Double affine Hecke algebras and Macdonald-Koornwinder polynomials

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2022/11/8 @RIMS 共同研究「組合せ論的表現論における最近の展開」

Abstract and contents

This talk is an introduction of Koornwinder polynomials, a family of multi-variable q-orthogonal polynomial which is regarded as the master class of Macdonald polynomials.

Based on the collaborations with Shintarou Yanagida:

[YY1] *Specializing Koornwinder polynomials to Macdonald polynomials of type B, C, D and BC*, J. Alg. Comb. (2022) published online; arXiv:2105.00936.

[YY2] *Bispectral difference Cherednik-Matsuo correspondences of rank* 1 in preparation.

1. Askey-Wilson and Koornwinder polynomials
2. Affine root system of type $(C_n^\vee, C_n)$
3. Various properties of Koornwinder polynomials
4. Specialization of parameters [YY1]
5. Bispectral difference CM correspondence of rank 1 and specialization [YY2]

Askey-Wilson polynomials [Askey-Wilson, 1985]

- $q \in \mathbb{C}$, $|q| < 1$: the q-shift parameter.
- For $k \in \mathbb{N} := \mathbb{Z}_{\geq 0}$ and $a, a_1, \ldots, a_m \in \mathbb{C}$,
  $(a;q)_k := \prod_{i=1}^{k}(1 - aq^{i-1})$, $\quad (a_1, \ldots, a_m; q)_k := (a_1;q)_k \cdots (a_m;q)_k$.
- The q-hypergeometric series ${}_{s+1}\phi_s\left[\begin{matrix} a_1, & \cdots, & a_{s+1} \\ b_1, & \cdots, & b_s \end{matrix}; q, \ z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_{s+1};q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k}$.

Askey-Wilson polynomial of degree $n \in \mathbb{N}$ and parameters $a, b, c, d \in \mathbb{C}$ is the one-variable q-hypergeometric polynomial given by

$$p_n(y; q, a, b, c, d) := \frac{a^{-n}(ab, ac, ad; q)_n}{(abcd; q)_n} \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, & q^{n-1}abcd, & ax, & a/x \\ & ab, & ac, & ad \end{matrix}; q, \ q\right]$$

with $y = (x + x^{-1})/2$. For $n = 0, 1, 2$, we have

$$p_0 = 1, \quad p_1 = 2y - \frac{s - s'\pi}{1 - \pi}, \quad p_2 = 4y^2 - \frac{2(1+q)(s - s'\pi q)}{1 - \pi q^2} y + \text{const.}$$

$$(\pi = abcd, \ s = a + b + c + d, \ s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}).$$

Recurrence relation: Askey-Wilson satisfies the 3-term recursive formula

$$2x\tilde{p}_n(y) = A_n\tilde{p}_{n+1}(y) + \left(a + a^{-1} - (A_n + C_n)\right)\tilde{p}_n(y) + C_n\tilde{p}_{n-1}(y),$$

$$\tilde{p}_n(y) := \frac{a^n(abcd;q)_n}{(ab,ac,ad;q)_n}p_n(y;q,a,b,c,d),$$

$$A_n := \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n := \frac{a(1-q^{n-1})(1-bcq^{n-1})(1-bdq^n)(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

Orthogonality: For generic parameters $a, b, c, d \in \mathbb{C}$,

$$\int_{-1}^{1} p_m(y)p_n(y)\frac{w(y)}{2\pi\sqrt{1-y^2}}dy = 0, \quad m \neq n,$$

where the weight function $w(y)$ is given by

$$w(y) := \frac{\prod_{k=0}^{\infty}(1-(2y^2-1)q^k+q^{2k})}{h(y,a)h(y,b)h(y,c)h(y,d)}, \quad h(y,\alpha) := \prod_{k=0}^{\infty}(1-2\alpha yq^k+\alpha^2q^{2k}).$$

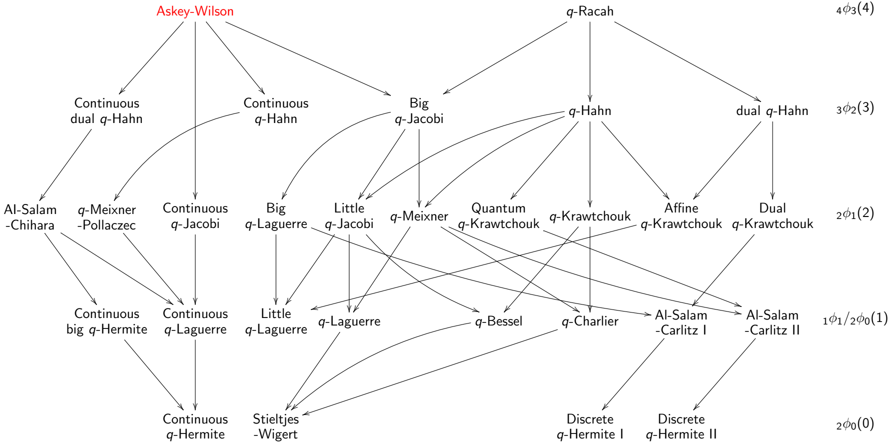


Figure: Askey scheme of q-hypergeometric orthogonal polynomials

Koornwinder polynomials

- $x = (x_1, \ldots, x_n)$, $A := \mathbb{C}[x^{\pm 1}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$: Laurent polynomials.
- $A \curvearrowleft W_{\mathrm{fin}} := \{\pm 1\}^n \rtimes \mathfrak{S}_n$: the finite Weyl group of type C_n.
- $A^{W_{\mathrm{fin}}} := \{f \in A \mid \forall w \in W_{\mathrm{fin}},\ w.f = f\}$: the W_{fin}-invariant Laurent polynomials.
- $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$: the weight lattice of type C_n.
- $\Lambda_+ := \left\{\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda \mid \text{dominant}\right\} = \{\text{partitions } \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$.

Koornwinder polynomial was introduced by [Koornwinder 1992] as a multi-variable version of the Askey-Wilson polynomial.

$$P_\lambda(x) = P_\lambda(x; q, t, a, b, c, d) \in A^{W_{\mathrm{fin}}}, \quad \lambda \in \Lambda_+.$$

If $n = 1$, it coincides with the Askey-Wilson polynomial.
Koornwinder polynomial can be regarded as Macdonald polynomial associated to the affine root system of type $(C_n^\vee, C_n)$.
[Noumi 1995], [Sahi 1999], [Stokman 2000]

Affine root system of type $(C_n^\vee, C_n)$

Root system of type C_n.

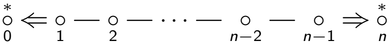
- $V = \bigoplus_{i=1}^n \mathbb{R}\epsilon_i$: n-dimensional real Euclidean space with orthogonal basis ϵ_i.
- $R = \{\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq n\} \subset V$: root system with simple roots α_i $(1 \leq i \leq n)$.
- $W_{\mathrm{fin}} = \{\pm 1\}^n \rtimes \mathfrak{S}_n \subset \mathrm{GL}(V)$: the finite Weyl group.

Affine root system of type $(C_n^\vee, C_n)$.

- $\widetilde{\Lambda} := \Lambda \oplus \mathbb{Z}\delta = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\delta$: extension by the null root δ.
- The non-reduced affine root system S of type $(C_n^\vee, C_n)$ is given by

  $$S := \{\pm c(\epsilon_i + \tfrac{k}{2}\delta), \pm\epsilon_i \pm \epsilon_j + k\delta \mid c = 1, 2,\ 1 \leq i < j \leq n,\ k \in \mathbb{Z}\} \subset \widetilde{V} := V \oplus \mathbb{R}\delta$$

  with simple roots α_i $(0 \leq i \leq n)$ and $\alpha_0^\vee, \alpha_n^\vee$.
  The Dynkin diagram is



The $*$ marks on the vertices 0 and n correspond to the extra simple roots $\alpha_0^\vee$ and $\alpha_n^\vee$.

- $W := \mathbf{t}(\Lambda) \rtimes W_{\mathrm{fin}} = \langle s_0, s_1, \ldots, s_n \rangle \subset \mathrm{GL}(\widetilde{V})$: the extended affine Weyl group.
- For $n \geq 2$, the system S has five W-orbits

  $$W.\alpha_i\ (i = 1, \ldots, n-1),\ W.\alpha_n,\ W.\alpha_n^\vee,\ W.\alpha_0,\ W.\alpha_0^\vee,$$

  corresponding to the Koornwinder parameters (t, a, b, c, d).
  If $n = 1$, S has 4 orbits, corresponding to the Askey-Wilson parameters (a, b, c, d).

Affine Hecke algebra of type $(C_n^\vee, C_n)$

Definition (the affine Hecke algebra H of type $(C_n^\vee, C_n)$)

- generators: $H := \mathbb{C}\langle T_0, T_1, \dots, T_n\rangle_{\mathbb{C}\text{-alg.}}$.
- relations:
  - $(T_i - t_i^{\frac{1}{2}})(T_i + t_i^{-\frac{1}{2}}) = 0 \quad (i = 0, 1, \dots, n)$
  - $T_i T_j = T_j T_i \quad (|i - j| > 1, (i, j) \notin \{(n, 0), (0, n)\})$
  - $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (i = 1, \dots, n-2)$
  - $T_i T_{i+1} T_i T_{i+1} = T_{i+1} T_i T_{i+1} T_i \quad (i = 0, n-1)$

H has a faithful representation called noumi representation. $\rho : H \longrightarrow \mathrm{End}(\mathbb{C}(x))$ s.t.

$$\rho(T_i) = t_i^{\frac{1}{2}} + t_i^{-\frac{1}{2}} \frac{1 - t_i x_i/x_{i+1}}{1 - x_i/x_{i+1}}(s_i - 1) \quad (i = 1, \dots, n-1),$$

$$\rho(T_0) = t_0^{\frac{1}{2}} + t_0^{-\frac{1}{2}} \frac{(1 - u_0^{\frac{1}{2}} t_0^{\frac{1}{2}} q^{\frac{1}{2}} x_1^{-1})(1 + u_0^{-\frac{1}{2}} t_0^{\frac{1}{2}} q^{\frac{1}{2}} x_1^{-1})}{1 - q x_1^{-2}}(s_0 - 1),$$

$$\rho(T_n) = t_n^{\frac{1}{2}} + t_n^{-\frac{1}{2}} \frac{(1 - u_n^{\frac{1}{2}} t_n^{\frac{1}{2}} x_n)(1 + u_n^{-\frac{1}{2}} t_n^{\frac{1}{2}} x_n)}{1 - x_n^2}(s_n - 1)$$

Affine Hecke algebras and Koornwinder polynomials

Set $Y_i := \rho(T_{i-1}^{-1} \cdots T_1^{-1} T_0 \cdots T_{n-1} T_n T_{n-1} \cdots T_i) \quad (i = 1, \ldots, n)$.

- (center of H) $= \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] \subset H$.

Fact (Koornwinder polynomials)

For $\lambda \in \Lambda_+$, there is a unique W_{fin}-invariant Laurent polynomial $P_\lambda(x)$ s.t.

- $\{P_\lambda(x)\}_{\lambda \in \Lambda_+}$ is a basis of $A^{W_{\text{fin}}}$.
- triangular expansion: $P_\lambda(x) = m_\lambda(x) + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu(x), \quad a_{\lambda\mu} \in \mathbb{C}$.
- q-diff. eqn: $f(Y)P_\lambda(x) = f(\gamma_\lambda)P_\lambda(x), \ \gamma_\lambda \in \mathbb{C}, \ f(Y) \in \mathbb{C}[Y^{\pm 1}]^{W_{\text{fin}}}$

q-difference equation

For each partition $\lambda \in \Lambda_+$, the Koornwinder polynomial $P_\lambda(x)$ satisfies

$$DP_\lambda(x) = c_\lambda P_\lambda(x)$$

where D is the Macdonald-Koornwinder q-difference operator

$$D := e_1(Y) \approx \sum_{k=1}^{n} \left(\gamma_k(x)(T_{q,x_k} - 1) + \gamma_k(x^{-1})(T_{q,x_k}^{-1} - 1)\right)$$

and $c_\lambda := \sum_{k=1}^{n}(abcdq^{-1}t^{2n-k-1}(q^{\lambda_k} - 1) + t^{k-1}(q^{-\lambda_k} - 1))$.

- $\gamma_k(x) := \frac{(1-ax_k)(1-bx_k)(1-cx_k)(1-dx_k)}{(1-x_k^2)(1-qx_k^2)} \prod_{j\neq k} \frac{(tx_k-x_j)(1-tx_kx_j)}{(x_k-x_j)(1-x_kx_j)}$.
- $T_{q,x_i}^{\pm1} f(x) = f(x_1, \ldots, q^{\pm1}x_j, \ldots, x_n)$: q-shift operator.
- $D(A^{W_{\mathrm{fin}}}) \subset A^{W_{\mathrm{fin}}}$.

If $n = 1$, the above eqn. = Askey-Wilson 2nd-order q-difference equation.

Orthogonality

If $0 < q < 1$ and $|a|, |b|, |c|, |d|, |t| < 1$, then the set $\{P_\lambda(x) \mid \lambda\text{: partitions}\}$ is an orthogonal basis of $A^{W_{\mathrm{fin}}}$ with respect to

$$\langle f(x), g(x)\rangle := \frac{1}{|W_{\mathrm{fin}}|}\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_T \overline{f(x)}g(x)\,|w(x)|^2\,\frac{dx_1\cdots dx_n}{x_1\cdots x_n},$$

where T is an n-dimensional real torus, and the weight function $w(x)$ is

$$w(x) := \prod_{k=1}^{n} \frac{(x_k^2; q)_\infty}{(ax_k, bx_k, cx_k, dx_k; q)_\infty} \prod_{1\le i<j\le n} \frac{(x_i/x_j, x_ix_j; q)_\infty}{(tx_i/x_j, tx_ix_j; q)_\infty}.$$

If $n = 1$, the inner product of Askey-Wilson polynomials.

Specialization to type B, C, D and BC

Theorem (Y.-Yanagida 1, Theorem 1)

Specializing parameters $(t, t_0, t_n, u_0, u_n) = (t, -ab/q, -cd, -a/b, -c/d)$ by the table below, we can recover Macdonald polynomials associated to subsystems of the affine root system of type $(C_n^\vee, C_n)$.

reduced	t	t_0	t_n	u_0	u_n	non-reduced	t	t_0	t_n	u_0	u_n
B_n	t_l	1	t_s	1	t_s	(BC_n, C_n)	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	t_l	t_s	t_s	t_s	t_s						
BC_n	t_m	t_l^2	t_s	1	t_s						
D_n	t	1	1	1	1						

As a corollary, we can re-derive various results on Macdonald polynomials of type B, C, D from properties of Koornwinder polynomials.

E.g. Ram-Yip formula of non-symmetric Macdonald polynomials of type B, C, D.
[A. Ram, M. Yip (2011)]

The rank one case

Affine root system of type $(C_1^\vee, C_1)$.

- $V := \mathbb{R}\epsilon$: one-dimensional real Euclidean space with orthogonal basis ϵ.
- $\Lambda := \mathbb{Z}\epsilon \subset V$: weight lattice of the root system C_1.
- The non-reduced affine root system S of type $(C_1^\vee, C_1)$ is given by

$$S := O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4,$$

$$O_1 := \{\pm\epsilon + c\delta \mid c \in \mathbb{Z}\}, \ O_2 := 2O_1, \ O_3 := O_1 + \frac{1}{2}\delta, \ O_4 := 2O_3.$$

- $W := \mathbf{t}(\Lambda) \rtimes W_{\text{fin}} = \langle s_0, s_1 \rangle \subset \mathrm{GL}(\widetilde{V})$: the extended affine Weyl group.

Specialization of the rank one case

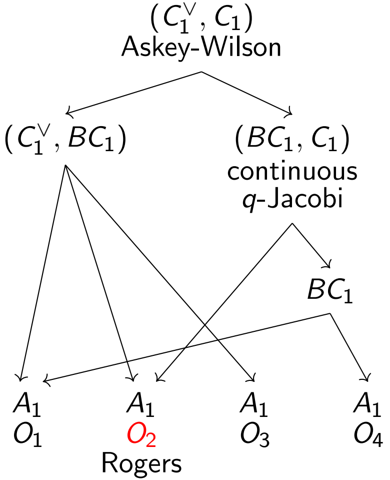


Figure: Root-theoretic degeneration scheme of Askey-Wilson polynomial

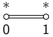
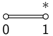
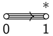
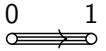
type	Dynkin	orbits	Noumi				Askey-Wilson			
			t_0	t_1	u_0	u_1	a	b	c	d
$(C_1^\vee, C_1)$ Askey-Wilson		$O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$	t_0	t_1	u_0	u_1	a	b	c	d
$(C_1^\vee, BC_1)$		$O_1 \sqcup O_2 \sqcup O_3$	t_s	$t_s t_l$	t_s	t_s/t_l	$q^{\frac{1}{2}} t_s$	$-q^{\frac{1}{2}}$	t_s	$-t_l$
(BC_1, C_1) cont. q-Jacobi		$O_1 \sqcup O_2 \sqcup O_4$	t_l^2	$t_s t_l$	1	t_s/t_l	$q^{\frac{1}{2}} t_l$	$-q^{\frac{1}{2}} t_l$	t_s	$-t_l$
BC_1		$O_1 \sqcup O_4$	t_l^2	t_s	1	t_s	$q^{\frac{1}{2}} t_l$	$-q^{\frac{1}{2}} t_l$	t_s	-1
A_1		O_1	1	t	1	t	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	t	-1
		O_3	t	1	t	1	$q^{\frac{1}{2}} t$	$-q^{\frac{1}{2}}$	1	-1
Rogers		O_2	1	t^2	1	1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	t	$-t$
		O_4	t^2	1	1	1	$q^{\frac{1}{2}} t$	$-q^{\frac{1}{2}} t$	1	-1

Table: Subsystems of $(C_1^\vee, C_1)$ and parameter specializations

Bispectral qKZ-equation

- $(k_1, k_0, l_1, l_0) := (t_1^{\frac{1}{2}}, t_0^{\frac{1}{2}}, u_1^{\frac{1}{2}}, u_0^{\frac{1}{2}})$
- $H_0 := \mathbb{C}T_e + \mathbb{C}T_{s_1} = \mathbb{C} + \mathbb{C}T_1 \subset H(W)$

Definition (The double affine Hecke algebra of type $(C_1^\vee, C_1)$)

The double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$, denoted as

$$\mathbb{H} = \mathbb{H}(k_1, k_0, l_1, l_0, q),$$

is defined to be the $\mathbb{C}$-subalgebra of $\mathrm{End}(\mathbb{C}[x^{\pm 1}])$ generated by $\mathbb{C}[x^{\pm 1}]$, H_0 and $\mathbb{C}[Y^{\pm 1}]$.

Fact (Cherednik involution)

$\mathbb{H}$ has a unique $\mathbb{C}$-algebra anti-involution determined by

$$T_1^* := T_1, \quad (Y^\lambda)^* := x^{-\lambda}, \quad (x^\lambda)^* := Y^{-\lambda}$$

for $\lambda \in \Lambda$ and

$$(k_1^*, k_0^*, l_1^*, l_0^*) := (k_1, l_1, k_0, l_0).$$

- $\mathbb{K}$: the space of meromorphic functions of variables x and $\xi = Y$
- $\mathbb{W} := (W \times W) \ltimes \langle \iota \rangle$, $\iota(w, w') = (w', w)$. $\exists$ action $\mathbb{W} \curvearrowright \mathbb{K}$.
- $H_0^{\mathbb{K}} := \mathbb{K} \otimes H_0 \ni f = f(x, \xi)$: regarded as function valued in H_0.

The bispectral qKZ equations of type $(C_1^{\vee}, C_1)$ for $f \in H_0^{\mathbb{K}}$:

$$\begin{cases} C_{1,0}(x,\xi) f(q^{-1}x, \xi) = f(x, \xi) \\ C_{0,1}(x,\xi) f(x, q\xi) = f(t, \xi) \end{cases}$$

where $C_{1,0}, C_{0,1}$ are the $\mathbb{W}$-cocycles valued in $\mathrm{GL}(H_0)$ given by

$$C_{1,0} = R_0^L(x_0) R_1^L(x_1'), \quad C_{0,1} = R_0^R(\xi_0') R_1^R(\xi_1'),$$

with $x_0 := qx^{-2}$, $x_1' := q^2 x^{-2}$, $\xi_0' := q\xi^2$, $\xi_1' := q^2\xi^2$ and

$$R_i^L(z) = k_i(1 - k_i l_i z^{1/2})^{-1}(1 + k_i l_i^{-1} z^{1/2})^{-1}\left((1-z)\eta_L(T_i) - (k_i - k_i^{-1}) - (l_i - l_i^{-1})z^{\frac{1}{2}}\right),$$
$$R_i^R(z) = k_i^*(1 - k_i^* l_i^* z^{1/2})^{-1}(1 + k_i^* (l_i^*)^{-1} z^{1/2})^{-1}\left((1-z)\eta_R(T_i^*) - (k_i^* - (k_i^*)^{-1}) - (l_i^* - l_i^{-1})z^{\frac{1}{2}}\right)$$

for $i = 0, 1$, using the Cherednik involution $*$.
Studied by [Takeyama 2010], [van Meer-Stokman '10], [van Meer '11]. [Stokman '14].

Bispectral Cherednik-Matsuo correspondence and specialization

Theorem (Y.-Yanagida2)

The specialization $O_2\colon (k_1, k_0, l_1, l_0) = (k, 1, 1, 1)$ yields the commutative diagram

$$\begin{array}{ccc}
\mathrm{SOL}_{\mathsf{bqKZ}}^{(C_1^\vee, C_1)}(k,1,1,1,q) & \xhookrightarrow{\chi_+^{(C_1^\vee, C_1)}} & \mathrm{SOL}_{\mathsf{bAW}}(k,1,1,1,q) \\
\mathsf{sp}\downarrow & & \downarrow\mathsf{sp} \\
\mathrm{SOL}_{\mathsf{bqKZ}}^{A_1}(k,q) & \xhookrightarrow{\chi_+^{(C_1^\vee, C_1)}} & \mathrm{SOL}_{\mathsf{bMR}}(k = t^{1/2}, q)
\end{array}$$

- $\mathrm{SOL}_{\mathsf{bqKZ}}^{X} := \{f \in H_0^{\mathbb{K}} \mid$ solutions of bispectral qKZ eqn. of type $X\}$.
- $\mathrm{SOL}_{\mathsf{bMR}} := \{$sols. of bispectral Macdonald-Ruijsenaars q-diff. eqn. of type $A_1\}$
- $\mathrm{SOL}_{\mathsf{bAW}} := \{$solution of bispectral Askey-Wilson q-difference equation$\}$
- $\chi_+\colon T_1^r \mapsto k^r$: bispectral version of difference Cherednik-Matsuo correspondence [van Meer-Stokman 2010], [van Meer 2011], [Stokman 2014]
- Among the 4 specializations, only O_2 is consistent with the Cherednik involution.

References

1. R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials*, Mem. Amer. Math. Soc., **54** (1985), no. 319.
2. T. H. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, Contemp. Math. **138** (1992), 189–204.
3. I. G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Math., **157**, Cambridge, 2003.
4. M. van Meer, *Bispectral quantum Knizhnik-Zamolodchikov equations for arbitrary root systems*, Sel. Math. New Ser., **17** (2011), 183–221.
5. M. van Meer, J. V. Stokman, *Double Affine Hecke Algebras and Bispectral Quantum Knizhnik-Zamolodchikov Equations*, Int. Math. Res. Not., **6** (2010), 969–1040.
6. M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings* (in Japanese), in *Various Aspects of Hypergeometric Functions* (Kyoto, 1994), RIMS Kôkyûroku, **919**, RIMS, Kyoto Univ., Kyoto, 1995, 44–55.
7. M. Noumi, J. Shiraishi, *A direct approach to the bispectral problem for the Ruijsenaars-Macdonald q-difference operators*, preprint (2012), arXiv:1206.5364.
8. A. Ram, M. Yip, *A combinatorial formula for Macdonald polynomials*, Adv. Math., **226** (2011), 309–331.
9. S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. Math., **150** (1999), 267–282.
10. J.V. Stokman, *Koornwinder Polynomials and Affine Hecke Algebras*, Int. Math. Res. Not., **19** (2000), 1005–1042.
11. J.V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, Ann. Math., **179** (2014), 253–299.
12. Y. Takeyama, *Differential equations compatible with boundary rational qKZ equation*, in *New trends in quantum integrable systems*, pp. 421–450 (2010); arXiv:0908.2288.
13. K.Y., S. Yanagida, *Specializing Koornwinder polynomials to Macdonald polynomials of type B,C,D and BC*, J. Alg. Comb. (2022) published online; arXiv:2105.00936.
14. K.Y., S. Yanagida, *Bispectral difference Cherednik-Matsuo correspondences of rank* 1, in preparation.

Thank you for your attention.