CLASSIFYING SUBCATEGORIES OF MODULES OVER A COMMUTATIVE NOETHERIAN RING

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Abstract. Let $R$ be a quotient ring of a commutative coherent regular ring by a finitely generated ideal. Hovey gave a bijection between the set of coherent subcategories of the category of finitely presented $R$-modules and the set of thick subcategories of the derived category of perfect $R$-complexes. Using this isomorphism, he proved that every coherent subcategory of finitely presented $R$-modules is a Serre subcategory. In this paper, it is proved that this holds whenever $R$ is a commutative noetherian ring. This paper also yields a module version of the bijection between the set of localizing subcategories of the derived category of $R$-modules and the set of subsets of Spec $R$ which was given by Neeman.

1. Introduction

Around 1990, Hopkins [6] and Neeman [9] gave a classification theorem of the thick subcategories of the derived category of perfect complexes (i.e. finite complexes of finitely generated projective modules) over a commutative noetherian ring in terms of the ring spectrum. After that, Thomason [10] generalized this classification theorem to quasi-compact and quasi-separated schemes, in particular, to arbitrary commutative rings. Let $D_{\text{perf}}(R)$ denote the derived category of perfect complexes over a commutative ring $R$. The classification theorem (for commutative rings) can be stated as follows.

Theorem (Hopkins-Neeman-Thomason). Let $R$ be a commutative ring. Then there is an isomorphism

$$
\{ \text{thick subcategories of } D_{\text{perf}}(R) \} \\
\cong \{ \text{complements of intersections of quasi-compact open subsets of Spec } R \}
$$

of lattices.

Here we recall the definitions of several subcategories of an abelian category. A coherent subcategory is defined to be a full subcategory which is closed under kernels, cokernels and extensions. A Serre subcategory is defined to be a coherent subcategory which is closed under subobjects. A torsion class is defined to be a Serre subcategory which is closed under arbitrary direct sums. Let $R$ be a commutative ring. We denote by $\text{Mod } R$ the category of $R$-modules and by $\text{mod } R$ the full subcategory of finitely presented $R$-modules. If $R$ is noetherian, then the lattice of

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Serre subcategories of \( \text{mod} \ R \), the lattice of torsion classes of \( \text{Mod} \ R \), and the lattice of subsets of \( \text{Spec} \ R \) which are closed under specialization are isomorphic to each other. Taking advantage of the Hopkins-Neeman-Thomason theorem, Garkusha and Prest \([3, 4]\) gave the following result very recently. (A torsion class \( \mathcal{X} \) of \( \mathcal{M} = \text{Mod} \ R \) is said to be of finite type if the inclusion functor \( \mathcal{M}/\mathcal{X} \to \mathcal{M} \), where \( \mathcal{M}/\mathcal{X} \) denotes the quotient category, preserves arbitrary direct sums.)

**Theorem** (Garkusha-Prest). Let \( R \) be a commutative ring. Then the following hold.

1. One has lattice isomorphisms
   \[
   \{ \text{thick subcategories of } \mathcal{D}_{\text{perf}}(R) \} \cong \{ \text{torsion classes of finite type of } \text{Mod} \ R \} \cong \{ \text{complements of intersections of quasi-compact open subsets of } \text{Spec} \ R \}.
   \]
2. Suppose that \( R \) is coherent. Then one has lattice isomorphisms
   \[
   \{ \text{thick subcategories of } \mathcal{D}_{\text{perf}}(R) \} \cong \{ \text{Serre subcategories of } \text{mod} \ R \} \cong \{ \text{complements of intersections of quasi-compact open subsets of } \text{Spec} \ R \}.
   \]

Also by using the Hopkins-Neeman-Thomason theorem, Hovey \([7]\) proved the following classification theorem of coherent subcategories.

**Theorem** (Hovey). Let \( R \) be a quotient ring of a commutative coherent regular ring by a finitely generated ideal. Then the following hold.

1. One has a lattice isomorphism
   \[
   \{ \text{thick subcategories of } \mathcal{D}_{\text{perf}}(R) \} \cong \{ \text{coherent subcategories of } \text{mod} \ R \}.
   \]
2. Every coherent subcategory of \( \text{mod} \ R \) is a Serre subcategory.

As Hovey pointed out as an interesting fact, there was no direct proof of the second assertion of the above theorem; it could not be proved without resorting to the rather difficult classification of thick subcategories of the derived category, namely the Hopkins-Neeman-Thomason theorem. Hovey conjectures that the isomorphism stated in the above theorem always holds for commutative coherent rings. (Recall that a commutative ring is called coherent if every finitely generated ideal is finitely presented.)

**Conjecture** (Hovey). Let \( R \) be a commutative coherent ring. Then one has a lattice isomorphism

\[
\{ \text{thick subcategories of } \mathcal{D}_{\text{perf}}(R) \} \cong \{ \text{coherent subcategories of } \text{mod} \ R \}.
\]

One of the main purposes of this paper is to prove that this conjecture is true if \( R \) is noetherian. First of all, we will directly prove that every coherent subcategory of \( \text{mod} \ R \) is Serre; in the proof, we will not apply the Hopkins-Neeman-Thomason theorem. Actually, we will not use the notion of a derived category in the proof. Furthermore, the proof we will give is much simpler than Hovey’s. After that, we
will prove that if \( R \) is noetherian, then all the lattices appearing in the above part are isomorphic to each other. Our first main theorem is the following.

**Theorem A.** Let \( R \) be a commutative noetherian ring. Then every coherent subcategory of \( \text{mod} \, R \) is a Serre subcategory, and one has the following isomorphisms of lattices:

\[
\begin{align*}
\{ \text{thick subcategories of } \mathcal{D}_{\text{perf}}(R) \} & \cong \{ \text{coherent subcategories of } \text{mod} \, R \} \\
& = \{ \text{Serre subcategories of } \text{mod} \, R \} \\
& \cong \{ \text{torsion classes of } \text{Mod} \, R \} \\
& \cong \{ \text{subsets of } \text{Spec} \, R \text{ closed under specialization} \} \\
& = \{ \text{complements of intersections of quasi-compact open subsets of } \text{Spec} \, R \}.
\end{align*}
\]

On the other hand, Neeman \[9\] showed the following theorem.

**Theorem (Neeman).** Let \( R \) be a commutative noetherian ring. Then one has an isomorphism

\[
\begin{align*}
\{ \text{localizing subcategories of } \mathcal{D}(R) \} & \cong \{ \text{subsets of } \text{Spec} \, R \}
\end{align*}
\]

of lattices. Moreover, this induces an isomorphism

\[
\begin{align*}
\{ \text{smashing subcategories of } \mathcal{D}(R) \} & \cong \{ \text{subsets of } \text{Spec} \, R \text{ closed under specialization} \}
\end{align*}
\]

of lattices.

Here, \( \mathcal{D}(R) \) denotes the derived category of \( \text{Mod} \, R \). A localizing subcategory is defined as a full triangulated subcategory which is closed under arbitrary direct sums, and a smashing subcategory is defined as a localizing subcategory such that Bousfield localization commutes with arbitrary direct sums. The second main purpose of this paper is to construct a module version of the above Neeman’s theorem:

**Theorem B.** Let \( R \) be a commutative noetherian ring. Then one has an isomorphism

\[
\begin{align*}
\{ \text{full subcategories of } \text{mod} \, R \text{ closed under submodules and extensions} \} & \cong \{ \text{subsets of } \text{Spec} \, R \}
\end{align*}
\]

of lattices. Moreover, this induces the isomorphism

\[
\begin{align*}
\{ \text{Serre subcategories of } \text{mod} \, R \} & \cong \{ \text{subsets of } \text{Spec} \, R \text{ closed under specialization} \}
\end{align*}
\]

of lattices given in Theorem A.

In Section 2, we will give the precise definitions of subcategories which are stated above, and study several basic properties of them. In Sections 3 and 4, we shall give proofs of Theorems A and B, respectively.
Remark. It is known that Hovey’s paper [7] contains an error, but it is not relevant to the results and arguments in this paper. The error has recently been corrected by Krause [8].

2. Basic properties

In this section, we will give some definitions and several basic results most of which are necessary to state and prove the main results of this paper.

Let $\mathcal{A}$ be an additive category. A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is said to be closed under isomorphisms (or replete) provided that if $X$ is an object of $\mathcal{X}$ and $Y$ is an object of $\mathcal{A}$ which is isomorphic to $X$, then $Y$ is also an object of $\mathcal{X}$. In this paper, by a subcategory we always mean a nonempty full subcategory which is closed under isomorphisms.

First of all, we recall the definitions of various types of closedness of a subcategory of an additive category.

**Definition 2.1.** Let $\mathcal{A}$ be an additive category, and let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We say that

1. $\mathcal{X}$ is closed under subobjects (resp. closed under quotient objects) provided that if $X$ is an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ is a subobject (resp. a quotient object) of $X$, then $Y$ is also an object of $\mathcal{X}$.
2. $\mathcal{X}$ is closed under direct summands (or closed under retracts) provided that if $X$ is an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ is a direct summand of $X$, then $Y$ is also an object of $\mathcal{X}$.
3. $\mathcal{X}$ is closed under finite direct sums (resp. closed under arbitrary direct sums) if all finite direct sums (resp. arbitrary direct sums) of objects of $\mathcal{X}$ are objects of $\mathcal{X}$.
4. $\mathcal{X}$ is closed under extensions provided that for any exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, if both $A$ and $C$ are objects of $\mathcal{X}$, then so is $B$.
5. $\mathcal{X}$ is closed under kernels (resp. closed under images, closed under cokernels) if the kernel (resp. the image, the cokernel) of every morphism of objects of $\mathcal{X}$ is also an object of $\mathcal{X}$.
6. $\mathcal{X}$ is closed under homologies if the homologies of every chain complex of objects of $\mathcal{X}$ are objects of $\mathcal{X}$.
7. $\mathcal{X}$ is closed under direct limits provided that if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a direct system of objects of $\mathcal{X}$ then the direct limit $\lim_{\rightarrow \lambda \in \Lambda} X_\lambda$ is an object of $\mathcal{X}$.

Here we study the relationships among the closed properties of a subcategory which are defined above.

**Proposition 2.2.** Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a subcategory of $\mathcal{A}$. Then the following hold.

1. If $\mathcal{X}$ is closed under kernels or cokernels, then $\mathcal{X}$ is closed under direct summands and contains the zero object of $\mathcal{A}$.
2. If $\mathcal{X}$ is closed under extensions, then $\mathcal{X}$ is closed under finite direct sums.
3. If $\mathcal{X}$ is closed under kernels and cokernels, then $\mathcal{X}$ is closed under images and homologies.
4. If $\mathcal{X}$ is closed under subobjects and cokernels, then $\mathcal{X}$ is closed under quotient objects.
5. If $\mathcal{X}$ is closed under arbitrary direct sums and quotient objects, then $\mathcal{X}$ is closed under direct limits.
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Proof. (1) Let $X$ be an object of $\mathcal{X}$ and $Y$ an object of $\mathcal{A}$ which is a direct summand of $X$. Then we can write $X = Y \oplus Z$ for some object $Z$ of $\mathcal{A}$. Considering the morphism $f : X \to X$ given by $(y, z) \mapsto (0, z)$ for $y \in Y$ and $z \in Z$, we see that both the kernel and the cokernel of $f$ are isomorphic to $Y$. By the assumption that $\mathcal{X}$ is closed under kernels or cokernels, the object $Y$ is in $\mathcal{X}$. Therefore $\mathcal{X}$ is closed under direct summands.

On the other hand, since $\mathcal{X}$ is nonempty, there exists an object $W \in \mathcal{X}$. Let $g : W \to W$ be the identity morphism. Then both the kernel and the cokernel of $g$ are the zero object. Since $\mathcal{X}$ is closed under kernels or cokernels, $\mathcal{X}$ contains the zero object.

(2) Let $X$ and $Y$ be two objects of $\mathcal{X}$. Then there exists a natural split exact sequence $0 \to X \to X \oplus Y \to Y \to 0$ in $\mathcal{A}$. Since $\mathcal{X}$ is closed under extensions, we see from this exact sequence that the direct sum $X \oplus Y$ is an object of $\mathcal{X}$. This argument shows that $\mathcal{X}$ is closed under finite direct sums.

(3) Let $f : X \to Y$ be a morphism of objects of $\mathcal{X}$. Then there is an exact sequence $0 \to \text{Im } f \to Y \xrightarrow{\pi} \text{Coker } f \to 0$ in $\mathcal{A}$. Since $\mathcal{X}$ is closed under cokernels, the object Coker $f$ is in $\mathcal{X}$. Noting that the object Im $f$ is the kernel of $\pi$ and $\mathcal{X}$ is closed under kernels, we see that Im $f$ is in $\mathcal{X}$. Therefore $\mathcal{X}$ is closed under images. That $\mathcal{X}$ is closed under kernels and images implies that $\mathcal{X}$ is closed under homologies.

(4) Let $X$ be an object of $\mathcal{X}$ and $Y \in \mathcal{A}$ a quotient object of $X$. Then there exists a subobject $Z \in \mathcal{A}$ of $X$ such that $Y = X/Z$. Since $\mathcal{X}$ is closed under subobjects, $Z$ is an object of $\mathcal{X}$. Let $i : Z \to X$ be the natural inclusion. Since $Y$ coincides with the cokernel of $i$ and $\mathcal{X}$ is closed under cokernels, we have $Y \in \mathcal{X}$. This says that $\mathcal{X}$ is closed under quotient objects.

(5) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a direct system of objects of $\mathcal{X}$, and let $X = \varinjlim_{\lambda \in \Lambda} X_\lambda$ be the direct limit. Then, by definition, $X$ is a quotient object of $Y := \bigoplus_{\lambda \in \Lambda} X_\lambda$. Since $\mathcal{X}$ is closed under arbitrary direct sums, $Y$ is an object of $\mathcal{X}$. Since $\mathcal{X}$ is closed under quotient objects, $X$ is an object of $\mathcal{X}$. Consequently, $\mathcal{X}$ is closed under direct limits. \qed

Next, we recall the definitions of a coherent subcategory, a Serre subcategory and a torsion class, which will play important roles throughout this paper.

Definition 2.3. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. Then

1. $\mathcal{X}$ is called a coherent subcategory of $\mathcal{A}$ if it is closed under kernels, cokernels and extensions.
2. $\mathcal{X}$ is called a Serre subcategory of $\mathcal{A}$ if it is a coherent subcategory which is closed under subobjects.
3. $\mathcal{X}$ is called a (hereditary) torsion class of $\mathcal{A}$ if it is a Serre subcategory which is closed under arbitrary direct sums.

Remark 2.4. The original definition of a coherent subcategory is as follows: let $\mathcal{A}$ be an abelian category, and let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. It is said that $\mathcal{X}$ is
coherent provided that for any exact sequence
\[ A \to B \to C \to D \to E \]
in \( \mathcal{A} \), if \( A, B, D \) and \( E \) are in \( \mathcal{A} \), then so is \( C \). One can easily check that this definition is equivalent to our definition.

A coherent subcategory, a Serre subcategory and a torsion class have the following properties, which immediately follow from Proposition 2.2.

**Corollary 2.5.** Let \( \mathcal{A} \) be an abelian category.

1. Let \( \mathcal{X} \) be a coherent subcategory of \( \mathcal{A} \). Then \( \mathcal{X} \) contains the zero object of \( \mathcal{A} \), and \( \mathcal{X} \) is closed under finite direct sums, direct summands, images and homologies.
2. Let \( \mathcal{X} \) be a Serre subcategory of \( \mathcal{A} \). Then \( \mathcal{X} \) is closed under quotient objects.
3. Let \( \mathcal{X} \) be a torsion class of \( \mathcal{A} \). Then \( \mathcal{X} \) is closed under direct limits.

Throughout the rest of this paper, let \( R \) be a commutative ring. We denote by \( \text{Mod} \, R \) the category of \( R \)-modules, and by \( \text{mod} \, R \) the full subcategory of \( \text{Mod} \, R \) consisting of finitely presented \( R \)-modules. Let us recall the definitions of a lattice and a homomorphism of lattices.

**Definition 2.6.**

1. Let \( \mathcal{L} \) be an ordered set.
   
   - Let \( x, y \in \mathcal{L} \) be elements. If the supremum (resp. infimum) of the set \( \{ x, y \} \) exists, then it is called the **join** (resp. **meet**) of \( x \) and \( y \), and denoted by \( x \lor y \) (resp. \( x \land y \)).
   
   - It is said that \( \mathcal{L} \) is a **lattice** if any two elements of \( \mathcal{L} \) have both the join and the meet.

2. A map \( f : \mathcal{L} \to \mathcal{L}' \) of lattices is called a **(lattice) homomorphism** if \( f(x \lor y) = f(x) \lor f(y) \) and \( f(x \land y) = f(x) \land f(y) \) for all \( x, y \in \mathcal{L} \). A bijective lattice homomorphism is called a **(lattice) isomorphism**.

This paper will deal with the following lattices of subcategories of modules.

**Definition 2.7.**

1. Let \( \text{coh}(R) \) be the coherent subcategory of \( \text{Mod} \, R \) generated by \( R \). We denote by \( \mathcal{L}_{\text{coh}}(\text{coh}(R)) \) the lattice of all coherent subcategories of \( \text{coh}(R) \).
2. Let \( \text{Serre}(R) \) be the Serre subcategory of \( \text{Mod} \, R \) generated by \( R \). We denote by \( \mathcal{L}_{\text{Serre}}(\text{Serre}(R)) \) the lattice of all Serre subcategories of \( \text{Serre}(R) \).
3. We denote by \( \mathcal{L}_{\text{tors}}(\text{Mod} \, R) \) the lattice of all torsion classes of \( \text{Mod} \, R \).

**Remark 2.8.** If \( R \) is noetherian, then one has \( \text{coh}(R) = \text{Serre}(R) = \text{mod} \, R \); see the first sentence and the latter half of Page 3185 of [7]. Hence, whenever \( R \) is a noetherian ring, one has \( \mathcal{L}_{\text{coh}}(\text{coh}(R)) = \mathcal{L}_{\text{coh}}(\text{mod} \, R) \) and \( \mathcal{L}_{\text{Serre}}(\text{Serre}(R)) = \mathcal{L}_{\text{Serre}}(\text{mod} \, R) \).

A **perfect** \( R \)-complex \( P_\bullet \) is defined to be an \( R \)-complex of the form
\[ P_\bullet = (0 \to P_s \to P_{s-1} \to \cdots \to P_{t+1} \to P_t \to 0), \]
where each \( P_i \) is a finitely generated projective \( R \)-module. We denote by \( \mathcal{D}(R) \) the derived category of the category \( \text{Mod} \, R \), and by \( \mathcal{D}_{\text{perf}}(R) \) the full subcategory of \( \mathcal{D}(R) \) consisting of \( R \)-complexes which are isomorphic to perfect \( R \)-complexes.
Remark 2.9. Recall that an object $X$ of an additive category $A$ is called small (or compact) if the functor $\text{Hom}_A(X, -)$ preserves arbitrary direct sums. It is well-known that a complex of $R$-modules is quasi-isomorphic to a perfect complex if and only if it is a small object of $\mathcal{D}(R)$ (cf. [2, 3.7]).

Definition 2.10. (1) Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ a subcategory of $\mathcal{T}$. Then we say that $\mathcal{X}$ is a thick subcategory if it satisfies the following two conditions.

(a) $\mathcal{X}$ is closed under direct summands.

(b) For any exact triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$ in $\mathcal{T}$, if two of the objects $A, B, C$ are in $\mathcal{X}$, then so is the third.

(2) We denote by $\mathcal{L}_{\text{thick}}(\mathcal{D}_{\text{perf}}(R))$ the lattice of all of the thick subcategories of $\mathcal{D}(R)$ whose objects are small, namely, the lattice of all thick subcategories of $\mathcal{D}_{\text{perf}}(R)$.

Recall that a subset $S$ of $\text{Spec } R$ is said to be closed under specialization provided that if $p$ is a prime ideal in $S$ and $q$ is a prime ideal containing $p$ then $q$ is also in $S$. Dually, $S$ is said to be closed under generalization provided that if $p$ is a prime ideal in $S$ and $q$ is a prime ideal contained in $p$ then $q$ is also in $S$. Note that every union of closed subsets of $\text{Spec } R$ is closed under specialization. Similarly, every intersection of open subsets of $\text{Spec } R$ is closed under generalization.

Definition 2.11. We denote by $\mathcal{L}_{\text{spcl}}(\text{Spec } R)$ the lattice of all subsets of $\text{Spec } R$ that are closed under specialization, and by $\mathcal{L}^0_{\text{spcl}}(\text{Spec } R)$ the sublattice of $\mathcal{L}_{\text{spcl}}(\text{Spec } R)$ consisting of all complements of arbitrary intersections of quasi-compact open subsets of $\text{Spec } R$.

Remark 2.12. An open subset $U$ of $\text{Spec } R$ is quasi-compact if and only if $U = D(I) := \{p \in \text{Spec } R \mid I \not\subseteq p\}$ for some finitely generated ideal $I$ of $R$; see the argument on the top of [5, Page 72]. Therefore, if $R$ is noetherian, then every open subset of $\text{Spec } R$ is quasi-compact, and $\mathcal{L}^0_{\text{spcl}}(\text{Spec } R)$ consists of all unions of closed subsets of $\text{Spec } R$.

Hovey [7] constructs the following order-preserving maps among the lattices which we defined above:

\[
\begin{align*}
\mathcal{L}_{\text{spcl}}(\text{Spec } R) & \xrightarrow{\tau} \mathcal{L}_{\text{tors}}(\text{Mod } R) & \mathcal{L}_{\text{tors}}(\text{Mod } R) & \xrightarrow{\nu} \mathcal{L}_{\text{Serre}}(\text{Serre}(R)) & \mathcal{L}_{\text{Serre}}(\text{Serre}(R)) & \xrightarrow{\beta} \mathcal{L}_{\text{coh}}(\text{coh}(R)) & \mathcal{L}_{\text{coh}}(\text{coh}(R)) & \xrightarrow{\alpha} \mathcal{L}_{\text{thick}}(\mathcal{D}_{\text{perf}}(R)) \xrightarrow{\delta} \mathcal{L}_{\text{thick}}(\mathcal{D}_{\text{perf}}(R))
\end{align*}
\]

The above maps are defined as follows.
For $S \in \mathcal{L}_{\text{spcl}}(\text{Spec } R)$, let $\sigma(S)$ be the full subcategory of $\text{Mod } R$ consisting of all $R$-modules $M$ with $\text{Supp } M \subseteq S$. For $X \in \mathcal{L}_{\text{tor}}(\text{Mod } R)$, let $\tau(X)$ be the union $\bigcup_{M \in X} \text{Supp } M$.

For $X \in \mathcal{L}_{\text{tors}}(\text{Mod } R)$, let $\mu(X)$ be the full subcategory of $\text{Mod } R$ consisting of all $R$-modules $M$ in the intersection of $X$ and $\text{Serre}(R)$. For $Y \in \mathcal{L}_{\text{Serre}}(\text{Serre}(R))$, let $\nu(Y)$ be the torsion class of $\text{Mod } R$ generated by $Y$.

For $Z \in \mathcal{L}_{\text{coh}}(\text{Serre}(R))$, let $\alpha(Z)$ be the full subcategory of $\text{Mod } R$ consisting of all $R$-modules $M$ in the intersection of $Z$ and $\text{coh}(R)$. For $W \in \mathcal{L}_{\text{thick}}(\text{D}_{\text{perf}}(R))$, let $f(W)$ be the full subcategory of $\text{D}(R)$ consisting of all complexes $X \cdot \in \text{D}_{\text{perf}}(R)$ such that the $i$th homology $H_i(X \cdot)$ is in $Z$ for any $i \in \mathbb{Z}$. For $W \in \mathcal{L}_{\text{thick}}(\text{D}_{\text{perf}}(R))$, let $g(W)$ be the coherent subcategory of $\text{Mod } R$ generated by \{ $H_i(X \cdot) \mid X \cdot \in W, i \in \mathbb{Z}$ \}.

We recall here the definition of an adjoint pair of order-preserving maps. Let $\phi : A \to B$ and $\psi : B \to A$ be two order-preserving maps between ordered sets. Then the pair $(\phi, \psi)$ is said to be an adjoint pair provided that $\phi(a) \leq b$ if and only if $a \leq \psi(b)$ for any $a \in A$ and $b \in B$. Concerning the above order-preserving maps among lattices, the following proposition holds.

**Proposition 2.13.**

1. (a) The pair $(\tau, \sigma)$ is an adjoint pair.
   (b) The composite map $\tau \sigma$ is the identity map.
   (c) If $R$ is noetherian, then $\sigma$ is a lattice isomorphism and $\tau$ is the inverse homomorphism.

2. (a) The pair $(\nu, \mu)$ is an adjoint pair.
   (b) The composite map $\nu \mu$ is the identity map.
   (c) If $R$ is noetherian, then $\mu$ is a lattice isomorphism and $\nu$ is the inverse homomorphism.

3. (a) The pair $(\beta, \alpha)$ is an adjoint pair.
   (b) If $R$ is noetherian, then the composite map $\beta \alpha$ is the identity map.

4. (a) The pair $(f, g)$ is an adjoint pair.
   (b) The composite map $fg$ is the identity map.

**Proof.** It is easy to check that every order-preserving bijective map between two lattices is a lattice isomorphism. Hence we see from the arguments in Page 3185 of [7] that the assertions (1), (2) and (3) hold. (Here, note that derived categories do not appear in those arguments.)

The assertion (a) and (b) in (4) are shown in [7, Proposition 1.4] and [7, Corollary 2.2], respectively. 

The above proposition especially says that one has the following relationships between two modules whose supports have inclusion relation.

**Corollary 2.14.** Let $R$ be a noetherian ring, and let $M, N$ be $R$-modules with $\text{Supp } M \subseteq \text{Supp } N$. Then $M$ belongs to the torsion class of $\text{Mod } R$ generated by $N$. If $M$ and $N$ are finitely generated, then $M$ belongs to the Serre subcategory of $\text{Mod } R$ generated by $N$.

**Proof.** Proposition 2.13 says that the maps $\sigma, \mu$ are isomorphisms whose inverse maps are $\tau, \nu$ respectively. Let $T$ be the torsion class of $\text{Mod } R$ generated by $N$. 

We have $\mathcal{T} = \sigma_T(\mathcal{T})$, which is the full subcategory of Mod $R$ consisting of all $R$-modules $K$ with $\text{Supp } K \subseteq \bigcup_{L \in \mathcal{T}} \text{Supp } L$. Since $N$ is in $\mathcal{T}$, we get $\text{Supp } M \subseteq \bigcup_{L \in \mathcal{T}} \text{Supp } L$. Therefore $M$ is in $\mathcal{T}$.

Suppose that both $M$ and $N$ are finitely generated. Let $S$ be the Serre subcategory of mod $R$ generated by $N$. Then we have $S = \mu \sigma \nu (S)$, which consists of all finitely generated $R$-modules $K$ with $\text{Supp } K \subseteq \bigcup_{L \in \mathcal{T}} \text{Supp } L (= \bigcup_{L \in S} \text{Supp } L)$. Hence $M$ is in $S$.

### 3. Coherent subcategories are Serre

Throughout this section, let $R$ be a commutative ring. We begin with proving the following theorem.

**Theorem 3.1.** Let $R$ be a noetherian ring. Let $\mathcal{X}$ be a full subcategory of mod $R$ which is closed under finite direct sums, kernels and cokernels. Then $\mathcal{X}$ is closed under submodules and quotient modules.

**Proof.** According to Proposition 2.2(4), it is enough to prove that $\mathcal{X}$ is closed under submodules. Assume that $X$ is not closed under submodules. Then there exist an $R$-module $X$ in $\mathcal{X}$ and an $R$-submodule $M$ of $X$ such that $M$ does not belong to $\mathcal{X}$. Since $R$ is noetherian and $X$ is a finitely generated $R$-module, $X$ is a noetherian $R$-module. Hence we can choose $M$ to be a maximal element, with respect to the inclusion relation, of the set of $R$-submodules $M'$ of $X$ such that $M'$ does not belong to $\mathcal{X}$. Since $M$ does not coincide with $X$, there is an element $x \in X - M$. Set $Y = M + Rx$. Note that $Y$ is an $R$-submodule of $X$ strictly containing $M$. By the maximality of $M$, the module $Y$ is in $\mathcal{X}$. Put $I = (M : x) := \{a \in R \mid ax \in M\}$. This is an ideal of $R$, and we easily see that the quotient $R$-module $Y/M$ is isomorphic to $R/I$. There is an exact sequence

$$0 \rightarrow M \rightarrow Y \xrightarrow{\pi} R/I \rightarrow 0$$

of $R$-modules. Since $M \notin \mathcal{X}$ and $Y \in \mathcal{X}$ and $\mathcal{X}$ is closed under kernels, we see from this exact sequence that $R/I$ must not be in $\mathcal{X}$.

On the other hand, the map $\pi$ in the exact sequence induces a surjective homomorphism

$$\pi : Y/IY \rightarrow R/I$$

of $R/I$-modules, which sends the residue class of $y \in Y$ in $Y/IY$ to $\pi(y)$. Of course $R/I$ is a projective $R/I$-module, so $\pi$ is a split epimorphism. Therefore $R/I$ is isomorphic to a direct summand of $Y/IY$. The noetherian property of $R$ implies that the ideal $I$ is finitely generated; write $I = (a_1, a_2, \ldots, a_n)R$ for some elements $a_1, a_2, \ldots, a_n \in R$. There is an exact sequence

$$R^{\oplus n} \xrightarrow{(a_1, \ldots, a_n)} R \longrightarrow R/I \longrightarrow 0$$

of $R$-modules. Tensoring the $R$-module $Y$ with this exact sequence yields another exact sequence of $R$-modules:

$$Y^{\oplus n} \xrightarrow{(a_1, \ldots, a_n)} Y \longrightarrow Y/IY \longrightarrow 0.$$

The assumption of the theorem says that $\mathcal{X}$ is closed under finite direct sums, cokernels, and direct summands; see Proposition 2.2(1). Hence the direct sum $Y^{\oplus n}$ belongs to $\mathcal{X}$, and so does the module $Y/IY$, and therefore so does $R/I$. This is a contradiction, which says that $\mathcal{X}$ is closed under submodules. Thus the proof of the theorem is completed. $\square$
Corollary 2.5(1) says that any coherent subcategory of mod $R$ is closed under finite direct sums, kernels and cokernels. Thus, according to Theorem 3.1, we obtain the following result, which is the former half part of Theorem A in the first section of this paper.

**Corollary 3.2.** Let $R$ be a noetherian ring. Then every coherent subcategory of mod $R$ is a Serre subcategory of mod $R$.

Now, let us check that the subset $L_{\text{spcl}}(\text{Spec } R)$ coincides with $L_{0}\text{spcl}(\text{Spec } R)$ if $R$ is a noetherian ring.

**Proposition 3.3.**

1. Let $Z$ be a subset of $\text{Spec } R$ which is closed under specialization. Then one has
   $$Z = \bigcup_{p \in Z} V(p).$$

2. Let $R$ be a noetherian ring. Then
   $$L_{\text{spcl}}(\text{Spec } R) = L_{0}\text{spcl}(\text{Spec } R).$$

**Proof.**

1. Let $q$ be a prime ideal in $Z$. Then $q$ is in $V(q)$, which is contained in $\bigcup_{p \in Z} V(p)$. As to the opposite inclusion relation, take a prime ideal $q$ in $\bigcup_{p \in Z} V(p)$. Then $q$ is in $V(p)$ for some $p \in Z$. Since $Z$ is closed under specialization, we get $q \in Z$, as required.

2. This immediately follows from Remark 2.12 and the assertion (1).

To prove our next result, we prepare here the following two lemmas.

**Lemma 3.4.** Let $R$ be a noetherian ring and $M$ a subcategory of mod $R$ which is closed under finite direct sums and cokernels. Let $M$ be a cyclic $R$-module in $M$. Then there exists a perfect $R$-complex $X_{\bullet}$ such that $H_0(X_{\bullet})$ is isomorphic to $M$ and that $H_j(X_{\bullet})$ belongs to $M$ for any $j \in \mathbb{Z}$.

**Proof.** Since $M$ is cyclic, there exists an ideal $I$ of $R$ such that $M$ is isomorphic to $R/I$. The noetherian property of $R$ implies that the ideal $I$ is finitely generated; let $x = x_1, x_2, \ldots, x_r$ be a system of generators of $I$. Consider the Koszul complex $K_{\bullet} := K_{\bullet}(x, R)$ of the sequence $x$. The complex $K_{\bullet}$ is a perfect $R$-complex, and the zeroth homology $H_0(K_{\bullet})$ is equal to $R/I$, which is isomorphic to $M$. Thus, to show the lemma, it suffices to check that the homology $H_j(K_{\bullet})$ belongs to $M$ for each $j \in \mathbb{Z}$. Note from [1, Proposition 1.6.5(b)] that the $R$-module $H_j(K_{\bullet}) = H_j(x, R)$ is annihilated by the ideal $I = xR$. Hence $H_j(K_{\bullet})$ can be regarded as an $R/I$-module. Since $R$ is noetherian, $H_j(K_{\bullet})$ is finitely generated, hence finitely presented as an $R/I$-module. It follows that there is an exact sequence
   $$(R/I)^{\oplus m} \to (R/I)^{\oplus m} \to H_j(K_{\bullet}) \to 0$$
   of $R/I$-modules. Since $M$ is closed under finite direct sums, the sum $(R/I)^{\oplus i} \cong M^{\oplus i}$ is an object of $M$ for any $i \geq 0$. Since $M$ is closed under cokernels, the module $H_j(K_{\bullet})$ is in $M$, as desired.

$\square$
Lemma 3.5. Let $M$ be a finitely generated $R$-module. Then there exist exact sequences

\[
\begin{align*}
0 & \to R/I_1 \to M_0 \to M_1 \to 0, \\
0 & \to R/I_2 \to M_1 \to M_2 \to 0, \\
& \vdots \\
0 & \to R/I_{n-1} \to M_{n-2} \to M_{n-1} \to 0, \\
0 & \to R/I_n \to M_{n-1} \to M_n \to 0
\end{align*}
\]

of $R$-modules such that $I_1, I_2, \ldots, I_{n-1}, I_n$ are ideals of $R$, and $M_0 = M$ and $M_n = 0$.

Proof. Let $x_1, x_2, \ldots, x_n$ be a system of generators of $M$. Set $M_0 = M$. We have an isomorphism $Rx_1 \cong R/I_1$, where $I_1 = \text{Ann}(x_1)$. Putting $M_1 = M_0/Rx_1$, we have an exact sequence

\[
0 \to R/I_1 \to M_0 \to M_1 \to 0
\]

of $R$-modules. Note that the $R$-module $M_1$ is generated by $n-1$ elements. By induction on $n$, we can obtain such a system of exact sequences as in the lemma. \(\Box\)

Now we are in a position to prove the following theorem, which is the latter half part of Theorem A in the first section. In the proof, we should note that all the isomorphisms in the theorem except $L_{\text{coh}}(\text{coh}(R)) \cong L_{\text{thick}}(\text{D}_{\text{perf}}(R))$ are obtained without using derived categories.

Theorem 3.6. Let $R$ be a noetherian ring. Then the homomorphisms $\sigma, \mu, \alpha, f$ (defined in the previous section) are lattice isomorphisms, and $\tau, \nu, \beta, g$ are their inverse homomorphisms, respectively. Consequently, one has

\[
L^0_{\text{spec}}(\text{Spec } R) = L_{\text{spec}}(\text{Spec } R) \\
\cong L_{\text{cors}}(\text{Mod } R) \\
\cong L_{\text{Serre}}(\text{Serre}(R)) \\
= L_{\text{coh}}(\text{coh}(R)) \\
\cong L_{\text{thick}}(\text{D}_{\text{perf}}(R)).
\]

Proof. The equality $L^0_{\text{spec}}(\text{Spec } R) = L_{\text{spec}}(\text{Spec } R)$ is already shown in Proposition 3.3. Proposition 2.13 says that the homomorphisms $\sigma, \mu$ are lattice isomorphisms and $\tau, \nu$ are the inverse homomorphisms of $\sigma, \mu$ respectively. Hence we have $L_{\text{spec}}(\text{Spec } R) \cong L_{\text{cors}}(\text{Mod } R) \cong L_{\text{Serre}}(\text{Serre}(R))$. It is seen from Corollary 3.2 and Remark 2.8 that both of the homomorphisms $\alpha$ and $\beta$ are the identity maps, and we have $L_{\text{Serre}}(\text{Serre}(R)) = L_{\text{coh}}(\text{coh}(R))$.

It remains to prove that $f$ is an isomorphism with the inverse homomorphism $g$. The composite map $fg$ is the identity homomorphism by Proposition 2.13(4)(b), and the subcategory $gf(X)$ is contained in $X$ for every $X \in L_{\text{coh}}(\text{coh}(R))$ by Proposition 2.13(4)(a). Let us show that $X$ is contained in $gf(X)$. Let $M$ be an $R$-module in $X$. Since Remark 2.8 guarantees that $X$ is a subcategory of mod $R$, $M$ is a finitely generated $R$-module, and we have a system of exact sequences as in Lemma 3.5. (In the following, we use the same notation.) Since $X$ is a coherent subcategory of mod $R$ by Remark 2.8, we see from Corollary 3.2 that $X$ is a Serre subcategory of mod $R$. Hence $X$ is closed under submodules and quotient modules in mod $R$ by Corollary 2.5(2). From the above exact sequences we easily see that the cyclic
$R$-module $R/I_i$ belongs to $X$ for every $1 \leq i \leq n$. Note by Corollary 2.5(1) that $X$ is closed under finite direct sums. Hence Lemma 3.4 shows that for each integer $1 \leq i \leq n$ there exists a perfect $R$-complex $X^{(i)}_\bullet$ such that $R/I_i$ is isomorphic to $H_0(X^{(i)}_\bullet)$ and that $H_j(X^{(i)}_\bullet)$ belongs to $X$ for any $j \in \mathbb{Z}$. It easily follows from the definitions of the homomorphisms $f, g$ that the $R$-module $R/I_i$ belongs to $gf(X)$. Since $gf(X)$ is a coherent subcategory, it is closed under extensions. Hence from the system of exact sequences, we see that $M$ belongs to $gf(X)$. Therefore $X$ is contained in $gf(X)$, and thus $gf$ is the identity homomorphism. This completes the proof of our theorem.

4. In relation to Neeman’s classification

In this section, we shall give a module version of Neeman’s classification theorem of localizing categories and smashing categories, which we stated in the first section of this paper. Throughout this section, let $R$ be a commutative noetherian ring.

We denote by $\mathcal{L}_{\text{subext}}(\text{mod } R)$ the lattice of all subcategories of $\text{mod } R$ which are closed under submodules and extensions, and by $\mathcal{L}(\text{Spec } R)$ the lattice of all subsets of $\text{Spec } R$. We define maps

$$
\begin{align*}
\Phi &: \mathcal{L}(\text{Spec } R) \rightarrow \mathcal{L}_{\text{subext}}(\text{mod } R) \\
\Psi &: \mathcal{L}_{\text{subext}}(\text{mod } R) \rightarrow \mathcal{L}(\text{Spec } R)
\end{align*}
$$

by $\Phi(S) = \{ M \in \text{mod } R \mid \text{Ass } M \subseteq S \}$ and $\Psi(M) = \bigcup_{M \in \mathcal{M}} \text{Ass } M$. It is easy to check that these maps are lattice homomorphisms. Let $\phi : \mathcal{L}_{\text{spcl}}(\text{Spec } R) \rightarrow \mathcal{L}_{\text{serre}}(\text{mod } R)$ and $\psi : \mathcal{L}_{\text{serre}}(\text{mod } R) \rightarrow \mathcal{L}_{\text{spcl}}(\text{Spec } R)$ be the composite maps $\mu \sigma$ and $\tau \nu$, where $\sigma, \tau, \mu, \nu$ are the homomorphisms defined in Section 2. Note that the maps $\phi, \psi$ are given by $\phi(S) = \{ M \in \text{mod } R \mid \text{Supp } M \subseteq S \}$ and $\psi(M) = \bigcup_{M \in \mathcal{M}} \text{Supp } M$ for $S \in \mathcal{L}_{\text{spcl}}(\text{Spec } R)$ and $M \in \mathcal{L}_{\text{serre}}(\text{mod } R)$.

Recall that $\phi$ is an isomorphism and $\psi$ is its inverse homomorphism since $R$ is noetherian. This section is mainly devoted to proving the following theorem.

**Theorem 4.1.** Let $R$ be a commutative noetherian ring. Then the homomorphism $\Phi$ is an isomorphism and $\Psi$ is its inverse homomorphism. Moreover, $\Phi$ and $\Psi$ induce the isomorphisms $\phi$ and $\psi$, respectively. Thus one has the following commutative diagram.

$$
\begin{array}{ccc}
\mathcal{L}_{\text{subext}}(\text{mod } R) & \xrightarrow{\psi} & \mathcal{L}(\text{Spec } R) & \xrightarrow{\phi} & \mathcal{L}_{\text{subext}}(\text{mod } R) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathcal{L}_{\text{serre}}(\text{mod } R) & \xrightarrow{\psi} & \mathcal{L}_{\text{spcl}}(\text{Spec } R) & \xrightarrow{\phi} & \mathcal{L}_{\text{serre}}(\text{mod } R)
\end{array}
$$

This theorem will be proved after showing the following two lemmas.

**Lemma 4.2.** Let $\mathcal{M}$ be a subcategory of $\text{mod } R$ which is closed under submodules and extensions, and let $M$ be a finitely generated $R$-module. Suppose that $M$ has a unique associated prime $p$. If $R/p$ is in $\mathcal{M}$, then so is $M$. 

Proof. Assume that $M$ is not in $\mathcal{M}$. Set $M_0 = M$, and let $f_{0,1}, \ldots, f_{0,s_0}$ be a system of generators of the $R$-module $\text{Hom}_R(M_0, R/p)$. There is an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow (R/p)^{\oplus s_0}$$

of $R$-modules. Since $M_0 = M$ is not in $\mathcal{M}$ and $(R/p)^{\oplus s_0}$ is in $\mathcal{M}$ and $\mathcal{M}$ is closed under submodules and extensions, it is easily seen that $M_1$ must not be in $\mathcal{M}$. In particular, $M_1 \neq 0$ and hence $p$ is the unique associated prime of $M_1$. Letting $f_{1,1}, \ldots, f_{1,s_1}$ be a system of generators of the $R$-module $\text{Hom}_R(M_1, R/p)$, we have an exact sequence

$$0 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow (R/p)^{\oplus s_1}.$$ 

Since $M_1$ is not in $\mathcal{M}$ and $(R/p)^{\oplus s_1}$ is in $\mathcal{M}$, we see that $M_2$ is not in $\mathcal{M}$, and that $p$ is the unique associated prime of $M_2$. Iterating this procedure, for each integer $i \geq 0$ we obtain an exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow (R/p)^{\oplus s_i},$$

where $f_{i,1}, \ldots, f_{i,s_i}$ is a system of generators of the $R$-module $\text{Hom}_R(M_i, R/p)$ and $p$ is the unique associated prime of $M_i$. Localizing the descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ at $p$ yields a descending chain

$$M_p = (M_0)_p \supseteq (M_1)_p \supseteq (M_2)_p \supseteq \cdots$$

of $R_p$-modules. Since the $R_p$-module $(M_i)_p$ has finite length for every $i$, there exists an integer $t$ such that $(M_t)_p = (M_{t+1})_p = (M_{t+2})_p = \cdots$. The exact sequence

$$0 \longrightarrow (M_{t+1})_p \longrightarrow (M_t)_p \longrightarrow (f_{t,s_t})_p \longrightarrow (R/p)^{\oplus s_t},$$

shows that $\text{Hom}_{R_p}((M_t)_p, \kappa(p)) = R_p (f_{t,1})_p + \cdots + R_p (f_{t,s_t})_p = 0$. Therefore $(M_t)_p = 0$. This is a contradiction since $p \in \text{Ass} M \subseteq \text{Supp} M$. Thus we conclude that $M$ is in $\mathcal{M}$.

Lemma 4.3. Let $\mathcal{M}$ be a subcategory of mod $R$ which is closed under submodules and extensions. Let $M$ be a finitely generated $R$-module. Suppose that $R/p$ belongs to $\mathcal{M}$ for every $p \in \text{Ass} M$. Then $M$ also belongs to $\mathcal{M}$.

Proof. Let $p_1, \ldots, p_s$ be the associated primes of $M$, and let

$$0 = N_1 \cap \cdots \cap N_s$$

be an irredundant primary decomposition of the zero submodule $0$ of $M$, where $N_i$ is a $p_i$-primary submodule of $M$ for $1 \leq i \leq s$. Then the natural homomorphism

$$M = M/N_1 \oplus \cdots \oplus M/N_s \rightarrow M/N_1 \oplus \cdots \oplus M/N_s$$
Let \( R \) be a noetherian ring. Let \( M \) and \( N \) be finitely generated \( R \)-modules with \( \text{Ass} \ M \subseteq \text{Ass} \ N \). Then \( M \) is in the full subcategory of \( \text{mod} \ R \) closed under submodules and extensions which is generated by \( N \).

**Proof.** Let \( \mathcal{E} \) be the full subcategory of \( \text{mod} \ R \) closed under submodules and extensions which is generated by \( N \). According to Lemma 4.3, we have only to show that the \( R \)-module \( R/p \) is in \( \mathcal{E} \) for every \( p \in \text{Ass} \ M \). Let \( p \) be a prime ideal in \( \text{Ass} \ M \). The assumption says that \( p \) is in \( \text{Ass} \ N \). Hence there exists an injective homomorphism \( R/p \to N \) of \( R \)-modules. Since \( N \) is in \( \mathcal{E} \) and \( \mathcal{E} \) is closed under submodules, \( R/p \) is also in \( \mathcal{E} \), as required. \( \square \)

In the following example, we will give several correspondences between subcategories of \( \text{mod} \ R \) which are closed under submodules and extensions and subsets of \( \text{Spec} \ R \), which are made by the isomorphisms \( \Phi \) and \( \Psi \). Before that, we
The isomorphisms Φ and Ψ make the following correspondences. In each correspondence, we denote by grade of elements of $M$ which are annihilated by some power of $I$. Recall that an $R$-module $M$ is called $I$-torsion if $\Gamma_I(M) = M$, and that $M$ is called $I$-torsionfree if $\Gamma_I(M) = 0$. It is well-known and easy to see that $M$ is $I$-torsion if and only if $Ass M \subseteq V(I)$, and that $M$ is $I$-torsionfree if and only if $Ass M \cap V(I) = \emptyset$. We set $grade(N,M) = \inf\{ i | Ext^i_R(N,M) \neq 0 \}$, $grade(I, M) = grade(R/I, M)$, $grade I = grade(I, R)$ and $grade M = grade(Ann M, R)$.

**Example 4.5.** The isomorphisms $\Phi$ and $\Psi$ make the following correspondences. Let $n$ be a nonnegative integer, $I$ an ideal of $R$ and $X$ a finitely generated $R$-module.

1. \{ $M \in \text{mod } R \mid M$ is $I$-torsion $\} \leftrightarrow V(I)$.
2. \{ $M \in \text{mod } R \mid grade(X, M) > 0 \} \leftrightarrow Spec R \setminus \text{Supp } X$.
3. \{ $M \in \text{mod } R \mid M$ is $I$-torsionfree $\} = \{ M \in \text{mod } R \mid grade(I, M) > 0 \} \leftrightarrow D(I)$.
4. \{ $M \in \text{mod } R \mid grade(M, X) \geq n \} \leftrightarrow \{ p \in Spec R \mid grade(p, X) \geq n \}$.
5. \{ $M \in \text{mod } R \mid rank M = 0 \} = \{ M \in \text{mod } R \mid grade M > 0 \} \leftrightarrow \{ p \in Spec R \mid grade p > 0 \}$.
6. \{ $M \in \text{mod } R \mid$ every $X$-regular element is $M$-regular $\} \leftrightarrow \{ p \in Spec R \mid grade(p, X) = 0 \}$.
7. \{ $M \in \text{mod } R \mid M$ is torsionfree $\} \leftrightarrow \{ p \in Spec R \mid grade p = 0 \}$.
8. \{ $M \in \text{mod } R \mid ht \text{Ann } M \geq n \} \leftrightarrow \{ p \in Spec R \mid ht p \geq n \}$.
9. \{ $M \in \text{mod } R \mid \text{dim } M \leq n \} \leftrightarrow \{ p \in Spec R \mid \text{dim } R/p \leq n \}$.
10. \{ $M \in \text{mod } R \mid \ell(M) < \infty \} \leftrightarrow \text{Max } R$.

**Proof.** In each correspondence, we denote by $M$ the left-hand subcategory of $\text{mod } R$, and by $S$ the right-hand subset of $\text{Spec } R$. Note that it is enough to check either that $S(M) = S$ or that $S(S) = M$ since $\Phi$ is an isomorphism with the inverse homomorphism $\Psi$.

1. The subcategory $M$ consists of all finitely generated $R$-modules $M$ with $Ass M \subseteq V(I)$, which coincides with $\Phi(V(I))$. Hence $\Phi(S) = M$.
2. Let $M$ be a finitely generated $R$-module. Note that $Ass M \subseteq Spec R \setminus \text{Supp } X$ if and only if $Ass M \cap \text{Supp } X = \emptyset$, if and only if $Ass \text{Hom}(X, M) = 0$ (cf. [1, Exercise 1.2.27]), if and only if $\text{Hom}(X, M) = 0$. Thus we have $\Phi(S) = M$.
3. The equality is well-known. Putting $X = R/I$ in (2), we obtain the correspondence.
4. Let $M$ be a finitely generated $R$-module with $grade(M, X) \geq n$, and let $p \in Ass M$. Then there is an $X$-regular sequence $a_1, \ldots, a_n$ in $\text{Ann } M$, and $p \in \text{Supp } M$. Hence $a$ is an $X$-regular sequence in $p$, and we have $grade(p, X) \geq n$. Therefore $\Psi(M)$ is contained in $S$. Conversely, if $p$ is a prime ideal with
grade(p, X) ≥ n, then R/p ∈ M and p ∈ Ass R/p. Hence p is in Ψ(M). Therefore S is contained in Ψ(M), and thus we get Ψ(M) = S.

(5) Let M be an R-module. We have that rank M = 0 if and only if M_\* = 0 for every p ∈ Ass R, if and only if Supp M ∩ Ass R = ∅, if and only if Ass Hom(M, R) = ∅, if and only if Hom(M, R) = 0, namely grade M > 0. Thus the equality holds. For the correspondence, put X = R and n = 1 in (4).

(6) Let p be a prime ideal in Ψ(M). Then there exists an R-module M ∈ M of which p is an associated prime. Assume that grade(p, X) > 0. Then there is an X-regular element a ∈ p, and this is also M-regular. This is a contradiction since p ∈ Ass M. Thus grade(p, X) = 0, namely p belongs to S. On the contrary, let p be a prime ideal with grade(p, X) = 0. Then there exists an associated prime q of X which contains p. Let a be an X-regular element. Then a is not in q, so is not in p. Hence a is an R/p-regular element, and R/p belongs to M. Since p ∈ Ass R/p, the prime ideal p is in Ψ(M), and it holds that Ψ(M) = S.

(7) Put X = R in (6), and we get this correspondence.

(8) If M is a finitely generated R-module with ht Ann M ≥ n, then ht p ≥ n for all p ∈ Supp M, hence for all p ∈ Ass M. Therefore Ψ(M) is contained in S. If p is a prime ideal of height at least n, then the ideal Ann R/p of R also has height at least n and p ∈ Ass R/p. Hence Ψ(M) contains S, and thus Ψ(M) = S.

(9) Let M be a finitely generated R-module. Then Ass M is contained in S if and only if dim R/p ≤ n for every p ∈ Ass M, if and only if dim M ≤ n. Therefore Φ(S) = M.

(10) Putting n = 0 in (9) yields this correspondence. □

Note that in the correspondences (1), (4), (5), (8), (9) and (10) in the above example, the left-hand subcategories of mod R are Serre subcategories and the right-hand subsets of Spec R are closed under specializations, hence those correspondences are in fact obtained by the isomorphisms φ and ψ.

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