Mehta conjecture*, Int. Math. Res. Not., **1997** (1997), no. 10, 449–467.

[C05] I. Cherednik, *Double affine Hecke algebras*, London Math. Soc. Lecture Note Series, **319**, Cambridge Univ. Press, Cambridge, 2005.

[C09] I. Cherednik, *Whittaker limits of difference spherical functions*, Int. Math. Res. Not., **2009** (2009), no. 20, 3793–3842.

[Chi21] M. Chihara, *Demazure slices of type $A_{2l}^{(2)}$*, Algebr. Represent. Theory, **25** (2022), 491–519.

[FH$^+$15] B. Feigin, A. Hoshino, M. Noumi, J. Shibahara, J. Shiraishi, *Tableau Formulas for One-Row Macdonald Polynomials of Types C_n and D_n*, SIGMA Symmetry Integrability Geom. Methods Appl., **11** (2015), Paper 100, 21 pp.

[GR04] G. Gasper, M. Rahman, *Basic hypergeometric series*, 2nd. ed., Encyclopedia of Mathematics and its Applications **96**, Cambridge University Press (2004).

[H06] M. Haiman, *Cherednik algebras, Macdonald polynomials and combinatorics*, International Congress of Mathematicians. Vol. III, 843–872, Eur. Math. Soc., Zürich, 2006.

[HS18] A. Hoshino, J. Shiraishi, *Macdonald polynomials of type Cn with one-column diagrams and deformed Catalan numbers*, SIGMA Symmetry Integrability Geom. Methods Appl., **14** (2018), Paper No. 101, 33 pp.

[HS20] A. Hoshino, J. Shiraishi, *Branching rules for Koornwinder polynomials with one column diagrams and matrix inversions*, SIGMA Symmetry Integrability Geom. Methods Appl., **16** (2020), Paper No. 084, 28 pp.

[I03] B. Ion, *Nonsymmetric Macdonald polynomials and Demazure characters*, Duke Math. J., **116** (2003), no. 2, 299–318.

[Ka90] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.

[KLS10] Koekoek, Lesky, Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2010

[Ko92] T. H. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, Contemp. Math., **138** (1992), 189–204.

[Ko] T. H. Koornwinder, *Charting the Askey and q-Askey schemes*, Lecture in *AMS Special Session on The Legacy of Dick Askey*, 6-9 January, 2021. The slide is available from https://staff.fnwi.uva.nl/t.h.koornwinder/art/sheets/

[L76] E. Looijenga, *Root systems and Elliptic curves*, Inv. Math., **38** (1976), 17–32.

[L89] G. Lusztig, *Affine Hecke algebras and their graded version*, J, Amer. Math. Soc., **2** (1989), 599–635.

[M71] I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Inv. Math., **15** (1972), 161–174.

[M87] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, preprint, 1987; typed and published in the Sḿinaire Lotharingien de Combinatoire, **45** (2000), B45a; available from arXiv:math/0011046.

[Ma95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ., Press, 1995.

[M98] I. G. Macdonald, *Symmetric functions and orthogonal polynomials*, University Lecture Series vol. **12**, Amer. Math. Soc., 1998.

[M03] I. G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics **157**, Cambridge Univ. Press, 2003.

[N95] M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings* (in Japanese), in *Various Aspects of Hypergeometric Functions* (Kyoto, 1994), RIMS Kôkyûroku **919**, Research Institute for Mathematical Sciences, Kyoto Univ., Kyoto, 1995, 44–55.

[NS04] M. Noumi, J. V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebraic approach*, in *Laredo Lectures on Orthogonal Polynomials and Special Functions* (R. Alvarez-Nodarse, F. Marcellan and W. Van Assche, Eds), pp. 111-144, Nova Science Publishers, 2004.

[NSh] M. Noumi, J. Shiraishi, *A direct approach to the bispectral problem for the Ruijsenaars-Macdonald q-difference operators*, preprint (2012), arXiv:1206.5364.

[OS18] D. Orr, M. Shimozono, *Specializations of nonsymmetric Macdonald-Koornwinder polynomials*, J. Algebraic Combin., **47** (2018), no. 1, 91–127.

[Ra06] A. Ram, *Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux*, Pure Appl. Math. Q., **2** (4) (2006), 963–1013.

[RY11] A. Ram, M. Yip, *A combinatorial formula for Macdonald polynomials*, Adv. Math., **226** (2011), 309–331.

[Ro21] H. Rosengren, *Proofs of some partition identities conjectured by Kanade and Russell*, Ramanujan J., (2021), published online; arXiv:1912.03689.

[R87] S. N. M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys., **110** (1987), 191–213.

[Sa99] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. Math., **150** (1999), 267–282.

[Sa00] S. Sahi, *Some properties of Koornwinder polynomials*, Contemp. Math., **254** (2000), 395–411.

[San00] Y. B. Sanderson, *On the Connection Between Macdonald Polynomials and Demazure Characters*, J. Alg. Comb., **11** (2000), 269–275.

[St00] J. V. Stokman, *Koornwinder Polynomials and Affine Hecke Algebras*, Int. Math. Res. Not., **19** (2000), 1005–1042.

[St02] J. V. Stokman, *An expansion formula for the Askey-Wilson function*, J. Approx. Theory, **114** (2002), 308–342.

[St03] J. V. Stokman, *Difference Fourier transforms for nonreduced root systems*, Selecta Math., **9** (2003), 409–494.

[St04] J. V. Stokman, *Lecture Notes on Koornwinder polynomials*, in *Laredo Lectures on Orthogonal Polynomials and Special Functions*, 145–207, Adv. Theory Spec. Funct. Orthogonal Polynomials, Nova Sci. Publ. Hauppauge, NY, 2004.

[St14] J. V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, Ann. Math., **179** (2014), 253–299.

[St20] J. V. Stokman, *Macdonald-Koornwinder polynomials*, in *Encyclopedia of Special Functions: The Askey-Bateman Project: Volume 2, Multivariable Special Functions*, Cambridge University Press, 2020; provisional version available from arXiv:1111.6112.

[T10] Y. Takeyama, *Differential equations compatible with boundary rational qKZ equation*, in *New trends in quantum integrable systems*, pp. 421–450 (2010); arXiv:0908.2288.

[vD96] J. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math., **126** (1996), no. 2, 319–339.

[vMS09] M. van Meer, J. V. Stokman, *Double Affine Hecke Algebras and Bispectral Quantum Knizhnik-Zamolodchikov Equations*, Int. Math. Res. Not., **6** (2010), 969–1040.

[vM11] M. van Meer, *Bispectral quantum Knizhnik-Zamolodchikov equations for arbitrary root systems*, Sel. Math. New Ser., **17** (2011), 183–221.

[Ya22] K. Yamaguchi, *A Littlewood–Richardson rule for Koornwinder polynomials*, J. Algebraic Combin., **56** (2022), 335–381, https://doi.org/10.1007/s10801-022-01114-5; arXiv:2009.13963.

[YY22] K. Yamaguchi, S. Yanagida, *Specializing Koornwinder polynomials to Macdonald polynomials of type B, C, D and BC*, J. Algebraic Combin., (2022), online published: https://doi.org/10.1007/s10801-022-01165-8

[YY] K. Yamaguchi, S. Yanagida, *A review of rank one bispectral correspondence of quantum affine KZ equations and Macdonald-type eigenvalue problems*, proceeding draft of "Recent developments in Combinatorial Representation Theory" at RIMS, Kyoto University, November 2022, arXiv:2211.13671.

[Yi12] M. Yip, *A Littlewood–Richardson rule for Macdonald polynomials*, Math. Z., **272** (2012), 1259–1290.

[青 13] 青本和彦, **直交多項式入門**, 数学書房, 2013.

[三 04] 三町勝久, **ダイソンからマクドナルドまで–マクドナルド多項式入門**–, 代数学百科 I 群論の進化 第 4 章, 335–437, 朝倉書店, 2004.