On Tensor Product Measures*

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Consider a spectral decomposition $U(t) = \int_{\mathbb{R}} e^{it\tau} E(d\tau)$ of a one-parameter unitary group $U(t)$ on a Hilbert space $\mathcal{H}$, where $E(\cdot)$ is a projection-valued measure on $\mathbb{R}$. In quantum mechanics, the dynamical behavior of a physical system is described by the associated automorphic action of $\mathbb{R}$ on the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators. The related transition probabilities are associated to $(\xi|U(t)TU(t)^*\eta) = (U(t)^*\xi|TU(t)^*\eta)$ ($\xi, \eta \in \mathcal{H}$, $T \in \mathcal{L}(\mathcal{H})$), which takes the form

$$(\int_{\mathbb{R}} e^{-it\tau} E(d\tau)\xi|T \int_{\mathbb{R}} e^{-it\tau'} E(d\tau')\eta)$$

in terms of the spectral measure. At first glance, it seems quite natural to rewrite this to the product measure form like

$$\int_{\mathbb{R}\times\mathbb{R}} e^{it(\tau-\tau')}(E(d\tau)\xi|TE(d\tau')\eta),$$

which means that we expect a complex-valued measure $(E(d\tau)\xi|TE(d\tau')\eta)$ to be well-defined on $\mathbb{R}^2$. This expectation is reasonably generalized to the following form: Let $T \in \mathcal{L}(\mathcal{H})$ and $\xi(\cdot), \eta(\cdot)$ be $\mathcal{H}$-valued measures on a σ-algebra $\mathcal{B}$ in a set S. It is immediate to check that the map $\mathcal{B} \times \mathcal{B} \ni A \times B \mapsto (\xi(A)|T\eta(B))$ is extended to a finitely additive function μ on the Boolean algebra $\mathcal{B} \otimes \mathcal{B}$ generated by $\mathcal{B} \times \mathcal{B}$. Is it then possible to extend μ to a complex measure on the σ-algebra generated by $\mathcal{B} \times \mathcal{B}$? When T is a finite rank operator, μ certainly admits an extension as a linear combination of product measures and with a litte more effort we can show that the question is yes even for a trace class operator. The general case, however, turns out to be negative: A bounded linear operator T has the measure extension property if and only if T is in the Hilbert-Schmidt class ([Swartz1976, Theorem 8]).

Our main purpose here is to review relevant results on vector-valued measures and try to present what seems to be core to this characterization of T.

Notation and Terminology

Let V be a Banach space and $\mathcal{B} \subset 2^S$ be a Boolean algebra in a set S. Consider a map $\phi : \mathcal{B} \to V$.

- ϕ is called a **semi-measure** if it is finitely additive.
- ϕ is said to be σ-additive if $A = \bigsqcup_{n\geq 1} A_n$ in $\mathcal{B}$ implies $\phi(A) = \sum_{n\geq 1} \phi(A_n)$ (norm-summable).

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- ϕ is called a **measure** if $\mathcal{B}$ is a σ-algebra and ϕ is σ-additive.
- Given a semi-measure ϕ, its **semi-variation** $|\phi| : \mathcal{B} \to [0, \infty]$ is defined by

$$|\phi|(A) = \sup\{\|\sum \alpha_j \phi(A_j)\|; A = \bigsqcup_{j=1}^{n} A_j \text{ in } \mathcal{B} \text{ and } \alpha_j \in \mathbb{C} \text{ satisfying } |\alpha_j| \leq 1\}.$$

- A semi-measure ϕ (or a semi-variation $|\phi|$) is said to be **squeezing** if $\bigsqcup A_n$ in $\mathcal{B}$ (not assumed $\bigsqcup A_n \in \mathcal{B}$) implies $\lim_{n\to\infty} \phi(A_n) = 0$ (or $\lim_{n\to\infty} |\phi|(A_n) = 0$).
- The complex vector space of simple functions based on a Boolean algebra $\mathcal{B}$ is denoted by $\mathbb{C}(\mathcal{B})$.

Facts:

- There is a one-to-one correspodence between semi-measures and linear maps $\mathbb{C}(\mathcal{B}) \to V$ so that $\|\phi\| = |\phi|(S)$.
- $|\phi|$ is monotone and subadditive: $|\phi|(A) \leq |\phi|(A \cup B) \leq |\phi|(A) + |\phi|(B)$ for $A, B \in \mathcal{B}$.
- $\forall B \in \mathcal{B}$, $\sup\{\|\phi(A)\|; A \subset B, A \in \mathcal{B}\} \leq |\phi|(B) \leq \pi \sup\{\|\phi(A)\|; A \subset B, A \in \mathcal{B}\}$.
- ϕ is a measure $\Longrightarrow$ (ϕ is squeezing $\iff$ $|\phi|$ is squeezing) $\Longrightarrow$ $\|\phi\| < \infty$.
- If ϕ is a complex measure, $|\phi|$ is a positive measure.
- For $A \in \mathcal{B}$, $|\phi|(A) = \sup\{|v^*(\phi)|(A); v^* \in V^*, \|v^*\| \leq 1\}$. Here $v^*(\phi) : \mathcal{B} \ni A \mapsto v^*(\phi(A)) \in \mathbb{C}$.

Example 0.1. A semi-measure $\phi : \mathcal{B} \to V$ with V a Hilbert space is said to be orthogonal if $\phi(A) \perp \phi(B)$ for $A \cap B = \emptyset$. For an orthogonal semi-measure ϕ, $|\phi|(A) = \|\phi(A)\|$, which is not additive unless ϕ is supported by an atomic set in $\mathcal{B}$.

Given a bounded set Λ of complex semi-measures on a σ-algebra $\mathcal{B}$, defined a semi-measure $\phi_\Lambda : \mathcal{B} \to \ell^\infty(\Lambda)$ by $\phi_\Lambda(A) = \{\lambda(A)\}_{\lambda\in\Lambda}$. Then $\|\phi_\Lambda\| = \sup\{\|\lambda\|; \lambda \in \Lambda\}$ and $|\phi_\Lambda| = |\phi_{|\Lambda|}|$ with $|\Lambda| = \{|\lambda|; \lambda \in \Lambda\}$.

Proposition 0.2. Consider the following conditions on a bounded set Λ of complex semi-measures on a Boolean algebra $\mathcal{B}$.

(i) ϕ_Λ is squeezing.
(ii) ϕ_Λ is a measure.
(iii) $\phi_{|\Lambda|}$ is squeezing.
(iv) $\phi_{|\Lambda|}$ is a measure.

(i) and (iii) are equivalent. If $\mathcal{B}$ is a σ-algebra and Λ consists of complex measures, all the conditions (i) $\sim$ (iv) are equivalent.

Let μ be a finite positive semi-measure on a Boolean-algebra $\mathcal{B}$. A vector semi-measure ϕ on $\mathcal{B}$ is said to be ***μ*-continuous** if $\forall \epsilon > 0, \exists \delta > 0, \forall A \in \mathcal{B}, \mu(A) \leq \delta \Longrightarrow \|\phi(A)\| \leq \epsilon$.

Theorem 0.3 (Pettis1938)**.** Suppose that both of ϕ and μ are measures ($\mathcal{B}$ being a σ-algebra necessarily). Then ϕ is μ-continuous if and only if $\mu(A) = 0$ ($A \in \mathcal{B}$) implies $\phi(A) = 0$.

Theorem 0.4 (Doubrovsky1947)**.** Let Λ be a bounded set of complex measures on a σ-algebra $\mathcal{B}$. If

$\phi_\Lambda : \mathcal{B} \to \ell^\infty(\Lambda)$ is a measure, there exists a positive measure $\mu \in L^1(\mathcal{B})$ for which ϕ_Λ is μ-continuous with a reverse inequality $\mu(A) \leq |\phi_\Lambda|(A) = \sup\{|\lambda|(A); \lambda \in \Lambda\}$.

Corollary 0.5 (Bartle-Dunford-Schwartz1955)**.** For a vector measure ϕ on a σ-algebra, we can find a finite positive measure μ so that ϕ is μ-continuous and μ is majorized by $|\phi|$.

Recall that countable additivity of a semi-measure $\mu : \mathcal{B} \to [0, \infty)$ is equivalent to the condition that, if $A_n \downarrow \emptyset$ in $\mathcal{B}$, then $\mu(A_n) \downarrow 0$. Let $\mathcal{B}^\sigma$ be the σ-algebra generated by $\mathcal{B}$. The classical extension theorem says that, if a finite positive semi-measure on $\mathcal{B}$ is σ-additive, it is uniquely extended to a positive measure on $\mathcal{B}^\sigma$.

In the framework of Daniell integral (see [8] for example), this can be explained in the following fashion: Let L be the vector lattice of real-valued simple functions on the base set S. Then a semi-measure μ can be interpreted as a positive linear functional $L \to \mathbb{R}$, which is also denoted by μ. Let $f_n \downarrow 0$ in L. Then, given $\epsilon > 0$, $[f_n \geq \epsilon] \downarrow \emptyset$ in $\mathcal{B}$ and $\mu(f_n) \leq \|f_1\|_\infty \mu([f_n \geq \epsilon]) + \epsilon\mu(S)$, together with the continuity of μ imply $\lim_n \mu(f_n) \leq \epsilon\mu(S)$. Thus, μ is continuous as a linear functional and we can apply the whole construction of Daniell integral to get a measure extension to $\mathcal{B}^\sigma$.

Theorem 0.6 (Kluvánek1961)**.** Let $\mu : \mathcal{B} \to [0, \infty)$ be a countably additive semi-measure on a Boolean algebra $\mathcal{B}$. Let $\phi : \mathcal{B} \to V$ be a semi-measure on a Boolean algebra $\mathcal{B}$. Then any μ-continuous semimeasure $\phi : \mathcal{B} \to V$ is extended to a measure $\mathcal{B}^\sigma \to V$ on the σ-algebra $\mathcal{B}^\sigma$ generated by $\mathcal{B}$.

Cross Norms

Let V and W be Banach spaces. The lower and maximal cross norms are introduced by the embeddings $V \otimes W \subset \mathcal{B}(V^*, W^*)$ and $\mathcal{B}(V, W)^*$ with the associated completions denoted by $V \underline{\otimes} W$ and $V \overline{\otimes} W$ respectively. Here $\mathcal{B}(V, W)$ denotes the Banach space of bounded bilinear forms on $V \times W$.

Here are more cross norms majorizing the least one: $\|\cdot\|_m = (\|\cdot\|_l + \|\cdot\|_r)/2$ (or any convex combination gives an equivalent cross norm) with

$$\|z\|_r = \inf\Big\{\sup\{\|\sum_i \alpha_i \|x_i\| y_i\|; |\alpha_i| \leq 1\}; z = \sum_{i=1}^n x_i \otimes y_i\Big\},$$
$$\|z\|_l = \inf\Big\{\sup\{\|\sum_i \alpha_i \|y_i\| x_i\|; |\alpha_i| \leq 1\}; z = \sum_{i=1}^n x_i \otimes y_i\Big\}.$$

Theorem 0.7 (Kluvánek1973)**.** Let $\varphi : \mathcal{A} \to V$ and $\psi : \mathcal{B} \to W$ be measures and $V \otimes_m W$ be the completion of $V \otimes W$ with respect to the cross norm $\|\cdot\|_m$. Then there exists a measure $\phi : \mathcal{A} \otimes_\sigma \mathcal{B} \to V \otimes_m W$ satisfying $\phi(A \times B) = \varphi(A) \otimes \psi(B)$ for $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Corollary 0.8 (Duchon-Kluvánek1967)**.** Tensor product measures are defined with respect to the least cross norm.

The obivious embedding of $\ell^p \otimes V$ into the Banach space $\ell^p(V) = \{(v_n); \sum_n \|v_n\|^p < \infty\}$ of ℓ^p-sequences in V has a dense range and $\ell^p(V)$ is identified with the completion $\ell^p \otimes V$ relative to a cross norm on $\ell^p \otimes V$: We have natural inclusions of Banach spaces $\ell^p \overline{\otimes} V \subset \ell^p(V) \subset \ell^p \underline{\otimes} V$.

A bounded linear map $T : V \to W$ is said to be ***p*-summing** if there exists a bounded linear map $\mathcal{L}^p(T) : \ell^p \underline{\otimes} V \to \ell^p(W)$ which makes the following diagram commutative.

$$\begin{array}{ccc} \ell^p(V) & \xrightarrow{\ell^p(T)} & \ell^p(W) \\ \downarrow & & \uparrow \mathcal{L}^p(T) \\ \ell^p \underline{\otimes} V & = & \ell^p \underline{\otimes} V \end{array}$$

Here $\ell^p(T)$ denotes the bounded linear map $(v_n) \mapsto (Tv_n)$. The set $\mathcal{L}^p(V, W)$ of p-summing linear maps is a Banach space with respect to the norm $\|\mathcal{L}^p(T)\|$.

Example 0.9. When V and W are Hilbert spaces, $\mathcal{L}^2(V, W) = \mathcal{C}_2(V, W)$ so that $\|\mathcal{L}^2(T)\|$ is equal to the Hilbert-Schmidt norm of $T : V \to W$.

Facts

- For $1 \leq p \leq q < \infty$, $\mathcal{L}^p(V, W) \subset \mathcal{L}^q(V, W)$ so that $\|\mathcal{L}^q(T)\| \leq \|\mathcal{L}^p(T)\|$ for $T \in \mathcal{L}^p(V, W)$.
- $\mathcal{L}(W, W') \circ \mathcal{L}^p(V, W) \circ \mathcal{L}(V', V) \subset \mathcal{L}^p(V', W')$ with $\|\mathcal{L}^p(bTa)\| \leq \|b\| \, \|\mathcal{L}^p(T)\| \, \|\|a\|$.
- The natural inclusion $\ell^1 \subset \ell^2$ is 1-summing (a rewriting of Khintchine's inequality).
- If V and W are Hilbert spaces, $\mathcal{L}^p(V, W) = \mathcal{L}^2(V, W)$ for $1 \leq p \leq 2$ and $\|\mathcal{L}^2(T)$ is exactly the Hilbert-Schmidt norm.

Theorem 0.10 (Swartz 1976)**.** A linear functional $\varphi : V \otimes W \to \mathbb{C}$ is $\| \cdot \|$-bounded if and only if the associated operator $\Phi : V \to W^*$ is 1-summing with $\|\varphi\|_l = \|\mathcal{L}^1(\Phi)\|$.

Corollary 0.11. Given measures $\varphi : \mathcal{A} \to V$ and $\psi : \mathcal{B} \to W$ with V and W Hilbert spaces, tensor product semi-measure $\mathcal{A} \otimes_\sigma \mathcal{B} \to V \otimes W$ is lifted to a measure $\mathcal{A} \otimes_\sigma \mathcal{B} \to V \otimes_2 W$, where $V \otimes_2 W$ denotes the ordinary Hilbert space tensor product.

Theorem 0.12 (Swartz?)**.** Let $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear map between Hilbert spaces. Then the following conditions are equivalent.

(i) Given an $\mathcal{H}$-valued measure ξ on $\mathcal{A}$ and a $\mathcal{K}$-valued measure η on $\mathcal{B}$, the semi-measure $(\xi|T\eta)$ on $\mathcal{A} \otimes \mathcal{B}$ is extended to a complex measure on $\mathcal{A} \otimes_\sigma \mathcal{B}$.

(ii) Given a $\overline{T\mathcal{H}}$-valued orthogonal measure ξ on $2^{\mathbb{N}}$ and a $\ker(T)^\perp$-valued orthogonal measure η on $2^{\mathbb{N}}$, the semi-measure $(\xi|T\eta)$ on $2^{\mathbb{N}} \otimes 2^{\mathbb{N}}$ is bounded.

(iii) T is in the Hilbert-Schmidt class.

Proof. (i) $\Longrightarrow$ (ii) is obvious, while (iii) $\Longrightarrow$ (i) is a consequence of the above Corollary. (ii) $\Longrightarrow$ (iii) can be checked by utilizing the lemma below. □

Lemma 0.13. Let $\mathcal{H}$ be a finite-dimensional Hilbert space and T be a positive operator on $\mathcal{H}$. Then we can find orthogonal measures $\xi, \eta : 2^{\mathbb{N}} \to \mathcal{H}$ satisfying $\|\xi\| = 1 = \|\eta\|$ and $\|(\xi|T\eta)\| = \|T\|_2$.

Here the complex semi-measure $(\xi|T\eta)$ on $2^{\mathbb{N}} \otimes 2^{\mathbb{N}} \subset 2^{\mathbb{N}\times\mathbb{N}}$ is specified by $(\xi|T\eta)(A \times B) = (\xi(A)|T\eta(B))$ for $A, B \in 2^{\mathbb{N}}$ and $\|T\|_2$ denotes the Hilbert-Schmidt norm of T.

Proof. Since the Boolean algebra $2^{\mathbb{N}} \otimes 2^{\mathbb{N}}$ is atomic,

$$\|(\xi|T\eta)\| = \sum_{j,k} |(\xi_j|T\eta_k)|, \quad \xi_j = \xi(\{j\}), \eta_k = \eta(\{k\}).$$

We now restrict ξ and η to be supported by the set $\{1, 2, \dots, \dim \mathcal{H}\} \subset \mathbb{N}$ and choose orthonormal bases $\{e_j\}$ and $\{f_j\}$ in $\mathcal{H}$ so that $\xi_j = \|\xi_j\| e_j$ and $\eta_j = \|\eta_j\| f_j$ for $1 \leq j \leq \dim \mathcal{H}$. Then, under the condition $\|\xi\| = \|\eta\| = 1$, orthogonal measures ξ and η are compactly parametrized and the problem is reduced to showing that $\|T\|_2$ is realized as

$$\max\{\sum_{j,k} \|\xi_j\| \, |(e_j|Tf_k)| \, \|\eta_k\|; \sum_j \|\xi_j\|^2 = 1 = \sum_k \|\eta_k\|^2\} = \|[e|Tf]\|$$

for some orthonomal bases $\{e_j\}$, $\{f_k\}$ of $\mathcal{H}$. Here $\|[e|Tf]\|$ denotes the operator norm of the matrix $[e|Tf] = (|(e_j|Tf_k)|)$.

Let $T = \sum_{1 \leq j \leq \dim \mathcal{H}} t_j |g_j)(g_j|$ be a spectral expression with $\{g_j\}$ an orthonormal basis. If we set $f_j = g_j$, then $|(e_j|Tf_k)| = |(e_j|g_k)| t_k$ and $(e_j|g_k)$ can be any unitary matrix, which allows us to choose $(e_j|g_k) = e^{2\pi ijk/n}/\sqrt{n}$ and get $\|[e|Tg]\| = \sqrt{t_1^2 + \cdots + t_n^2} = \|T\|_2$:

$$[e|Tg] = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} t_1 & \cdots & t_n \end{pmatrix}$$

with the norm of the last matrix equal to $\|(t_1, \cdots, t_n)\| = \sqrt{t_1^2 + \cdots + t_n^2}$. □

Remark 1. For a real Hilbert space of $\dim \mathcal{H} = 2^m$, the conclusion of Lemma remains true by taking $(e_j|e_k)$ to be the m-times tensor product of two-dimensional reflection (or rotation) matrix by an angle $\pi/4$ as utilized in [Dudley-Pakula1972].

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