

# Koornwinder 多項式の Littlewood-Richardson 係数

(A Littlewood-Richardson rule for Koornwinder polynomials)

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K.Y., *A Littlewood-Richardson rule for Koornwinder polynomials*, arXiv:2009.13963.

# Introduction —Koornwinder polynomials—

- Macdonald polynomials  $P_\lambda(x|q, t_i)$  are multivariate  $q$ -orthogonal polynomials associated to affine root systems.
- **Koornwinder polynomials**  $P_\lambda(x|q, t, t_0, t_n, u_0, u_n)$  are  $q$ -orthogonal polynomials, multivariate analogue of Askey-Wilson polynomials [Koornwinder, 1992].
- Koornwinder polynomials are understood as **Macdonald polynomials** associated to the affine root system **of type**  $(C_n^\vee, C_n)$  [野海, 1995; Sahi 1999; Stokman 2000].
- Specializing parameters  $(t, t_0, t_n, u_0, u_n)$ , we can recover Macdonald polynomials of type  $B_n$ ,  $C_n$  and  $BC_n$ .

# Main result —Koornwinder LR-coefficients—

## Notations

- $x = (x_1, \dots, x_n)$  : variables of Koornwinder polynomials.
- $\mathfrak{h}_{\mathbb{Z}}^* := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$  : weight lattice of type  $C_n$ .
- $(\mathfrak{h}_{\mathbb{Z}}^*)_+ := \{\lambda = (\lambda_i) \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}$  : set of dominant weights.
- $\mathbb{K} := \mathbb{Q}(q^{1/2}, t^{1/2}, t_0^{1/2}, t_n^{1/2}, u_0^{1/2}, u_n^{1/2})$ : base field.
- $W_0$ : Weyl group of type  $C_n$ .

Denote Koornwinder polynomial for  $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$  by

$$P_{\lambda}(x) := P_{\lambda}(x|q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}]^{W_0}$$

Consider **Littlewood Richardson** (LR) **coefficients** for Koornwinder polynomials, i.e., the structure constants  $c_{\lambda\mu}^{\nu}$  of the product

$$P_{\lambda}(x)P_{\mu}(x) = \sum_{\nu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+} c_{\lambda\mu}^{\nu} P_{\nu}(x).$$

## Main result (cont.)

### Theorem (Y., Theorem 3.4.2)

For dominant weights  $\lambda, \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$ , we have

$$P_{\lambda}(x)P_{\mu}(x) = \frac{1}{t_{w_{\lambda}}^{-\frac{1}{2}} W_{\lambda}(t)} \sum_{\nu \in W^{\mu}} \sum_{\rho \in \Gamma_2^C(\overrightarrow{w(\lambda)})^{-1}, (\nu \cdot w(\mu))^{-1}} A_{\rho} B_{\rho} C_{\rho} P_{-w_0 \text{wt}(\rho)}(x).$$

### Main properties

- $\rho$  runs over a set  $\Gamma_2^C(\overrightarrow{w(\lambda)})^{-1}, (\nu \cdot w(\mu))^{-1}$  of **colored alcove walks**.
- The terms  $A_{\rho}$ ,  $B_{\rho}$  and  $C_{\rho}$  are factored (shown in the next pages).

### Minor notations

- $w_{\lambda}$ : the longest element of the stabilizer  $W_{\lambda} \subset W_0$  of  $\lambda$ .
- $W_{\lambda}(t)$ : the Poincaré polynomial of  $W_{\lambda}$  with variables  $t, t_n$ .
- $W^{\mu} \subset W_0$ : a complete system of representatives of  $W_0/W_{\mu}$ .
- $w_0 \in W_0$ : the longest element.
- $\text{wt}(\rho) \in (\mathfrak{h}_{\mathbb{Z}})^*$ : a certain element determined by the colored alcove walk  $\rho$ .

## Main result (cont.)

The terms  $A_p$  and  $B_p$  are given by

$$A_p := \prod_{\alpha \in \mathcal{L}((v \cdot w(\mu))^{-1}, t(-w_0 \cdot \mu))} \rho(\alpha), \quad B_p := \prod_{\alpha \in \mathcal{L}(t(\text{wt}(p))w_0, e(p))} \rho(\alpha).$$

The factor  $\rho(\alpha)$  is given by

$$\rho(\alpha) := \begin{cases} t^{\frac{1}{2}} \frac{1 - t^{-1} q^{\text{sh}(-\alpha)} t^{\text{ht}(-\alpha)}}{1 - q^{\text{sh}(-\alpha)} t^{\text{ht}(-\alpha)}} & (\alpha \notin W \cdot \alpha_n) \\ t_n^{\frac{1}{2}} \frac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2} \text{sh}(-\alpha)} t^{\frac{1}{2} \text{ht}(-\alpha)}) (1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2} \text{sh}(-\alpha)} t^{\frac{1}{2} \text{ht}(-\alpha)})}{1 - q^{\text{sh}(-\alpha)} t^{\text{ht}(-\alpha)}} & (\alpha \in W \cdot \alpha_n) \end{cases}$$

$$q^{\text{sh}(\alpha)} := q^{-k}, \quad t^{\text{ht}(\alpha)} := \prod_{\gamma \in R_+^s} t^{\frac{1}{2} \langle \gamma^\vee, \beta \rangle} \prod_{\gamma \in R_+^\ell} (t_0 t_n)^{\frac{1}{2} \langle \gamma^\vee, \beta \rangle} \quad (\alpha = \beta + k\delta \in S),$$

with  $R_+^s := \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$  and  $R_+^\ell := \{2\epsilon_i \mid 1 \leq i \leq n\}$ .

- The factor  $\rho$  looks similar to Pieri coefficients for Macdonald polynomials of type  $A_n$ .

# Main result (cont.)

The term  $C_p$  is given by  $C_p := \prod_{k=1}^r C_{p,k}$  with

$$C_{p,k} := \begin{cases} 1 & \text{the } k\text{-th step of } p \text{ is a positive crossing} \\ \prod_{k \in \xi_{\text{des}}(p)} n_{i_k} (q^{\text{sh}(-h_k(p))} t^{\text{ht}(-h_k(p))}) & \text{a negative crossing} \\ -\psi_{i_k}^+ (q^{\text{sh}(-h_k(p))} t^{\text{ht}(-h_k(p))}) & \text{a gray positive folding} \\ -\psi_{i_k}^- (q^{\text{sh}(-h_k(p))} t^{\text{ht}(-h_k(p))}) & \text{a gray negative folding} \\ -\psi_{i_k}^+ (q^{\text{sh}(-\beta_k)} t^{\text{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is positive} \\ -\psi_{i_k}^- (q^{\text{sh}(-\beta_k)} t^{\text{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is negative} \end{cases},$$

where the factor  $\psi^\pm(z)$ ,  $n_i(z)$  are given by the following:

$$\psi_i^\pm(z) := \mp \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - z^{\pm 1}} \quad (i = 1, \dots, n-1),$$

$$\psi_0^\pm(z) := \mp \frac{(u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}}) + z^{\pm \frac{1}{2}} (u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}, \quad \psi_n^\pm(z) := \mp \frac{(t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}}) + z^{\pm \frac{1}{2}} (t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})}{1 - z^{\pm 1}},$$

$$n_i(z) := \frac{1 - tz}{1 - z} \frac{1 - t^{-1}z}{1 - z} \quad (\beta \in W.\alpha_i, i = 1, \dots, n-1),$$

$$n_0(z) := \frac{(1 - u_n^{\frac{1}{2}} u_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + u_n^{\frac{1}{2}} u_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \frac{(1 + u_n^{-\frac{1}{2}} u_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 - u_n^{-\frac{1}{2}} u_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \quad (\beta \in W.\alpha_0),$$

$$n_n(z) := \frac{(1 - t_n^{\frac{1}{2}} t_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 + t_n^{\frac{1}{2}} t_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \frac{(1 + t_n^{-\frac{1}{2}} t_0^{\frac{1}{2}} z^{\frac{1}{2}})(1 - t_n^{-\frac{1}{2}} t_0^{-\frac{1}{2}} z^{\frac{1}{2}})}{1 - z} \quad (\beta \in W.\alpha_n).$$

# Affine root system of type $(C_n^\vee, C_n)$

Root system of type  $C_n$ .

- $\mathfrak{h}_{\mathbb{Z}}^* = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ : weight lattice.
- $R = \{\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq n\}$ : root system with simple roots  $\alpha_i$  ( $1 \leq i \leq n$ ).
- $W_0 = \{\pm 1\}^n \rtimes \mathfrak{S}_n = \langle s_1, s_2, \dots, s_n \rangle \subset \text{GL}(\mathfrak{h}_{\mathbb{R}}^*)$ : the finite Weyl group.
- $W_0 \curvearrowright \mathfrak{h}_{\mathbb{R}}^*$  by  $s_i \cdot \epsilon_j = \begin{cases} \epsilon_{i+1} & (j = i) \\ \epsilon_j & (j = i + 1) \\ \epsilon_j & (j = n) \end{cases}$ ,  $s_n \cdot \epsilon_j = \begin{cases} -\epsilon_n & (j = n) \\ \epsilon_j & (j \neq n) \end{cases}$ .

Affine root system of type  $(C_n^\vee, C_n)$ .

- $\widetilde{\mathfrak{h}}_{\mathbb{Z}}^* := \mathfrak{h}_{\mathbb{Z}}^* \oplus \mathbb{Z}\delta$ : extension by the null root  $\delta$ .
- The (non-reduced) affine root system  $S$  of type  $(C_n^\vee, C_n)$ :

$$S := \left\{ \pm\epsilon_i + \frac{k}{2}\delta, \pm 2\epsilon_i + k\delta \mid k \in \mathbb{Z}, 1 \leq i \leq n \right\} \\ \cup \left\{ \pm\epsilon_i \pm \epsilon_j + k\delta \mid k \in \mathbb{Z}, 1 \leq i < j \leq n \right\} \subset \widetilde{\mathfrak{h}}_{\mathbb{R}}^*.$$

Simple roots are  $\alpha_i$  and  $\alpha_i^\vee$  ( $0 \leq i \leq n$ ).

- $W := t(P) \rtimes W_0 \subset \text{GL}(\widetilde{\mathfrak{h}}_{\mathbb{R}}^*)$ : the **extended affine Weyl group**. ( $t(P)$ : the group ring of  $P$ .)
- $W = \langle s_0, s_1, \dots, s_n \rangle$  with  $s_0 \cdot \epsilon_i = \begin{cases} \delta - \epsilon_1 & (i = 1) \\ \epsilon_i & (i \neq 1) \end{cases}$ .

# Koornwinder polynomials

## Notations of parameters

- $\{t_\alpha \mid \alpha \in S\}$  : a family of parameters with  $t_\alpha = t_\beta$  iff  $W.\alpha = W.\beta$ .
- Five orbits for the  $W$ -action on  $S$  (if  $n \geq 2$ ):

$$W.\alpha_i = W.\alpha_i^\vee \quad (i = 1, \dots, n-1), \quad W.\alpha_n, \quad W.\alpha_n^\vee, \quad W.\alpha_0, \quad W.\alpha_0^\vee.$$

- $(t_{\alpha_0}, t_{\alpha_i} = t_{\alpha_i^\vee}, t_{\alpha_n}, t_{\alpha_n^\vee}, t_{\alpha_0^\vee}) = (t_0, t, t_n, u_0, u_n)$ .
- $\mathbb{K} := \mathbb{Q}(q^{1/2}, t^{1/2}, t_0^{1/2}, t_n^{1/2}, u_0^{1/2}, u_n^{1/2})$ : the base field.

## Fact (Koornwinder, 1992)

There is a family  $\{P_\lambda(x) \mid \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_+\}$  of **Koornwinder polynomials**

$$P_\lambda(x) = P_\lambda(x \mid q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}]^{W_0}$$

satisfying the following conditions.

- $P_\lambda(x)$ 's are orthogonal with respect to a certain inner product.
- $P_\lambda(x)$  is an eigenfunction of a certain  $q$ -difference operator.
- $P_\lambda(x)$  is monic with respect to the dominance order.

The family  $\{P_\lambda(x)\}$  is a  $\mathbb{K}$ -basis of  $\mathbb{K}[x^{\pm 1}]^{W_0}$ .



# Alcove walks

Recall the main statement:

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{\nu \in W^\mu} \sum_{\rho \in \Gamma_2^C(\vec{w}(\lambda)^{-1}, (\nu \cdot w(\mu))^{-1})} A_\rho B_\rho C_\rho P_{-w_0 \text{wt}(\rho)}(x).$$

Definition of alcove walks.

- $A := \{x = (x_1, \dots, x_n) \in \mathfrak{h}_{\mathbb{R}}^* \mid \frac{1}{2} \geq x_1 \geq \dots \geq x_n \geq 0\}$ : the fundamental alcove.
- For each alcove  $wA$ , we give properly the signs  $\pm$  to the two sides on edges of  $wA$ .
- Let  $w \in W$  and take a reduced expression  $w = s_{i_1} \cdots s_{i_r} \in W$ .  
For  $z \in W$  and  $b = (b_1, \dots, b_r) \in \{0, 1\}^r$ , we call a sequence

$$\rho = (zA, z s_{i_1}^{b_1} A, \dots, z s_{i_1}^{b_1} \cdots s_{i_r}^{b_r} A)$$

an **alcove walk of type  $\vec{w} = (i_1, \dots, i_r)$  beginning at  $z$** .

- $\Gamma(\vec{w}, z)$ : the set of alcove walks of type  $\vec{w}$  beginning at  $z$ .

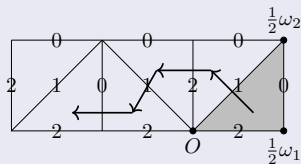
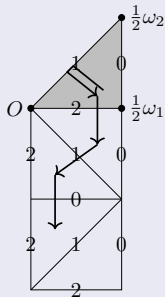
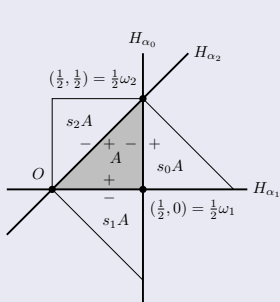
# Alcove walks (cont.)

## Examples of alcove walks in rank 2

$w = s_1 s_2 s_1 s_0$ ,  $z = e \in W$ .

$p_1 := (A, A, s_2 A, s_2 s_1 A, s_2 s_1 s_0 A)$ ,

$p_2 := (A, s_1 A, s_1 s_2 A, s_1 s_2 s_1 A, s_1 s_2 s_1 s_0 A) \in \Gamma(\vec{w}, z)$



# Colored alcove walks

## Crossings and foldings.

- For  $p = (p_0, \dots, p_r) \in \Gamma(\vec{w}, z)$ , the  $k$ -th step means  $p_{k-1} \mapsto p_k$ .
- Each step is classified into four types:

positive crossing	negative crossing	positive folding	negative folding
$\begin{array}{c c} - & + \\ \hline \xrightarrow{\quad} & \\ p_{k-1} & p_k \end{array}$	$\begin{array}{c c} + & - \\ \hline \xrightarrow{\quad} & \\ p_{k-1} & p_k \end{array}$	$\begin{array}{c c} + & - \\ \hline \xleftarrow{\quad} & \\ p_{k-1} = p_k & v_{k-1}s_{i_k}A \end{array}$	$\begin{array}{c c} - & + \\ \hline \xleftarrow{\quad} & \\ p_{k-1} = p_k & v_{k-1}s_{i_k}A \end{array}$

## Definition of colored alcove walks.

- $C := \{x = (x_1, \dots, x_n) \in \mathfrak{h}_{\mathbb{R}}^* \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$  : the dominant chamber.
- $\Gamma^C(\vec{w}, z) := \{p = (p_0, \dots, p_r) \in \Gamma(\vec{w}, z) \mid p_k \subset C, \forall k\}$ .
- $\Gamma_2^C(\vec{w}, z)$  : the set of alcove walks  $p \in \Gamma^C(\vec{w}, z)$  with **coloring of all the foldings by either black or gray.**

# Outline of the proof

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{\nu \in W^\mu} \sum_{\rho \in \Gamma_2^C(\overrightarrow{w(\lambda)})^{-1}, (\nu \cdot w(\mu))^{-1}} A_\rho B_\rho C_\rho P_{-w_0 \text{wt}(\rho)}(x)$$

$E_\mu(x) \in \mathbb{K}[X^{\pm 1}]$ : the non-symmetric Koornwinder polynomials [Sahi].

- $E_\mu(x)$  is an eigenfunction of Dunkl operators in the affine Hecke algebra of type  $C_n$ .
- $\{E_\lambda(x) \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$  is a  $\mathbb{K}$ -basis of  $\mathbb{K}[X^{\pm 1}]$ .
- $P_\lambda(x)$  is obtained by symmetrizing  $E_\lambda(x)$ .

Our proof is a  $(C_n^V, C_n)$ -type analogue of [Yip, 2012].

① For  $\lambda, \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$ , we derive an expansion formula  $x^\mu E_\lambda(x) = \sum_{\rho \in \Gamma^C} c_\rho E_{\varpi(\rho)}(x)$ .

② Use Ram-Yip type formula  $E_\mu(x) = \sum_{\rho \in \Gamma} f_\rho t_{d(\rho)}^{\frac{1}{2}} x^{\text{wt}(\rho)}$ .

This formula was derived by [Orr, Shimozono 2018], based on [Ram, Yip, 2008] for the untwisted affine root systems.

③ Using (1) and (2) we can derive  $E_\mu(x)P_\lambda(x) = \sum_{\nu \in W^\lambda} \sum_{\rho \in \Gamma_2^C} A_\rho C_\rho E_{\varpi(\rho)}(x)$ .

④ Symmetrize (3) and switch  $\lambda \leftrightarrow \mu$ .

# Main result (recollection)

For dominant weights  $\lambda, \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$ , we have

$$P_{\lambda}(x)P_{\mu}(x) = \frac{1}{t_{w_{\lambda}}^{-\frac{1}{2}} W_{\lambda}(t)} \sum_{v \in W^{\mu}} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)})^{-1}, (v \cdot w(\mu))^{-1}} A_p B_p C_p P_{-w_0 \text{wt}(p)}(x)$$

where

- $p$  runs over a set  $\Gamma_2^C(\overrightarrow{w(\lambda)})^{-1}, (v \cdot w(\mu))^{-1}$  of colored alcove walks, and
- the terms  $A_p$ ,  $B_p$  and  $C_p$  are factored.

In the preprint we studied some **specializations** of these LR-coefficients.

- Rank  $n = 1$  case  $\iff$  **Askey-Wilson polynomials**: discussed below.
- Macdonald polynomials of type  $C_n$ .
- Hall-Littlewood limit  $q \rightarrow 0$ .

# The case of Askey-Wilson polynomials

One-variable Koornwinder polynomials = Askey-Wilson polynomials.

- $P_l(x) := P_{l\epsilon}(x; q, t_0, t_1, u_0, u_1)$  : Askey-Wilson polynomial for  $l\epsilon \in \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\epsilon$ .

Consider **Pieri coefficients**, i.e., LR coefficients for minuscule weights.

- $\epsilon \in (\mathfrak{h}_{\mathbb{Z}}^*)_+ = \mathbb{N}\epsilon$  : the unique minuscule weight of type  $C_n$ .
- Pieri coefficients are  $c_m$  in  $P_1(x)P_l(x) = \sum_m c_m P_m(x)$ .

## Corollary ([Y., Proposition 4.1.3])

For a dominant weight  $\lambda = l\epsilon \in (\mathfrak{h}_{\mathbb{Z}}^*)_+ = \mathbb{N}\epsilon$ , we have

$$P_1(x)P_l(x) = P_{l+1}(x) + F_l P_l(x) + G_l P_{l-1}(x),$$

$$F_l := -\rho(2l\delta - \alpha_1)(\psi_0^-(q^{2l+1}t_0t_1) + \psi_0^-(qt_0t_1) + \psi_0^+(q^{2l-1}t_0t_1) + \psi_0^+(qt_0t_1)),$$

$$G_l := -\rho(2l\delta - \alpha_1)\rho(2(l-1)\delta - \alpha_1)n_0(q^{2l-1}t_0t_1).$$

# Askey-Wilson case (cont.)

$v = s_1$				
$p^*$	$p$	$A_p$	$B_p$	$C_p$
		1	1	1
		1	$\rho(2l\delta - \alpha_1)$	$-\psi_0^-(q^{2l+1}t_0t_1)$
		1	$\rho(2l\delta - \alpha_1)$	$-\psi_0^-(qt_0t_1)$
$v = e$				
$p^*$	$p$	$A_p$	$B_p$	$C_p$
		$\rho(2l\delta - \alpha_1)$	$\rho((2l-2)\delta - \alpha_1)$	$n_0(q^{2l-1}t_0t_1)$
		$\rho(2l\delta - \alpha_1)$	1	$-\psi_0^+(q^{2l-1}t_0t_1)$
		$\rho(2l\delta - \alpha_1)$	1	$-\psi_0^+(qt_0t_1)$

## Askey-Wilson case (cont.)

Correspondence between Askey-Wilson and Koornwinder parameters:

$$(q, a, b, c, d) = (q, q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, t_1^{\frac{1}{2}} u_1^{\frac{1}{2}}, -t_1^{\frac{1}{2}} u_1^{-\frac{1}{2}}).$$

The equation  $P_1(x)P_l(x) = P_{l+1}(x) + F_l P_l(x) + G_l P_l(x)$  is rewritten as

$$2z p_l(z) = h_l p_{l+1}(z) + f_l p_l(z) + g_l p_{l-1}(z), \quad p_l((x + x^{-1})/2) = \gamma_l P_l(x)$$

with

$$h_l := \frac{1 - q^{l-1}\pi}{(1 - q^{2l-1}\pi)(1 - q^{2l}\pi)},$$

$$f_l := q^{l-1} \frac{(1 + q^{2l-1}\pi)(qs + \pi s') - q^{l-1}(1 + q)\pi(s + qs')}{(1 - q^{2l-2}\pi)(1 - q^{2l}\pi)},$$

$$g_l := (1 - q^l) \frac{(1 - q^{l-1}ab)(1 - q^{l-1}ac)(1 - q^{l-1}ad)(1 - q^{l-1}bc)(1 - q^{l-1}bd)(1 - q^{l-1}cd)}{(1 - q^{2l-2}\pi)(1 - q^{2l-1}\pi)},$$

$$\pi := abcd, \quad s := a + b + c + d, \quad s' := a^{-1} + b^{-1} + c^{-1} + d^{-1},$$

$$\gamma_l := (q^{l-1}\pi; q)_l = (1 - q^{l-1}\pi)(1 - q^l\pi) \cdots (1 - q^{2l-2}\pi).$$

The rewritten equation coincides with the original recurrence relation of Askey-Wilson polynomials  $p_l(z)$  in [Askey, Wilson, 1985].



# Conclusions and Remarks

## Conclusions:

- We derived an alcove-walk formula of LR coefficients for Koornwinder polynomials, which is a  $(C_n^\vee, C_n)$ -type analogue of Yip's formula for Macdonald polynomials of untwisted affine root systems.
- We also obtained some specializations of the formula. In particular, in the rank 1 case, we recovered the recurrence formula of Askey-Wilson polynomials.

## Problems:

- Specialization to Macdonald polynomials of type  $B_n$  and  $BC_n$ .
- Simplified formula of Pieri coefficients for Koornwinder polynomials, and its relation to Pieri formulas of type  $B_n$ ,  $C_n$  and  $BC_n$ .
- Relation to tableaux formula for Koornwinder polynomials.

*Thank you for your attention.*

# References

- 1 R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomials*, Mem. Amer. Math. Soc., **54** (1985), no. 319.
- 2 D. Orr, M. Shimozono, *Specializations of nonsymmetric Macdonald-Koornwinder polynomials*, J. Algebraic Combin., **47** (2018), no. 1, 91–127.
- 3 A. Ram, M. Yip, *A combinatorial formula for Macdonald polynomials*, Adv. Math., **226** (2011), 309–331.
- 4 S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. Math., **150** (1999), 267–282.
- 5 J.V. Stokman, *Koornwinder Polynomials and Affine Hecke Algebras*, Int. Math. Res. Not., **19** (2000), 1005–1042.
- 6 M. Yip, *A Littlewood–Richardson rule for Macdonald polynomials*, Math. Z., **272** (2012), 1259–1290.
- 7 野海 正俊, *Macdonald-Koornwinder 多項式と affine Hecke 環*, 数理解析研究所講究録, **919** (1995), 44–55.

## Rank 2 case

Consider the case  $\lambda = \omega_1$ ,  $\mu = \omega_2$ .

- $t(\omega_1) = s_0 s_1 s_2 s_1$ ,  $t(\omega_2) = s_0 s_1 s_0 s_2 s_1 s_2$ .
- $w(\omega_1) = s_0$ ,  $w(\omega_2) = s_0 s_1 s_0$ .
- $W^\mu = W^{\omega_2} = \{e, s_2, s_1 s_2, s_2 s_1 s_2\}$ .

### Proposition (Y., Proposition 4.4.1)

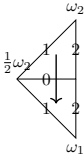
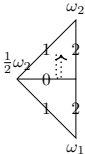
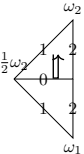
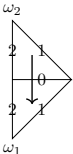
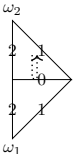
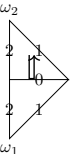
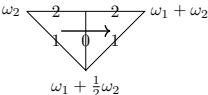
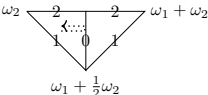
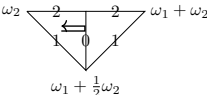
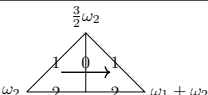
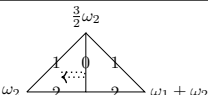
For Koornwinder polynomials of rank 2, we have

$$P_{\omega_1}(x)P_{\omega_2}(x) = \frac{1}{t_2^{-\frac{1}{2}} + t_2^{\frac{1}{2}}} (FP_{\omega_1+\omega_2}(x) + GP_{\omega_2}(x) + HP_{\omega_1}(x)),$$

$$F := 2\rho(2\delta - 2\epsilon_1),$$

$$G := -2\rho(2\delta - 2\epsilon_2)\rho(2\delta - (\epsilon_1 + \epsilon_2))\rho(2\delta - 2\epsilon_1)\rho(\epsilon_1 - \epsilon_2) \\ \times (\psi_0^-(qt_0 t_2) + \psi_0^+(qt_0 t_2) + \psi_0^-(qt_0 t_2) + \psi_0^-(q^3 t_0 t_2)),$$

$$H := 2\rho(2\delta - 2\epsilon_1)\rho(2\delta - (\epsilon_1 + \epsilon_2))^2\rho(2\delta - 2\epsilon_2)\rho(2\epsilon_2)n_0(qt_0 t_2).$$

$X_{11}$ 	$X_{12}$ 	$X_2$ 
$Y_{11}$ 	$Y_{12}$ 	$Y_2$ 
$Z_{11}$ 	$Z_{12}$ 	$Z_2$ 
$W_{11}$ 	$W_{12}$ 	$W_2$ 