[ArXiv:2001.03490]

D. Orr, "Equivariant K-theory of the semi-finite flag manifold as a Nil-DAHA module"

の紹介.

山口 航平 (名大多元)

K理論的シューベルト・カルキュラスにおける最近の発展 @ zoom

§.1. Setting

§.2. Def. of $K^{I \times \mathbb{C}^*}(\mathcal{Q}^{rat})$

§.3. $K^{I \times \mathbb{C}^*}(\mathcal{Q}^{rat})$ is $(DH_0(W), \mathcal{H})$-bimodule)

§.4. Construction of $\rho_0 : DH_0(W) \longrightarrow \mathrm{Mat}_{W \times W}(\mathcal{H})$

- - -

§1.1. Setting.

G : conn. & simply. cmp. lin. alg. group.

$\cup$ ($SL_n(\mathbb{C})$, ...)

B : Borel.

$\cup$

T : maximal Torus.

U : unipotent radical of B

$$Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset P = \bigoplus_{i\in I} \mathbb{Z}\varpi_i$$

$Q^\vee$, $P^\vee$: coroot, weight lattice.

$P_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq} \varpi_i$: dominant integral weight

W : (finite) Weyl group of G.

$\quad = \langle S_i := S_{\alpha_i} \mid i \in I \rangle$

$V(\lambda)$: irr. G-module w/ h.w. $\lambda \in P_+$

$V(\lambda)_\mu \subset V(\lambda)$: μ-wt. sp. $(\mu \in P)$

$\dim(V(\lambda)_\lambda) = 1$

$\mathbb{Z}[P] = \left\{ \sum_{\mu \in P} c_\mu e^\mu \,\middle|\, c_\mu \in \mathbb{Z} \right\}$: group alg. of P

V : T-module & $\dim < \infty$

$$\mathrm{ch}(V) = \sum_{\mu:\,\mathrm{wt.}} \dim(V_\mu)\, e^{\mu} \in \mathbb{Z}[P]$$

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \quad : \quad (T \times \mathbb{C}^*)\text{-module}$$

s.t.
- $\dim(V_i) < \infty$
- $V_i = \{0\}$ $\quad i \gg 0$ or $i \ll 0$
- $\mathbb{C}^* \curvearrowright V_i$ by $t \cdot v_i = q^i v_i$
  $(t \in \mathbb{C}^*,\ v_i \in V_i)$

$$\mathrm{gch}(V) = \sum_{i \in \mathbb{Z}} q^i\, \mathrm{ch}(V_i) \in \mathbb{Z}[P]((q^{-1}))$$

③ Affine Weyl group.

$$t(Q^\vee) = \{ t(\beta) \mid \beta \in Q^\vee \}$$

$$W_{af} := W \ltimes t(Q^\vee)$$

$\circ\ t(\beta)t(\beta') = t(\beta+\beta')$

$$= \langle s_i ,\ s_0 = s_\theta t(-\theta^\vee) \mid i \in I \rangle$$

highest root of G

$W_{af} \curvearrowright P_{af} = P \oplus \mathbb{Z}\delta$

by $\quad wt(\beta)(\mu + k\delta) = w(\mu - \langle \beta, \mu \rangle \delta + k\delta)$

$$= w(\mu) + (k - \langle \beta, \mu \rangle)\delta$$

$\langle\ ,\ \rangle : Q^\vee \times P \longrightarrow \mathbb{Z}$: cannonical pairing.

$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij} \quad (i,j \in I)$

$\langle \rho^\vee, \rho \rangle \subset \frac{1}{e}\mathbb{Z}$

最小 $a\, e \in \mathbb{Z}_{>0}$

$$\ell^{\frac{\infty}{2}} : W_{af} \longrightarrow \mathbb{Z}$$
$$\tilde{w} = w\,t(\beta) \longmapsto \ell^{\frac{\infty}{2}}(\tilde{w}) = \ell(w) + 2\langle \beta, \rho \rangle$$

$w, w' \in W_{af}$

$$w \preceq w' \overset{\text{def}}{\Longleftrightarrow} {}^{\exists} w = w_0 \xrightarrow{\beta_1} w_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_r} w_r = w'$$

$\beta_i \in \Delta^+_{af}$

s.t. $\ell^{\frac{\infty}{2}}(s_{\beta_i} w_i) = \ell^{\frac{\infty}{2}}(w_{i-1}) + 1$

$(i = 1, \ldots, r)$

Semi-infinite Bruhat order

§.2. Def. of $K^{I \rtimes \mathbb{C}^*}(Q^{rat})$.

- $\mathcal{K} := \mathbb{C}((z)) \supset R := \mathbb{C}[[z]]$

For V : finite vector sp.

$$V((z)) := V \underset{\mathbb{C}}{\otimes} \mathcal{K} \supset V[[z]] := V \underset{\mathbb{C}}{\otimes} R$$

- $\mathbb{P}(V((z))) := (V((z)) - 0) / \mathbb{C}^*$ infinite-type proj. sch. w/ homog. coor ring $S(V[z]^*)$

$$i_m : \mathbb{P}(V[[z]]) \underset{\times z^m}{\hookrightarrow} \mathbb{P}(V[[z]])$$

$m \in \mathbb{Z}$.

$$Q := \left\{ (\ell_\lambda)_{\lambda\in P_+} \in \prod_{\lambda\in P_+} \mathbb{P}(V(\lambda)[[z]]) \;\middle|\; \begin{array}{c} \ell_\lambda \otimes \ell_\mu = \ell_{\lambda+\mu} \\ \forall \lambda,\mu\in P_+ \end{array} \right\}$$ infinite-type sch.

$$\Psi_{\lambda,\mu} : \quad V(\lambda)\otimes V(\mu) \hookrightarrow V(\lambda+\mu)$$

DP:

$$Q \ni (\ell_\lambda)_{\lambda\in P_+} \longmapsto (\ell_{\omega_i})_{i\in I} \in \mathbb{P} := \prod_{i\in I} \mathbb{P}(V(\omega_i)[[z]])$$

$$\hookrightarrow$$

Drinfeld-Plücker emb.

For $\beta \in Q^\vee_+$, $\quad i_\beta := \prod_{i\in I} i_{\langle\beta,\omega_i\rangle} :$
$$\begin{array}{ccc} \mathbb{P} & \hookrightarrow & \mathbb{P} \\ \cup & & \cup \\ Q & \hookrightarrow & Q \end{array}$$

$$\{Q_\alpha\ (\equiv Q)\}_{\alpha\in Q^\vee_+}$$

$$i_{\alpha\beta} := i_{\beta-\alpha} \qquad ({}^{\forall}\alpha \leqq \beta)$$

$$:\Longleftrightarrow \beta-\alpha \in Q^{\vee}_{+}$$

$$Q^{\mathrm{rat}} := \varinjlim_{\beta \in Q^{\vee}_{+}} Q_\beta \qquad \text{ind-infinite sch.}$$

$$= G(\mathcal{K}) / (T(\mathbb{C}) \cdot U(\mathcal{K}))$$

level of $\mathbb{C}$-point

Iwahori subgroup.

$$G = \bigsqcup_{w \in W} B \dot{w} B \implies G(\mathcal{R}) = \bigsqcup_{w \in W} \mathbb{I} \dot{w} \mathbb{I}$$

$$\left(\begin{array}{l} \mathrm{ev}_0 : G(\mathcal{R}) \longrightarrow G \\ \\ \mathbb{I} := \mathrm{ev}_0^{-1}(B) \subset G(\mathcal{R}) \end{array} \right)$$

$$Q^{rat} = \bigsqcup_{\tilde{w} \in W_{af}} \mathbb{O}(\tilde{w}) \qquad (\mathcal{I}\text{-orbit decomp.})$$

$$Q(\tilde{w}) := \overline{\mathbb{O}(\tilde{w})} \subset Q^{rat}$$

$$\left(\begin{array}{l} \operatorname{codim}_{Q^{rat}} Q(\tilde{w}) = \ell(w) + 2\langle \beta, \rho \rangle \\ (\tilde{w} = w t(\beta) \in W_{af}) \end{array} \right)$$

Infinite-Schubert var.

$$\mathcal{O}_{\tilde{w}} = \mathcal{O}_{Q(\tilde{w})} : \text{ str. sheaf of } Q(\tilde{w})$$

$$\tilde{w} \lesssim \tilde{v} \iff Q(\tilde{w}) \supset Q(\tilde{v})$$

$$(Q(e) = Q)$$

$$\mathcal{O}_{\tilde{w}}(\lambda) = \mathcal{O}_{Q(\tilde{w})} \otimes \mathcal{O}(\lambda)$$

$\mathcal{O}(\lambda)$: line bdle of Q $\Longleftrightarrow$ line bdl $\boxtimes_{i\in I} \mathcal{O}(m_i)$ on $\mathbb{P}$

$\lambda = \sum_{i\in I} m_i \omega_i \in P$

$$DP^*\left(\boxtimes_{i\in I} \mathcal{O}(m_i)\right)$$

<u>Example</u> $G = Sl_2(\mathbb{C})$

$$Q^{rat} = \mathbb{P}(\underbrace{V(\omega)}_{\cong\ \mathbb{C}^2}((z))) = \varinjlim_{\beta\in Q^\vee_+} \mathbb{P}(z^{-\langle\beta,\omega\rangle} \underbrace{V(\omega)}_{\cong\ \mathbb{C}^2}[[z]])$$

$q \in R(\mathbb{C}^*)$ $\quad q : \mathbb{C}^* \longrightarrow \mathbb{C}^*$, $a \longmapsto a^{-1}$ $\quad Q = \mathbb{P}(\mathbb{C}^2[[z]])$

$$K^{I \rtimes \mathbb{C}^*}(\mathbf{Q}) := \left\{ \sum_{\lambda \in P} f_\lambda [\mathcal{O}(\lambda)] \;\middle|\; \begin{array}{l} f_\lambda \in \mathbb{Z}[P][\![q^{-1}]\!] \text{ s.t.} \\ \sum_{\lambda \in P} |f_\lambda| \operatorname{gch}(H^0(\mathbf{Q}, \mathcal{O}(\lambda+\mu))) \quad \forall \mu \in P \\ \in \mathbb{Z}_{\geq 0}[P][\![q^{-1}]\!] \end{array} \right\} \Big/ L$$

infinite sum

$$L := \left\{ \sum_{\lambda \in P} f_\lambda [\mathcal{O}(\lambda)] \;\middle|\; \begin{array}{l} \sum_{\lambda \in P} f_\lambda \operatorname{gch}(H^0(\mathbf{Q}, \mathcal{O}(\lambda+\mu))) = 0 \\ \text{for all sufficiently dominant } \mu \in P_+ \end{array} \right\}$$

$$f = \sum_{k \geq 0} f_k q^{-k} \in \mathbb{Z}[P][\![q^{-1}]\!]$$

$$|f| := \sum_{k \geq 0} |f_k| q^{-k} \qquad \text{where } |f_k| = \sum_{\nu \in P} |c_\nu| e^\nu$$

For $\beta \in Q^\vee_+$

$$(\iota_\beta)_* : K^{I \rtimes \mathbb{C}^*}(\mathbf{Q}) \hookrightarrow K^{I \rtimes \mathbb{C}^*}(\mathbf{Q})$$

$$(\tilde{\iota}_\beta)_* ([\mathcal{O}(\lambda)]) = q^{\langle \beta, \lambda \rangle} [\mathcal{O}_{t(\beta)}(\lambda)] \quad (\forall \lambda \in P)$$

$$(\tilde{\iota}_\beta)_* ([\mathcal{O}_{\tilde{w}}(\lambda)]) = q^{\langle \beta, \lambda \rangle} [\mathcal{O}_{\tilde{w} t(\beta)}(\lambda)] \quad \begin{cases} \tilde{w} \in W_{af} \\ \beta \in Q^\vee_+ \end{cases}$$

$$\{ K_\alpha \; (\equiv K^{I \rtimes \mathbb{C}^*}(Q)) \}_{\alpha \in Q^\vee_+} \qquad (\tilde{\iota}_{\alpha\beta})_* = (\tilde{\iota}_{\beta-\alpha})_*$$

$$(\text{for } \alpha \leqq \beta)$$

$$K^{I \rtimes \mathbb{C}^*}(Q^{rat}) := \mathbb{Z}[P]((q^{-1})) \otimes_{\mathbb{Z}[P][[q^{-1}]]} \varinjlim_{\alpha \in Q^\vee_+} K_\alpha$$

$\lrcorner$