A Littlewood-Richardson rule for Koornwinder polynomials

山口 航平 (Kohei Yamaguchi)

Nagoya Univ.

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Based on the preprint

K.Y., *A Littlewood-Richardson rule for Koornwinder polynomials*, arXiv:2009.13963.

K.Y., S. Yanagida, *Specializing Koornwinder polynomials to Macdonald polynomials of type B, C, D and BC*, arXiv: 2105.00936.

Introduction —Koornwinder polynomials—

- Macdonald polynomials $P_\lambda(x|q, t_i)$ are multivariate q-orthogonal polynomials associated to affine root systems.
- Koornwinder polynomials $P_\lambda(x|q, t, t_0, t_n, u_0, u_n)$ are q-orthogonal polynomials, multivariate analogue of Askey-Wilson polynomials [Koornwinder, 1992].
- Koornwinder polynomials are understood as Macdonald polynomials associated to the affine root system of type $(C_n^\vee, C_n)$ [野海, 1995; Sahi 1999; Stokman 2000].
- Specializing parameters (t, t_0, t_n, u_0, u_n), we can recover Macdonald polynomials of type B_n, C_n and BC_n.

Main result —Koornwinder LR-coefficients—

Notations

- $x = (x_1, \dots, x_n)$: variables of Koornwinder polynomials.
- $\mathfrak{h}_{\mathbb{Z}}^* := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$: weight lattice of type C_n.
- $(\mathfrak{h}_{\mathbb{Z}}^*)_+ := \{\lambda = (\lambda_i) \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$: set of dominant weights.
- $\mathbb{K} := \mathbb{Q}(q^{1/2}, t^{1/2}, t_0^{1/2}, t_n^{1/2}, u_0^{1/2}, u_n^{1/2})$: base field.
- W_0: Weyl group of type C_n.

Denote Koornwinder polynomial for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$ by

$$P_\lambda(x) := P_\lambda(x|q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}]^{W_0}$$

Consider Littlewood Richardson (LR) coefficients for Koornwinder polynomials, i.e., the structure constants $c_{\lambda\mu}^{\nu}$ of the product

$$P_\lambda(x) P_\mu(x) = \sum_{\nu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+} c_{\lambda\mu}^{\nu} P_\nu(x).$$

Main result (cont.)

Theorem (Y., Theorem 3.4.2)

For dominant weights $\lambda, \mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_+$, we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x).$$

Main properties

- p runs over a set $\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})$ of colored alcove walks.
- The terms A_p, B_p and C_p are factored (shown in the next pages).

Minor notations

- w_λ: the longest element of the stabilizer $W_\lambda \subset W_0$ of λ.
- $W_\lambda(t)$: the Poincare polynomial of W_λ with variables t, t_n.
- $W^\mu \subset W_0$: a complete system of representatives of W_0/W_μ.
- $w_0 \in W_0$: the longest element.
- $\mathrm{wt}(p) \in (\mathfrak{h}_{\mathbb{Z}})^*$: a certain element determined by the colored alcove walk p.

Main result (cont.)

The terms A_p and B_p are given by

$$A_p := \prod_{\alpha \in \mathcal{L}((v.w(\mu))^{-1}, t(-w_0.\mu))} \rho(\alpha), \quad B_p := \prod_{\alpha \in \mathcal{L}(t(\mathrm{wt}(p))w_0, e(p))} \rho(-\alpha).$$

The factor $\rho(\alpha)$ is given by

$$\rho(\alpha) := \begin{cases} t^{\frac{1}{2}} \dfrac{1 - t^{-1} q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}}{1 - q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}} & (\alpha \notin W.\alpha_n) \\ t_n^{\frac{1}{2}} \dfrac{(1 + t_0^{\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-\alpha)} t^{\frac{1}{2}\mathrm{ht}(-\alpha)})(1 - t_0^{-\frac{1}{2}} t_n^{-\frac{1}{2}} q^{\frac{1}{2}\mathrm{sh}(-\alpha)} t^{\frac{1}{2}\mathrm{ht}(-\alpha)})}{1 - q^{\mathrm{sh}(-\alpha)} t^{\mathrm{ht}(-\alpha)}} & (\alpha \in W.\alpha_n) \end{cases}$$

$$q^{\mathrm{sh}(\alpha)} := q^{-k}, \; t^{\mathrm{ht}(\alpha)} := \prod_{\gamma \in R_+^s} t^{\frac{1}{2}\langle \gamma^\vee, \beta \rangle} \prod_{\gamma \in R_+^\ell} (t_0 t_n)^{\frac{1}{2}\langle \gamma^\vee, \beta \rangle} \quad (\alpha = \beta + k\delta \in S),$$

with $R_+^s := \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$ and $R_+^\ell := \{2\epsilon_i \mid 1 \le i \le n\}$.

- The factor ρ looks similar to Pieri coefficients for Macdonald polynomials of type A_n.

Main result (cont.)

The term C_p is given by $C_p := \prod_{k=1}^{r} C_{p,k}$ with

$$C_{p,k} := \begin{cases} 1 & \text{the } k\text{-th step of } p \text{ is a positive crossing} \\ \prod_{k\in\xi_{\mathrm{des}}(p)} n_{i_k}(q^{\mathrm{sh}(-h_k(p))}t^{\mathrm{ht}(-h_k(p))}) & \text{a negative crossing} \\ -\psi^{+}_{i_k}(q^{\mathrm{sh}(-h_k(p))}t^{\mathrm{ht}(-h_k(p))}) & \text{a gray positive folding} \\ -\psi^{-}_{i_k}(q^{\mathrm{sh}(-h_k(p))}t^{\mathrm{ht}(-h_k(p))}) & \text{a gray negative folding} \\ \psi^{+}_{i_k}(q^{\mathrm{sh}(-\beta_k)}t^{\mathrm{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is positive} \\ \psi^{-}_{i_k}(q^{\mathrm{sh}(-\beta_k)}t^{\mathrm{ht}(-\beta_k)}) & \text{a black folding and the } k\text{-th step of } p^* \text{ is negative} \end{cases},$$

where the factor $\psi^{\pm}(z)$, $n_i(z)$ are given by the following:

$$\psi^{\pm}_i(z) := \mp\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{1 - z^{\pm 1}} \qquad (i = 1, \dots, n-1),$$

$$\psi^{\pm}_0(z) := \mp\frac{(u_n^{\frac{1}{2}} - u_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(u_0^{\frac{1}{2}} - u_0^{-\frac{1}{2}})}{1 - z^{\pm 1}}, \quad \psi^{\pm}_n(z) := \mp\frac{(t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}}) + z^{\pm\frac{1}{2}}(t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}})}{1 - z^{\pm 1}},$$

$$n_i(z) := \frac{1 - tz}{1 - z}\,\frac{1 - t^{-1}z}{1 - z} \qquad (\beta \in W.\alpha_i,\ i = 1, \dots, n-1),$$

$$n_0(z) := \frac{(1 - u_n^{\frac{1}{2}}u_0^{\frac{1}{2}}z^{\frac{1}{2}})(1 + u_n^{\frac{1}{2}}u_0^{-\frac{1}{2}}z^{\frac{1}{2}})}{1 - z}\,\frac{(1 + u_n^{-\frac{1}{2}}u_0^{\frac{1}{2}}z^{\frac{1}{2}})(1 - u_n^{-\frac{1}{2}}u_0^{-\frac{1}{2}}z^{\frac{1}{2}})}{1 - z} \quad (\beta \in W.\alpha_0),$$

$$n_n(z) := \frac{(1 - t_n^{\frac{1}{2}}t_0^{\frac{1}{2}}z^{\frac{1}{2}})(1 + t_n^{\frac{1}{2}}t_0^{-\frac{1}{2}}z^{\frac{1}{2}})}{1 - z}\,\frac{(1 + t_n^{-\frac{1}{2}}t_0^{\frac{1}{2}}z^{\frac{1}{2}})(1 - t_n^{-\frac{1}{2}}t_0^{-\frac{1}{2}}z^{\frac{1}{2}})}{1 - z} \qquad (\beta \in W.\alpha_n).$$

Affine root system of type $(C_n^\vee, C_n)$

Root system of type C_n.

- $\mathfrak{h}_{\mathbb{Z}}^* = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$: weight lattice.
- $R = \{\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq n\}$: root system with simple roots α_i $(1 \leq i \leq n)$.
- $W_0 = \{\pm 1\}^n \rtimes \mathfrak{S}_n = \langle s_1, s_2, \ldots, s_n \rangle \subset \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$: the finite Weyl group.
- $W_0 \curvearrowright \mathfrak{h}_{\mathbb{R}}^*$ by $s_i.\epsilon_j = \begin{cases} \epsilon_{i+1} & (j = i) \\ \epsilon_i & (j = i+1) \\ \epsilon_j & (j = n) \end{cases}$, $s_n.\epsilon_j = \begin{cases} -\epsilon_n & (j = n) \\ \epsilon_j & (j \neq n) \end{cases}$.

Affine root system of type $(C_n^\vee, C_n)$.

- $\widetilde{\mathfrak{h}_{\mathbb{Z}}^*} := \mathfrak{h}_{\mathbb{Z}}^* \oplus \mathbb{Z}\delta$: extension by the null root δ.
- The (non-reduced) affine root system S of type $(C_n^\vee, C_n)$:

$$S := \{\pm\epsilon_i + \frac{k}{2}\delta, \pm 2\epsilon_i + k\delta \mid k \in \mathbb{Z},\ 1 \leq i \leq n\}$$
$$\cup \{\pm\epsilon_i \pm \epsilon_j + k\delta \mid k \in \mathbb{Z}, 1 \leq i < j \leq n\} \subset \widetilde{\mathfrak{h}_{\mathbb{R}}^*}.$$

  Simple roots are α_i and $\alpha_i^\vee$ $(0 \leq i \leq n)$.

- $W := t(P) \rtimes W_0 \subset \mathrm{GL}(\widetilde{\mathfrak{h}_{\mathbb{R}}^*})$: the extended affine Weyl group.

  ($t(P) := P$ as additive group.)

- $W = \langle s_0, s_1, \ldots, s_n \rangle$ with $s_0.\epsilon_i = \begin{cases} \delta - \epsilon_1 & (i = 1) \\ \epsilon_i & (i \neq 1) \end{cases}$.

Koornwinder polynomials

Notations of parameters

- $\{t_\alpha \mid \alpha \in S\}$: a family of parameters with $t_\alpha = t_\beta$ iff $W.\alpha = W.\beta$.
- Five orbits for the W-action on S (if $n \geq 2$):

$$W.\alpha_i = W.\alpha_i^\vee \ (i = 1, \ldots, n-1), \ W.\alpha_n, \ W.\alpha_n^\vee, \ W.\alpha_0, \ W.\alpha_0^\vee.$$

- $(t_{\alpha_0}, t_{\alpha_i} = t_{\alpha_i^\vee}, t_{\alpha_n}, t_{\alpha_0^\vee}, t_{\alpha_n^\vee}) = (t_0, t, t_n, u_0, u_n)$.
- $\mathbb{K} := \mathbb{Q}(q^{1/2}, t^{1/2}, t_0^{1/2}, t_n^{1/2}, u_0^{1/2}, u_n^{1/2})$: the base field.

Fact (Koornwinder, 1992)

There is a family $\{P_\lambda(x) \mid \lambda \in (\mathfrak{h}_\mathbb{Z}^*)_+\}$ of Koornwinder polynomials

$$P_\lambda(x) = P_\lambda(x \mid q, t, t_0, t_n, u_0, u_n) \in \mathbb{K}[x^{\pm 1}]^{W_0}$$

satisfying the following conditions.

- $P_\lambda(x)$'s are orthogonal with respect to a certain inner product.
- $P_\lambda(x)$ is an eigenfunction of a certain q-difference operator.
- $P_\lambda(x)$ is monic with respect to the dominance order.

The family $\{P_\lambda(x)\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x^{\pm 1}]^{W_0}$.

Alcove walks

Recall the main statement:

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(v.w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x).$$

Definition of alcove walks.

- $A := \left\{x = (x_1, \dots, x_n) \in \mathfrak{h}_\mathbb{R}^* \mid \frac{1}{2} \geq x_1 \geq \cdots \geq x_n \geq 0\right\}$: the fundamental alcove.
- For each alcove wA, we give properly the signs $\pm$ to the two sides on edges of wA.
- Let $w \in W$ and take a reduced expression $w = s_{i_1} \cdots s_{i_r} \in W$. For $z \in W$ and $b = (b_1, \dots, b_r) \in \{0, 1\}^r$, we call a sequence

$$p = (zA, zs_{i_1}^{b_1}A, \dots, zs_{i_1}^{b_1} \cdots s_{i_r}^{b_r}A)$$

  an alcove walk of type $\overrightarrow{w} = (i_1, \dots, i_r)$ beginning at z.
- $\Gamma(\overrightarrow{w}, z)$: the set of alcove walks of type $\overrightarrow{w}$ beginning at z.

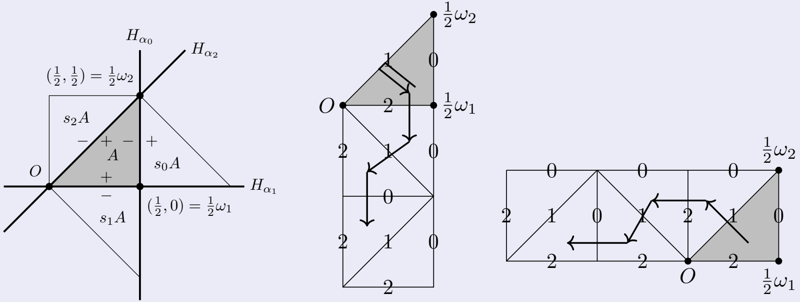
Alcove walks (cont.)

Examples of alcove walks in rank 2

$w = s_1 s_2 s_1 s_0, z = e \in W.$

$p_1 := (A, A, s_2A, s_2s_1A, s_2s_1s_0A),$

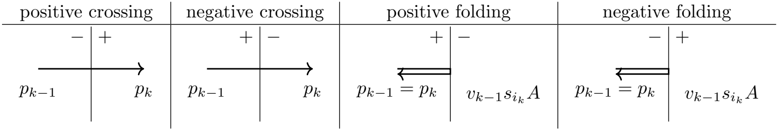
$p_2 := (A, s_1A, s_1s_2A, s_1s_2s_1A, s_1s_2s_1s_0A) \in \Gamma(\overrightarrow{w}, z)$



Colored alcove walks

Crossings and foldings.

- For $p = (p_0, \dots, p_r) \in \Gamma(\overrightarrow{w}, z)$, the k-th step means $p_{k-1} \mapsto p_k$.
- Each step is classified intro four types:



Definition of colored alcove walks.

- $C := \{x = (x_1, \dots, x_n) \in \mathfrak{h}^*_{\mathbb{R}} \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$: the dominant chamber.
- $\Gamma^C(\overrightarrow{w}, z) := \{p = (p_0, \dots, p_r) \in \Gamma(\overrightarrow{w}, z) \mid p_k \subset C, \forall k\}$.
- $\Gamma^C_2(\overrightarrow{w}, z)$: the set of alcove walks $p \in \Gamma^C(\overrightarrow{w}, z)$ with coloring of all the foldings by either black or gray.

Outline of the proof

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x)$$

$E_\mu(x) \in \mathbb{K}[X^{\pm 1}]$: the non-symmetric Koornwinder polynomials [Sahi].

- $E_\mu(x)$ is an eigenfunction of Dunkl operators in the affine Hecke algebra of type C_n.
- $\{E_\lambda(x) \mid \lambda \in \mathfrak{h}_\mathbb{Z}^*\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x^{\pm 1}]$.
- $P_\lambda(x)$ is obtained by symmetrizing $E_\lambda(x)$.

Our proof is a $(C_n^\vee, C_n)$-type analogue of [Yip, 2012].

1. For $\lambda, \mu \in (\mathfrak{h}_\mathbb{Z}^*)_+$, we derive an expansion formula $x^\mu E_\lambda(x) = \sum_{p \in \Gamma^C} c_p E_{\varpi(p)}(x)$.
2. Use Ram-Yip type formula $E_\mu(x) = \sum_{p \in \Gamma} f_p t_{\mathrm{d}(p)}^{\frac{1}{2}} x^{\mathrm{wt}(p)}$.
   This formula was derived by [Orr, Shimozono 2018], based on [Ram, Yip, 2008] for the untwisted affine root systems.
3. Using (1) and (2) we can derive $E_\mu(x)P_\lambda(x) = \sum_{v \in W^\lambda} \sum_{p \in \Gamma_2^C} A_p C_p E_{\varpi(p)}(x)$.
4. Symmetrize (3) and switch $\lambda \leftrightarrow \mu$.

Main result (recollection)

For dominant weights $\lambda, \mu \in (\mathfrak{h}^*_{\mathbb{Z}})_+$, we have

$$P_\lambda(x)P_\mu(x) = \frac{1}{t_{w_\lambda}^{-\frac{1}{2}} W_\lambda(t)} \sum_{v \in W^\mu} \sum_{p \in \Gamma_2^C(\overrightarrow{w(\lambda)}^{-1},(v.w(\mu))^{-1})} A_p B_p C_p P_{-w_0 \mathrm{wt}(p)}(x)$$

where

- p runs over a set $\Gamma_2^C(\overrightarrow{w(\lambda)}^{-1}, (v.w(\mu))^{-1})$ of colored alcove walks, and
- the terms A_p, B_p and C_p are factored.

In the preprint we studied some specializations of these LR-coefficients.

- Rank $n = 1$ case $\iff$ Askey-Wilson polynomials: discussed below.
- Hall-Littlewood limit $q \to 0$.

The case of Askey-Wilson polynomials

One-variable Koornwinder polynomials = Askey-Wilson polynomials.

- $P_l(x) := P_{l\epsilon}(x; q, t_0, t_1, u_0, u_1)$: Askey-Wilson polynomial for $l\epsilon \in \mathfrak{h}_{\mathbb{Z}}^* = \mathbb{Z}\epsilon$.

Consider Pieri coefficients, i.e., LR coefficients for minuscule weights.

- $\epsilon \in (\mathfrak{h}_{\mathbb{Z}}^*)_+ = \mathbb{N}\epsilon$: the unique minuscule weight of type C_n.
- Pieri coefficients are c_m in $P_1(x)P_l(x) = \sum_m c_m P_m(x)$.

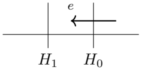
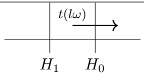
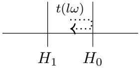
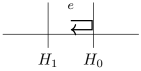
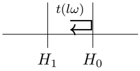
Corollary ([Y., Proposition 4.1.3])

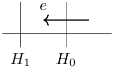
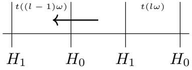
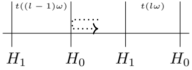
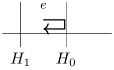
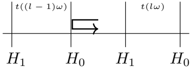
For a dominant weight $\lambda = l\epsilon \in (\mathfrak{h}_{\mathbb{Z}}^*)_+ = \mathbb{N}\epsilon$, we have

$$P_1(x)P_l(x) = P_{l+1}(x) + F_l P_l(x) + G_l P_{l-1}(x),$$

$$\begin{aligned} F_l &:= \rho(-2l\delta + \alpha_1)(-\psi_0^-(q^{2l+1}t_0t_1) + \psi_0^-(qt_0t_1)) \\ &\qquad + \rho(2l\delta - \alpha_1)(-\psi_0^+(q^{2l-1}t_0t_1) + \psi_0^+(qt_0t_1)), \\ G_l &:= \rho(2l\delta - \alpha_1)\rho(-2(l-1)\delta + \alpha_1)n_0(q^{2l-1}t_0t_1). \end{aligned}$$

Askey-Wilson case (cont.)

$v = s_1$				
$p*$	p	A_p	B_p	C_p
		1	1	1
		1	$\rho(-2l\delta + \alpha_1)$	$-\psi_0^-(q^{2l+1}t_0t_1)$
		1	$\rho(-2l\delta + \alpha_1)$	$\psi_0^-(qt_0t_1)$

$v = e$				
$p*$	p	A_p	B_p	C_p
		$\rho(2l\delta - \alpha_1)$	$\rho(-(2l-2)\delta + \alpha_1)$	$n_0(q^{2l-1}t_0t_1)$
		$\rho(2l\delta - \alpha_1)$	1	$-\psi_0^+(q^{2l-1}t_0t_1)$
		$\rho(2l\delta - \alpha_1)$	1	$\psi_0^+(qt_0t_1)$

Askey-Wilson case (cont.)

Correspondence between Askey-Wilson and Koornwinder parameters:

$$(q, a, b, c, d) = (q, q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{\frac{1}{2}}, -q^{\frac{1}{2}} t_0^{\frac{1}{2}} u_0^{-\frac{1}{2}}, t_1^{\frac{1}{2}} u_1^{\frac{1}{2}}, -t_1^{\frac{1}{2}} u_1^{-\frac{1}{2}}).$$

The equation $P_1(x)P_l(x) = P_{l+1}(x) + F_l P_l(x) + G_l P_l(x)$ is rewritten as

$$2zp_l(z) = h_l p_{l+1}(z) + f_l p_l(z) + g_l p_{l-1}(z), \quad p_l((x + x^{-1})/2) = \gamma_l P_l(x)$$

with

$$h_l := \frac{1 - q^{l-1}\pi}{(1 - q^{2l-1}\pi)(1 - q^{2l}\pi)},$$

$$f_l := q^{l-1} \frac{(1 + q^{2l-1}\pi)(qs + \pi s') - q^{l-1}(1 + q)\pi(s + qs')}{(1 - q^{2l-2}\pi)(1 - q^{2l}\pi)},$$

$$g_l := (1 - q^l) \frac{(1 - q^{l-1}ab)(1 - q^{l-1}ac)(1 - q^{l-1}ad)(1 - q^{l-1}bc)(1 - q^{l-1}bd)(1 - q^{l-1}cd)}{(1 - q^{2l-2}\pi)(1 - q^{2l-1}\pi)},$$

$$\pi := abcd, \quad s := a + b + c + d, \quad s' := a^{-1} + b^{-1} + c^{-1} + d^{-1},$$

$$\gamma_l := (q^{l-1}\pi; q)_l = (1 - q^{l-1}\pi)(1 - q^l\pi) \cdots (1 - q^{2l-2}\pi).$$

The rewritten equation coincides with the original recurrence relation of Askey-Wilson polynomials $p_l(z)$ in [Askey, Wilson, 1985].

Specialization to type B, C, and BC

Specializing parameters (t, t_0, t_n, u_0, u_n), we can recover Macdonald polynomials associated to subsystems of the affine root system of type $(C_n^\vee, C_n)$.

reduced	t	t_0	t_n	u_0	u_n	non-reduced	t	t_0	t_n	u_0	u_n
B_n	t_l	1	t_s	1	t_s	(BC_n, C_n)	t_m	t_l^2	$t_s t_l$	1	t_s/t_l
$B_n^\vee$	t_s	1	t_l^2	1	1	$(C_n^\vee, BC_n)$	t_m	t_s	$t_s t_l$	t_s	t_s/t_l
C_n	t_s	t_l^2	t_l^2	1	1	$(B_n^\vee, B_n)$	t_m	1	$t_s t_l$	1	t_s/t_l
$C_n^\vee$	t_l	t_s	t_s	t_s	t_s						
BC_n	t_m	t_l^2	t_s	1	t_s						
D_n	t	1	1	1	1						

Table: Specialization table

We can recover Ram and Yip' result (2011) in the particular case of type B, C, D by specializing Ram-Yip type formula for (non-symmetric) Koornwinder polynomials as follows.

- type B: $t_n = u_n, t_0 = u_0 = 1$
- type C: $t_0 = u_0 = u_n = 1$.
- type D: $t_n = u_n = t_0 = u_0 = 1$.

However, pay attention that the realization of affine root systems in Ram-Yip (2011) is different from our default one in Macdonald (2003).

Conclusions and Remarks

Conclusions:

- We derived an alcove-walk formula of LR coefficients for Koornwinder polynomials, which is a $(C_n^\vee, C_n)$-type analogue of Yip's formula for Macdonald polynomials of untwisted affine root systems.
- We also obtained some specializations of the formula. In particular, in the rank 1 case, we recovered the recurrence formula of Askey-Wilson polynomials.

Problems:

- Simplified formula of Pieri coefficients for Koornwinder polynomials, and its relation to Pieri formulas of type B_n, C_n and BC_n.
- Relation to tableaux formula for Koornwinder polynomials.

Thank you for your attention.

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Rank 2 case

Consider the case $\lambda = \omega_1$, $\mu = \omega_2$.

- $t(\omega_1) = s_0 s_1 s_2 s_1$, $t(\omega_2) = s_0 s_1 s_0 s_2 s_1 s_2$.
- $w(\omega_1) = s_0$, $w(\omega_2) = s_0 s_1 s_0$.
- $W^{\mu} = W^{\omega_2} = \{e, s_2, s_1 s_2, s_2 s_1 s_2\}$.

Proposition (Y., Proposition 4.4.1)

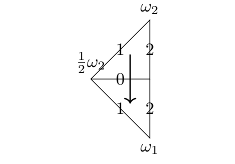
For Koornwinder polynomials of rank 2, we have

$$P_{\omega_1}(x) P_{\omega_2}(x) = P_{\omega_1+\omega_2}(x) + F P_{\omega_2}(x) + G P_{\omega_1}(x),$$

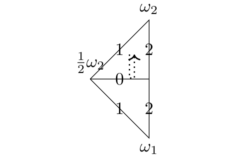
$$F := \rho(-2\delta + \epsilon_1 + \epsilon_2)\rho(-2\delta + 2\epsilon_1)\rho(-\epsilon_1 + \epsilon_2)(-\psi_0^-(q^3 t_0 t_1) + \psi_0^-(q t_0 t_1)),$$

$$G := \rho(2\delta - \epsilon_1 - \epsilon_2)\rho(2\delta - 2\epsilon_2)\rho(-2\epsilon_2)\rho(-\delta + \epsilon_1 + \epsilon_2) n_0(q t_0 t_1)$$

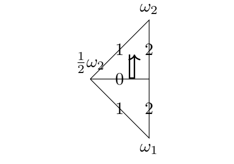
X_{11}



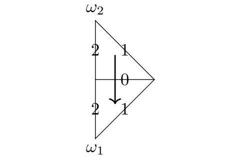
X_{12}



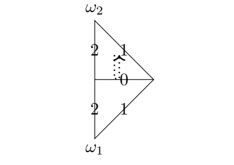
X_2



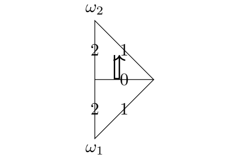
Y_{11}



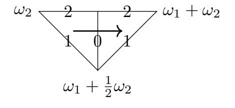
Y_{12}



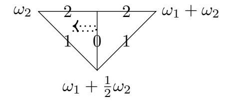
Y_2



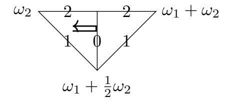
Z_{11}



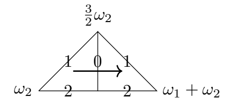
Z_{12}



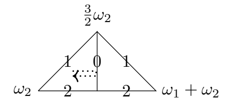
Z_2



W_{11}



W_{12}



W_2