Macdonald-Koornwinder polynomials as multivariate q-orthogonal functions

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Abstract and contents

This talk is an introduction of Koornwinder polynomials,
a family of multi-variable q-orthogonal polynomial which is regarded as the master class of Macdonald polynomials.

Based on the papers

[Y] K.Y.,
A Littlewood-Richardson rule for Koornwinder polynomials,
arXiv:2009.13963; to appear in J. Alg. Comb.

[YY] K.Y., S. Yanagida,
Specializing Koornwinder polynomials to Macdonald polynomials of type B, C, D and BC, arXiv: 2105.00936.

1. Askey-Wilson and Koornwinder polynomials
2. Affine root system of type $(C_n^\vee, C_n)$
3. Various properties of Koornwinder polynomials
4. Littlewood-Richardson coefficients [Y]
5. Specialization of parameters [YY]

Askey-Wilson polynomials

- $q \in \mathbb{C}$, $|q| < 1$: the q-shift parameter.
- For $k \in \mathbb{N} := \mathbb{Z}_{\geq 0}$ and $a, a_1, \ldots, a_m \in \mathbb{C}$,
  $(a; q)_k := \prod_{i=1}^{k}(1 - aq^{i-1})$, $\quad (a_1, \ldots, a_m; q)_k := (a_1; q)_k \cdots (a_m; q)_k$.
- The q-hypergeometric series ${}_{s+1}\phi_s\left[\begin{matrix} a_1, & \cdots, & a_{s+1} \\ b_1, & \cdots, & b_s \end{matrix}; q, \ z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_{s+1};q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k}$.

Askey-Wilson polynomial of degree $l \in \mathbb{N}$ and parameters $a, b, c, d \in \mathbb{C}$ is the one-variable q-hypergeometric polynomial given by

$$p_l(y; q, a, b, c, d) := \frac{a^{-l}(ab, ac, ad; q)_l}{(abcd; q)_l} \cdot {}_4\phi_3\left[\begin{matrix} q^{-l}, & q^{l-1}abcd, & ax, & a/x \\ & ab, & ac, & ad \end{matrix}; q, \ q\right]$$

with $y = (x + x^{-1})/2$. For $n = 0, 1, 2$, we have

$$p_0 = 1, \quad p_1 = 2y - \frac{s - s'\pi}{1 - \pi}, \quad p_2 = 4y^2 - \frac{2(1+q)(s - s'\pi q)}{1 - \pi q^2} y + \text{const.}$$

$$(\pi = abcd, \ s = a + b + c + d, \ s' = a^{-1} + b^{-1} + c^{-1} + d^{-1}).$$

Recurrence relation: Askey-Wilson satisfies the 3-term recursive formula

$$2x\tilde{p}_l(y) = A_l\tilde{p}_{l+1}(y) + \left(a + a^{-1} - (A_l + C_l)\right)\tilde{p}_l(y) + C_l\tilde{p}_{l-1}(y),$$

where

$$\tilde{p}_l(y) := \frac{a^n(abcd;q)_l}{(ab,ac,ad;q)_l}p_l(y;q,a,b,c,d)$$

and

$$A_l := \frac{(1-abq^l)(1-acq^l)(1-adq^l)(1-abcdq^{l-1})}{a(1-abcdq^{2l-1})(1-abcdq^{2l})},$$

$$C_l := \frac{a(1-q^{l-1})(1-bcq^{l-1})(1-bdq^l)(1-cdq^{l-1})}{(1-abcdq^{2l-2})(1-abcdq^{2l-1})}.$$

Orthogonality: For generic parameters $a, b, c, d \in \mathbb{C}$,

$$\int_{-1}^{1} p_l(y)p_m(y)\frac{w(y)}{2\pi\sqrt{1-y^2}}dy = 0, \quad l \neq m,$$

where the weight function $w(y)$ is given by

$$w(y) := \frac{\prod_{k=0}^{\infty}(1-(2y^2-1)q^k+q^{2k})}{h(y,a)h(y,b)h(y,c)h(y,d)}, \quad h(y,\alpha) := \prod_{k=0}^{\infty}(1-2\alpha yq^k+\alpha^2q^{2k}).$$

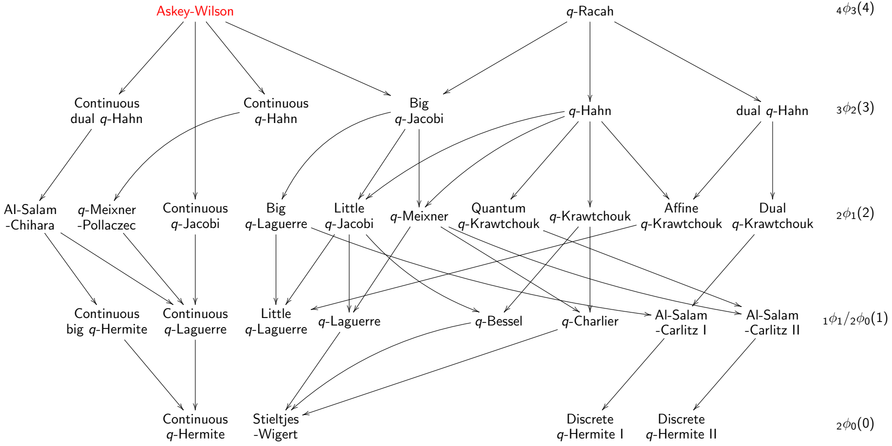


Figure: Askey scheme of *q*-hypergeometric orthogonal polynomials

Koornwinder polynomials

- $x = (x_1, \dots, x_n)$, $A := \mathbb{C}[x^{\pm 1}] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$: Laurent polynomials.
- $A \curvearrowleft W_{\text{fin}} := \{\pm 1\}^n \rtimes \mathfrak{S}_n$: the finite Weyl group of type C_n.
- $A^{W_{\text{fin}}} := \{f \in A \mid \forall w \in W_{\text{fin}},\ w.f = f\}$: the W_{fin}-invariant Laurent polynomials.
- $\mathfrak{h}_{\mathbb{Z}}^* = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$: the weight lattice of type C_n.
- $(\mathfrak{h}_{\mathbb{Z}}^*)_+ := \left\{\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\right\} = \{\text{partitions}\}$.

Koornwinder polynomial was introduced by Koornwinder (1992) as a multi-variable version of the Askey-Wilson polynomial.

$$P_\lambda(x) = P_\lambda(x; q, t, a, b, c, d) \in A^{W_{\text{fin}}}, \quad \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_+.$$

If $n = 1$, it coincides with the Askey-Wilson polynomial.

$$(\mathfrak{h}_{\mathbb{Z}}^*)_+ = \mathbb{N}, \quad \lambda = l, \quad P_\lambda(x) = p_l(x).$$

Koornwinder polynomial can be regarded as Macdonald polynomial associated to the affine root system of type $(C_n^\vee, C_n)$.
[Noumi (1995), Sahi (1999), Stokman (2000)]

Affine root system of type $(C_n^\vee, C_n)$

Root system of type C_n.

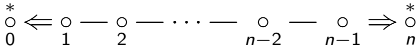
- $R = \{\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq n\}$: root system with simple roots α_i ($1 \leq i \leq n$).
- $W_{\mathrm{fin}} = \{\pm 1\}^n \rtimes \mathfrak{S}_n \subset \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$: the finite Weyl group.

Affine root system of type $(C_n^\vee, C_n)$.

- $\widetilde{\mathfrak{h}_{\mathbb{Z}}^*} := \mathfrak{h}_{\mathbb{Z}}^* \oplus \mathbb{Z}\delta = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\delta$: extension by the null root δ.
- The non-reduced affine root system S of type $(C_n^\vee, C_n)$ is given by

$$S := \{\pm c(\epsilon_i + \tfrac{k}{2}\delta), \pm\epsilon_i \pm \epsilon_j + k\delta \mid c = 1, 2,\ 1 \leq i < j \leq n,\ k \in \mathbb{Z}\} \subset \widetilde{\mathfrak{h}_{\mathbb{R}}^*}$$

with simple roots α_i ($0 \leq i \leq n$) and $\alpha_0^\vee, \alpha_n^\vee$. The Dynkin diagram is



The $*$ marks on the vertices 0 and n correspond to the extra simple roots $\alpha_0^\vee$ and $\alpha_n^\vee$.

- $W := t(\mathfrak{h}_{\mathbb{Z}}^*) \rtimes W_{\mathrm{fin}} \subset \mathrm{GL}(\widetilde{\mathfrak{h}_{\mathbb{R}}^*})$: the extended affine Weyl group.
- For $n \geq 2$, the system S has five W-orbits

$$W.\alpha_i\ (i = 1, \ldots, n-1),\ W.\alpha_n,\ W.\alpha_n^\vee,\ W.\alpha_0,\ W.\alpha_0^\vee,$$

corresponding to the Koornwinder parameters (t, a, b, c, d).
If $n = 1$, S has 4 orbits, corresponding to the Askey-Wilson parameters (a, b, c, d).

Basis, triangular property, and characterizations

Basic properties of Koornwinder polynomial $P_\lambda(x)$.

- $\{P_\lambda \mid \lambda\text{: partitions}\}$ is a basis of W_{fin}-inv. Laurent polynomials $A^{W_{\text{fin}}}$.
- P_λ has a triangular expansion

$$P_\lambda(x) = m_\lambda(x) + \sum_{\mu<\lambda} a_{\lambda\mu} m_\mu(x), \quad a_{\lambda\mu} \in \mathbb{C}$$

with $m_\lambda := \sum_{\mu \in W_{\text{fin}}.\lambda} x^\mu \in A^{W_{\text{fin}}}$ the orbit sum and $\mu \le \lambda$ the dominance ordering ($:\Leftrightarrow \sum_{i=1}^k \mu_k \le \sum_{i=1}^k \lambda_k$ for all $k = 1, \ldots, n$).

Characterizations of Koornwinder polynomial P_λ

- P_λ is characterized by $\begin{cases} P_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{C} m_\mu & \text{triangular property} \\ D P_\lambda(x) = c_\lambda P_\lambda(x) & q\text{-difference equation (page 9)} \end{cases}$.
- Another characterization: $\begin{cases} P_\lambda \in m_\lambda + \sum_{\mu<\lambda} \mathbb{C} m_\mu & \text{triangular property} \\ \langle P_\lambda, P_\mu \rangle \propto \delta_{\lambda,\mu} & \text{orthogonality (page 10)} \end{cases}$.

q-difference equation

For each partition $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_+$, the Koornwinder polynomial $P_\lambda(x)$ satisfies

$$DP_\lambda(x) = c_\lambda P_\lambda(x)$$

where D is the Macdonald q-difference operator of type $(C_n^\vee, C_n)$

$$D := \sum_{k=1}^{n} \left(\gamma_k(x)(T_{q,x_k} - 1) + \gamma_k(x^{-1})(T_{q,x_k}^{-1} - 1)\right)$$

and $c_\lambda := \sum_{k=1}^n (abcdq^{-1}t^{2n-k-1}(q^{\lambda_k} - 1) + t^{k-1}(q^{-\lambda_k} - 1))$.

- $\gamma_k(x) := \frac{(1-ax_k)(1-bx_k)(1-cx_k)(1-dx_k)}{(1-x_k^2)(1-qx_k^2)} \prod_{j\neq k} \frac{(tx_k-x_j)(1-tx_kx_j)}{(x_k-x_j)(1-x_kx_j)}$.
- $T_{q,x_i}^{\pm 1} f(x) = f(x_1, \ldots, qx_j, \ldots, x_n)$: q-shift operator.
- $D(A^{W_{\mathrm{fin}}}) \subset A^{W_{\mathrm{fin}}}$.

If $n = 1$, the above eqn. $=$ the 3-term recurrence relation of Askey-Wilson polynomial.

Orthogonality

If $0 < q < 1$ and $|a|, |b|, |c|, |d|, |t| < 1$, then the set $\{P_\lambda(x) \mid \lambda\text{: partitions}\}$ is an orthogonal basis of $A^{W_{\mathrm{fin}}}$ with respect to

$$\langle f(x), g(x) \rangle := \frac{1}{|W_{\mathrm{fin}}|} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_T \overline{f(x)} g(x) \, |w(x)|^2 \, \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n},$$

where T is an n-dimensional real torus, and the weight function $w(x)$ is

$$w(x) := \prod_{k=1}^{n} \frac{(x_k^2; q)_\infty}{(ax_k, bx_k, cx_k, dx_k; q)_\infty} \prod_{1 \le i < j \le n} \frac{(x_i/x_j, x_i x_j; q)_\infty}{(tx_i/x_j, tx_i x_j; q)_\infty}.$$

If $n = 1$, it coincides with the inner product of Askey-Wilson polynomials.

Koornwinder LR-coefficients

Hereafter we explain our new results on Koornwinder polynomials.

- $x = (x_1, \dots, x_n)$, $\quad A \coloneqq \mathbb{C}[x^{\pm 1}] = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \curvearrowleft W_{\mathrm{fin}} \coloneqq \{\pm 1\}^n \rtimes \mathfrak{S}_n$.
- $A^{W_{\mathrm{fin}}} \coloneqq \{f \in A \mid \forall w \in W_{\mathrm{fin}},\ w.f = f\}$.
- $\mathfrak{h}_{\mathbb{Z}}^* = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \supset (\mathfrak{h}_{\mathbb{Z}}^*)_+ \coloneqq \left\{\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\right\} = \{\text{partitions}\}$.
- $P_\lambda(x) \coloneqq P_\lambda(x; q, t, a, b, c, d) \in A^{W_{\mathrm{fin}}}$: Koornwinder polynomial for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$.

The product of two Koornwinder polynomials $P_\lambda(x)$ and $P_\mu(x)$ belongs to $A^{W_{\mathrm{fin}}}$, which can be expanded by the basis $\{P_\mu(x) \mid \nu\colon \text{partitions}\}$.

$$P_\lambda(x) P_\mu(x) = \sum_{\nu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+} c_{\lambda\mu}^{\nu} P_\nu(x), \quad c_{\lambda\mu}^{\nu} \in \mathbb{Q}(q, t, a, b, c, d).$$

The coefficients $c_{\lambda\mu}^{\nu}$ are called the Littlewood-Richardson (LR) coefficients for Koornwinder polynomials.

Koornwinder LR-coefficients (Cont.)

Theorem (Y., Theorem 3.4.2)

For partitions $\lambda, \mu \in (\mathfrak{h}^*_{\mathbb{Z}})_+$, we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x).$$

Main properties

- p runs over a set $\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})$ of colored alcove walks.
- The terms A_p, B_p and C_p are factored (shown in the next pages).

Minor notations

- w_λ: the longest element of the stabilizer $W_\lambda \subset W_{\mathrm{fin}}$ of λ.
- $W_\lambda(t)$: the Poincare polynomial of W_λ with variables t, t_n.
- $W^\mu \subset W_{\mathrm{fin}}$: a complete system of representatives of W_{fin}/W_μ.
- $\mathrm{wt}(p) \in (\mathfrak{h}_{\mathbb{Z}})^*$: a certain element determined by the colored alcove walk p.

Koornwinder LR-coefficients (Cont.)

The terms A_p and B_p are given by

$$A_p := \prod_{\alpha \in \mathcal{L}((v.w(\mu))^{-1}, t(-w_0.\mu))} \rho(\alpha), \quad B_p := \prod_{\alpha \in \mathcal{L}(t(\mathrm{wt}(p))w_0, e(p))} \rho(-\alpha).$$

The factor $\rho(\alpha)$ is given by

$$\rho(\alpha) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}}{1 - q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}} & (\alpha \notin W.\alpha_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-\alpha)} t^{\frac{1}{2}\mathrm{ht}(-\alpha)})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-\alpha)} t^{\frac{1}{2}\mathrm{ht}(-\alpha)})}{1 - q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}} & (\alpha \in W.\alpha_n) \end{cases}$$

$$q^{\mathrm{sh}(\alpha)} := q^{-k}, \ t^{\mathrm{ht}(\alpha)} := \prod_{\gamma \in R_+^s} t^{\frac{1}{2}\langle \gamma^\vee, \beta\rangle} \prod_{\gamma \in R_+^\ell} (t_0 t_n)^{\frac{1}{2}\langle \gamma^\vee, \beta\rangle} \quad (\alpha = \beta + k\delta \in S),$$

with $R_+^s := \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$ and $R_+^\ell := \{2\epsilon_i \mid 1 \le i \le n\}$.

- The factor ρ looks similar to Pieri coefficients for Macdonald polynomials of type A_n.

Koornwinder LR-coefficients (cont.)

The term C_p is given by $C_p := \prod_{k=1}^{r} C_{p,k}$ with

$$C_{p,k} := \begin{cases} 1 & \text{the } k\text{-th step of } p \text{ is a positive crossing} \\ \prod_{k \in \xi_{\mathrm{des}}(p)} n_{i_k}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a negative crossing} \\ -\psi_{i_k}^{+}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a gray positive folding} \\ -\psi_{i_k}^{-}(q^{\mathrm{sh}(-h_k(p))} t^{\mathrm{ht}(-h_k(p))}) & \text{a gray negative folding} \\ \psi_{i_k}^{+}(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is positive} \\ \psi_{i_k}^{-}(q^{\mathrm{sh}(-\beta_k)} t^{\mathrm{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is negative} \end{cases},$$

where the factor $\psi^{\pm}(z)$, $n_i(z)$ are given by the following:

$$\psi_i^{\pm}(z) := \mp \frac{t^{1/2} - t^{-1/2}}{1 - z^{\pm 1}} \qquad (i = 1, \dots, n-1),$$

$$\psi_0^{\pm}(z) := \mp \frac{(u_n^{1}/2 - u_n^{-1/2}) + z^{\pm 1/2}(u_0^{1}/2 - u_0^{-1/2})}{1 - z^{\pm 1}}, \quad \psi_n^{\pm}(z) := \mp \frac{(t_n^{1}/2 - t_n^{-1/2}) + z^{\pm 1/2}(t_0^{1}/2 - t_0^{-1/2})}{1 - z^{\pm 1}},$$

$$n_i(z) := \frac{1 - tz}{1 - z}\,\frac{1 - t^{-1}z}{1 - z} \qquad (\beta \in W.\alpha_i,\ i = 1, \dots, n-1),$$

$$n_0(z) := \frac{(1 - u_n^{1}/2u_0^{1}/2z^{1}/2)(1 + u_n^{1}/2u_0^{-1/2}z^{1}/2)}{1 - z}\,\frac{(1 + u_n^{-1/2}u_0^{1}/2z^{1}/2)(1 - u_n^{-1/2}u_0^{-1/2}z^{1}/2)}{1 - z} \qquad (\beta \in W.\alpha_0),$$

$$n_n(z) := \frac{(1 - t_n^{1}/2t_0^{1}/2z^{1}/2)(1 + t_n^{1}/2t_0^{-1/2}z^{1}/2)}{1 - z}\,\frac{(1 + t_n^{-1/2}t_0^{1}/2z^{1}/2)(1 - t_n^{-1/2}t_0^{-1/2}z^{1}/2)}{1 - z} \qquad (\beta \in W.\alpha_n).$$

Specialization to type B, C, and BC

Theorem (Y.-Yanagida, Theorem 1)

Specializing parameters $(t, t_0, t_n, u_0, u_n) = (t, -ab/q, -cd, -a/b, -c/d)$ by the table below, we can recover Macdonald polynomials associated to subsystems of the affine root system of type $(C_n^\vee, C_n)$.

reduced	t	t_0	t_n	u_0	u_n	non-reduced	t	t_0	t_n	u_0	u_n
B_n	t_l	1	t_s	1	t_s	(BC_n, C_n)	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	t_l	t_s	t_s	t_s	t_s						
BC_n	t_m	t_l^2	t_s	1	t_s						
D_n	t	1	1	1	1						

As a corollary, we can re-derive various results on Macdonald polynomials of type B, C, D from properties of the "master" Koornwinder polynomials.
E.g. Ram-Yip formula of non-symmetric Macdonald polynomials of type B, C, D.
[A. Ram, M. Yip (2011)]

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Thank you for your attention.