COHEN–MACAULAY MODULES OVER GORENSTEIN LOCAL RINGS

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Introduction

Goal. The structure theorem of (maximal) Cohen–Macaulay modules over commutative Gorenstein local rings

Throughout.

- R : a commutative Noetherian ring with 1
- $\begin{cases} A = k[[X,Y]]/(XY) \\ B = k[[X,Y]]/(X^2, XY) \end{cases}$ $\quad k$: a field
- $x := \overline{X},\ y := \overline{Y}$

1. Associated primes

Def 1.1. Let $I \subsetneq R$ be an ideal of R

(1) I is a <u>maximal ideal</u> if there exists no ideal J of R with $I \subsetneq J \subsetneq R$
(2) I is a <u>prime ideal</u> if ($ab \in I \Rightarrow a \in I$ or $b \in I$)
(3) $\operatorname{Spec} R := \{\text{Prime ideals of } R\}$

Ex 1.2.

$$\operatorname{Spec} A = \{(x), (y), (x,y)\}, \quad \operatorname{Spec} B = \{(x), (x,y)\}$$

Prop 1.3. $I \subseteq R$ an ideal

(1) I is prime iff R/I is an integral domain
(2) I is maximal iff R/I is a field

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(3) Every maximal ideal is prime

Throughout the rest of this section, let M be an R-module.

Def 1.4.
$\mathfrak{p} \in \operatorname{Spec} R$ is an <u>associated prime</u> of M if

$$\exists\, x \in M \quad \text{s.t} \quad \mathfrak{p} = \operatorname{ann}(x)$$

$\operatorname{Ass}_R M := \{\text{Associated primes of } M\}$

Ex 1.5.

$$\operatorname{Ass}_A A = \{(x), (y)\}, \quad \operatorname{Ass}_B B = \{(x), (x, y)\}$$

Prop 1.6. TFAE for $\mathfrak{p} \in \operatorname{Spec} R$
(1) $\mathfrak{p} \in \operatorname{Ass}_R M$
(2) $\exists\, f : R/\mathfrak{p} \to M$ an injective homomorphism

Proof.
(1) $\Rightarrow$ (2)
$\exists\, x \in M$ s.t $\mathfrak{p} = \operatorname{ann}(x)$ Define $f : R/\mathfrak{p} \to M$ by $f(\overline{a}) = ax$
(2) $\Rightarrow$ (1)
$x := f(\overline{1}) \in M \quad \mathfrak{p} = \operatorname{ann}(x)$ ■

Prop 1.7.

$$\max\{\operatorname{ann}(x) \mid 0 \neq x \in M\} \subseteq \operatorname{Ass}_R M$$

Proof. $\mathfrak{p} := \operatorname{ann}(x)$ <u>ETS</u> $\mathfrak{p} \in \operatorname{Spec} R$
$x \neq 0 \Rightarrow 1 \notin \mathfrak{p} \Rightarrow \mathfrak{p} \subsetneq R$
Suppose $a, b \in R$, $ab \in \mathfrak{p}$, $a \notin \mathfrak{p}$. Then $abx = 0$

$$\therefore\ \mathfrak{p} = \operatorname{ann}(x) \subsetneq \operatorname{ann}(bx) \quad (a \in \operatorname{ann}(bx) - \mathfrak{p})$$

The maximality of $\mathfrak{p}$ shows $\operatorname{ann}(bx) = R$ $\therefore bx = 0$ $\therefore x \in \mathfrak{p}$ ■

Cor 1.8.

$$M \neq 0 \iff \operatorname{Ass}_R M \neq \emptyset$$

Proof.
($\Leftarrow$) Trivial
($\Rightarrow$) As R is Noetherian, LHS in Prop 1.7 is nonempty. ■

Def 1.9. $x \in R$, M an R-module
(1) x is a <u>zerodivisor</u> (ZD) on M if

$$0 \neq \exists\, m \in M \quad \text{s.t} \quad xm = 0$$

(2) x is a <u>nonzerodivisor</u> (NZD) on M if x is not a ZD on M

Cor 1.10.

$$\{\text{ZDs on } M\} = \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$$

Proof.
($\supseteq$) Let $a \in \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$
$\Rightarrow a \in \exists\, \mathfrak{p} \in \operatorname{Ass} M \Rightarrow \exists\, x \in M$ s.t $\mathfrak{p} = \operatorname{ann}(x)$
$\therefore ax = 0$, $x \neq 0$ $\therefore$ a is a ZD on M
($\subseteq$) Let $a \in R$ be a ZD on M
$\Rightarrow 0 \neq \exists\, z \in M$ s.t $az = 0 \Rightarrow a \in \operatorname{ann}(z)$

$$\begin{aligned} \operatorname{ann}(z) &\subseteq \exists\, \mathfrak{q} \in \max\{\operatorname{ann}(x) \mid 0 \neq x \in M\} \\ &\subseteq \operatorname{Ass} M \text{ by Prop 1.7} \end{aligned}$$

$\therefore a \in \mathfrak{q} \in \operatorname{Ass} M$ $\therefore a \in \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p}$ ■

Ex 1.11. (1) $\{\text{ZDs on } A\} = (x) \cup (y)$ $\therefore x - y$ is a NZD on A
(2) $\{\text{ZDs on } B\} = (x, y)$ $\therefore$ All nonunits of B are ZDs on B

Thm 1.12.
If M is finitely generated, then $\# \operatorname{Ass}_R M < \infty$

Sketch. (Details: Appendix A)
(1) $\exists\, 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ submodules s.t.

$$M_i/M_{i-1} \cong R/\mathfrak{p}_i, \quad \mathfrak{p}_i \in \operatorname{Spec} R$$

(2) If $0 \to X \to Y \to Z \to 0$ is an exact seq of R-modules, then

$$\operatorname{Ass} Y \subseteq \operatorname{Ass} X \cup \operatorname{Ass} Z$$

(3) $\operatorname{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$ $\forall\, \mathfrak{p} \in \operatorname{Spec} R$
$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0$
$\rightsquigarrow$ $\operatorname{Ass} M_i \subseteq \operatorname{Ass} M_{i-1} \cup \operatorname{Ass} R/\mathfrak{p}_i = \operatorname{Ass} M_{i-1} \cup \{\mathfrak{p}_i\}$
$\rightsquigarrow$ $\operatorname{Ass} M \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ ■

2. Depth

Def 2.1. R is <u>local</u> if R has a unique maximal ideal $\mathfrak{m}$
$k := R/\mathfrak{m}$ is the <u>residue field</u> of R
We say $(R, \mathfrak{m}, k)$ is a local ring

Ex 2.2.
(1) $(A, (x, y)A, k)$ is local
(2) $(B, (x, y)B, k)$ is local

Def 2.3. $(R, \mathfrak{m}, k)$: local M: a f.g R-mod
The <u>depth</u> of M is:

$$\operatorname{depth}_R M = \inf\{i \geqslant 0 \mid \operatorname{Ext}^i_R(k, M) \neq 0\}$$

Prop 2.4. $(R, \mathfrak{m}, k)$: local

$$0 \to L \to M \to N \to 0$$

an exact seq of f.g R-modules Then
(1) $\operatorname{depth} L \geqslant \inf\{\operatorname{depth} M, \operatorname{depth} N + 1\}$

(2) $\operatorname{depth} M \geqslant \inf\{\operatorname{depth} L, \operatorname{depth} N\}$
(3) $\operatorname{depth} N \geqslant \inf\{\operatorname{depth} M, \operatorname{depth} L - 1\}$

Proof. (1) $n := \inf\{\operatorname{depth} M, \operatorname{depth} N + 1\}$
$\Rightarrow \operatorname{depth} M \geqslant n$, $\operatorname{depth} N \geqslant n - 1$

$$\begin{array}{ccccccc} & & & & & & \operatorname{Ext}^{n-2}(k, N) \\ & \longrightarrow & \operatorname{Ext}^{n-1}(k, L) & \longrightarrow & \operatorname{Ext}^{n-1}(k, M) & \longrightarrow & \operatorname{Ext}^{n-1}(k, N) \\ & \longrightarrow & \operatorname{Ext}^{n}(k, L) & \longrightarrow & \operatorname{Ext}^{n}(k, M) & \longrightarrow & \operatorname{Ext}^{n}(k, N) \end{array}$$

$\therefore \operatorname{depth} L \geqslant n$
(2)(3) Similar to (1) ■

Cor 2.5 (Depth Lemma)**.** $(R, \mathfrak{m}, k)$: local
$0 \to L \to M \to N \to 0$ an exact seq of f.g R-modules

$$\operatorname{depth} M > \operatorname{depth} N \implies \operatorname{depth} L = \operatorname{depth} N + 1$$

Proof. $\operatorname{depth} M \geqslant \operatorname{depth} N + 1$
By Prop 2.4(1),

$$\operatorname{depth} L \geqslant \inf\{\operatorname{depth} M, \operatorname{depth} N + 1\} = \operatorname{depth} N + 1$$

Assume $\operatorname{depth} L > \operatorname{depth} N + 1$. Then by Prop 2.4(3)

$$\operatorname{depth} N \geqslant \inf\{\operatorname{depth} M, \operatorname{depth} L - 1\} > \operatorname{depth} N$$

This contradiction shows $\operatorname{depth} L = \operatorname{depth} N + 1$ ■

Prop 2.6. $(R, \mathfrak{m}, k)$: local M: a f.g R-mod

$$\mathfrak{m} \in \operatorname{Ass}_R M \iff \operatorname{depth}_R M = 0$$

Proof.
($\Rightarrow$) Prop 1.6 $\Rightarrow k = R/\mathfrak{m} \hookrightarrow M \Rightarrow \operatorname{Hom}_R(k, M) \neq 0$
($\Leftarrow$) $\operatorname{Hom}_R(k, M) \neq 0 \Rightarrow \exists\, f : k \to M,\ f(\overline{1}) = x \neq 0$

$$\overline{a} \neq \overline{0} \Rightarrow a \notin \mathfrak{m} \Rightarrow a \in R^{\times} \Rightarrow ax \neq 0$$

$\therefore f$ is injective Prop 1.6 implies $\mathfrak{m} \in \operatorname{Ass} M$ ■

Ex 2.7. (1) $\operatorname{depth}_A A > 0$ since $(x, y) \notin \operatorname{Ass}_A A$
(2) $\operatorname{depth}_B B = 0$ since $(x, y) \in \operatorname{Ass}_B B$

Lem 2.8 (Prime Avoidance)**.**
$I \subseteq R$ an ideal $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec} R$

$$I \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \implies I \subseteq \mathfrak{p}_\ell\ (\exists\, \ell)$$

Proof. Induction on n
<u>$n = 1$</u> Trivial

<u>$n \geqslant 2$</u> May assume $\nexists$ inclusion relation among $\mathfrak{p}_1, \dots, \mathfrak{p}_n$
Suppose $I \nsubseteq \mathfrak{p}_i\ (\forall\, i)$
Ind hyp $\Rightarrow I \nsubseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1} \Rightarrow \exists\, x \in I - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1})$

As $I \subseteq (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) \cup \mathfrak{p}_n$, we have $x \in \mathfrak{p}_n$

$$\begin{aligned} &\exists\, y \in I - \mathfrak{p}_n, \\ &\exists\, z_i \in \mathfrak{p}_i - \mathfrak{p}_n \ (1 \leqslant \forall\, i \leqslant n-1), \\ &w := x + yz_1 \cdots z_{n-1} \end{aligned}$$

(1) $x, y \in I \Rightarrow w \in I \subseteq (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}) \cup \mathfrak{p}_n$
(2) $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1},\ z_1 \in \mathfrak{p}_1 \Rightarrow w \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{n-1}$
(3) $x \in \mathfrak{p}_n,\ y, z_1, \ldots, z_{n-1} \notin \mathfrak{p}_n \Rightarrow w \notin \mathfrak{p}_n$

(1), (2), (3) yield a contradiction ■

Prop 2.9. $(R, \mathfrak{m}, k)$ local M a f.g R-mod TFAE
(1) $\operatorname{depth}_R M > 0$
(2) $\exists\, x \in \mathfrak{m}$ a NZD on M

Proof. By Prop 2.6, $\operatorname{depth} M > 0 \Leftrightarrow \mathfrak{m} \notin \operatorname{Ass} M$
(2) $\Rightarrow$ (1)
Cor 1.10 $\Rightarrow x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p} \Rightarrow x \notin \forall\, \mathfrak{p} \in \operatorname{Ass} M$
As $x \in \mathfrak{m}$, we must have $\mathfrak{m} \notin \operatorname{Ass} M$
(1) $\Rightarrow$ (2)
$\mathfrak{m} \notin \operatorname{Ass} M \Rightarrow \mathfrak{m} \nsubseteq \forall\, \mathfrak{p} \in \operatorname{Ass} M$
Thm 1.12 implies $\# \operatorname{Ass} M < \infty$
Lem 2.8 and Cor 1.10 yield

$$\mathfrak{m} \nsubseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p} = \{\text{ZDs on } M\}$$

$\therefore \exists\, x \in \mathfrak{m}$ a NZD on M ■

Ex 2.10.
(1) $\begin{cases} \operatorname{depth} A > 0 \\ x - y \text{ is a NZD on } A \end{cases}$
(2) $\begin{cases} \operatorname{depth} B = 0 \\ \nexists\, \text{NZD on } B \text{ in the maximal ideal of } B \end{cases}$

Lem 2.11.
M, N: R-modules $x \in \operatorname{ann} N$: a NZD on R, M Then

$$\operatorname{Ext}_R^{i+1}(N, M) \cong \operatorname{Ext}_{R/(x)}^{i}(N, M/xM) \ (\forall\, i \in \mathbb{Z})$$

Proof. $T^i := \operatorname{Ext}_R^{i+1}(-, M)$ ETS:
(1) $\forall\, 0 \to X \to Y \to Z \to 0$ exact seq of $R/(x)$-modules
$\exists\, 0 \to T^0(Z) \to T^0(Y) \to T^0(X) \to T^1(Z) \to \cdots$ exact
(2) $T^0 \cong \operatorname{Hom}_{R/(x)}(-, M/xM)$
(3) $T^i(P) = 0$ ($\forall\, P$ projective R-mod, $\forall\, i > 0$)
by Axioms for (contravariant) Ext [5, Theorem 6.64] ■

Prop 2.12.
$(R, \mathfrak{m}, k)$ local M a f.g R-mod $x \in \mathfrak{m}$ a NZD on M

(1) $\operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1$
(2) If x a NZD on R, then $\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M - 1$

Proof. (1) $t := \operatorname{depth}_R M$ From the exact seq

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

we get an exact seq

$$\begin{array}{ccccc}
\longrightarrow & \operatorname{Ext}_R^{t-2}(k, M) & \xrightarrow[0]{x} & \operatorname{Ext}_R^{t-2}(k, M) & \longrightarrow \operatorname{Ext}_R^{t-2}(k, M/xM) \\
\longrightarrow & \operatorname{Ext}_R^{t-1}(k, M) & \xrightarrow[0]{x} & \operatorname{Ext}_R^{t-1}(k, M) & \longrightarrow \operatorname{Ext}_R^{t-1}(k, M/xM) \\
\longrightarrow & \operatorname{Ext}_R^{t}(k, M) & \xrightarrow[0]{x} & \operatorname{Ext}_R^{t}(k, M) & \longrightarrow \operatorname{Ext}_R^{t}(k, M/xM)
\end{array}$$

$$\therefore \begin{cases} \operatorname{Ext}_R^{<(t-1)}(k, M/xM) = 0, \\ \operatorname{Ext}_R^{t-1}(k, M/xM) \cong \operatorname{Ext}_R^{t}(k, M) \neq 0 \end{cases}$$

$\therefore \operatorname{depth}_R M/xM = t - 1$
(2) Lem 2.11 implies

$$\operatorname{Ext}^i_{R/(x)}(k, M/xM) \cong \operatorname{Ext}_R^{i+1}(k, M)$$

for $i \in \mathbb{Z}$ ■

Ex 2.13. $x - y$ is a NZD on A By Prop 2.12,

$$\operatorname{depth}_{A/(x-y)} A/(x-y) = \operatorname{depth}_A A - 1$$

$A/(x-y) \cong k[X]/(X^2)$ $k \hookrightarrow k[X]/(X^2)$, $\overline{1} \mapsto x$
$\therefore \operatorname{depth}_{A/(x-y)} A/(x-y) = 0$ $\therefore \operatorname{depth}_A A = 1$

3. Krull dimension

Lem 3.1. $(R, \mathfrak{m}, k)$ local
(1) k is a simple R-module
(2) If M is a simple R-module, then $M \cong k$

Prop 3.2 (Nakayama's Lemma (NAK))**.**
$(R, \mathfrak{m}, k)$ local M a f.g R-mod $I \subsetneq R$ an ideal

$$IM = M \implies M = 0$$

Proof. May assume $M \neq 0$
As M is Noetherian, $\exists\, L \subsetneq M$ a maximal submod
The maximality shows M/L is a simple R-mod
Lemma 3.1 $\Rightarrow M/L \cong k \Rightarrow \mathfrak{m}(M/L) = 0 \Rightarrow \mathfrak{m}M \subseteq L$

$$M = IM \subseteq \mathfrak{m}M \subseteq L \subsetneq M$$

This contradiction shows the proposition ■

Def 3.3. The <u>(Krull) dimension</u> of R is:

$$\dim R = \sup\{n \geqslant 0 \mid \exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec} R\}$$

Fact 3.4. [1, Theorem 11.14]
If $(R, \mathfrak{m}, k)$ is local, then

$$\dim R = \inf \left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} \exists\, x_1, \dots, x_n \in \mathfrak{m} \text{ s.t} \\ R/(x_1, \dots, x_n) \text{ is Artinian} \end{array} \right\}$$

Ex 3.5. $\dim A = \dim B = 1$

$$A/(x-y), B/(y) \text{ are Artinian}$$

Cor 3.6. $(R, \mathfrak{m}, k)$ local TFAE
(1) R is Artinian
(2) $\mathfrak{m}^n = 0$ $(\exists\, n \geqslant 0)$
(3) $\operatorname{Spec} R = \{\mathfrak{m}\}$
(4) $\dim R = 0$

Proof. (1) $\Rightarrow$ (2)

$$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \Rightarrow \exists\, n \geqslant 0 \text{ s.t } \mathfrak{m}^n = \mathfrak{m}^{n+1}$$

By Prop 3.2, $\mathfrak{m}^n = 0$
(2) $\Rightarrow$ (3)

$$\begin{aligned} \mathfrak{p} \in \operatorname{Spec} R \;&\Rightarrow\; n \geqslant 0 \text{ s.t } \mathfrak{m}^n = 0 \subseteq \mathfrak{p} \\ &\Rightarrow\; \mathfrak{m} \subseteq \mathfrak{p} \;\Rightarrow\; \mathfrak{p} = \mathfrak{m} \end{aligned}$$

(3) $\Rightarrow$ (4) By definition
(4) $\Rightarrow$ (1) By Fact 3.4 ■

Prop 3.7. $(R, \mathfrak{m}, k)$ local M a f.g R-module

$$\operatorname{depth}_R M \leqslant \dim R/\mathfrak{p} \quad (\forall\, \mathfrak{p} \in \operatorname{Ass}_R M)$$

Proof. Induction on $t := \operatorname{depth}_R M$
<u>$t = 0$</u> NTS
<u>$t \geqslant 1$</u> By Prop 2.9, $\exists\, x \in \mathfrak{m}$ a NZD on M. By Prop 2.12(1),

$$\operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1 = t - 1$$

From the exact seq $0 \to M \xrightarrow{x} M \to M/xM \to 0$ we get an exact seq

$$0 \to \operatorname{Hom}_R(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Hom}_R(R/\mathfrak{p}, M) \xrightarrow{f} \operatorname{Hom}_R(R/\mathfrak{p}, M/xM)$$

As $\mathfrak{p} \in \operatorname{Ass} M$, $\operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0$. By Prop 3.2,

$$\operatorname{Im} f \cong \operatorname{Hom}_R(R/\mathfrak{p}, M)/x \operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0$$

$\therefore \operatorname{Hom}_R(R/\mathfrak{p}, M/xM) \neq 0$. Take $0 \neq f \in \operatorname{Hom}_R(R/\mathfrak{p}, M/xM)$

$$f : R/\mathfrak{p} \to M/xM, \quad f(\overline{1}) =: \overline{z} \neq 0$$

$$a \in \mathfrak{p} \;\Rightarrow\; \overline{a} = \overline{0} \text{ in } R/\mathfrak{p} \;\Rightarrow\; a \cdot \overline{z} = \overline{az} = \overline{0} \text{ in } M/xM$$

$\therefore \mathfrak{p} \subseteq \operatorname{ann}_R(\overline{z})$

$$\mathfrak{p} \subsetneq \operatorname{ann}_R(\overline{z}) \subseteq \exists\, \mathfrak{q} \in \operatorname{Ass}_R(M/xM),$$

where
($\subsetneq$) holds as $x \in \operatorname{ann}_R(\overline{z}) - \mathfrak{p}$ by Cor 1.10, and

($\subseteq$) follows from Prop 1.7.

In particular, $\mathfrak{p} \subsetneq \mathfrak{q}$

$$\therefore \dim R/\mathfrak{p} > \dim R/\mathfrak{q} \geqslant \operatorname{depth}_R(M/xM) = t-1,$$

where ($\geqslant$) is by ind hyp. Thus $t \leqslant \dim R/\mathfrak{p}$ ■

Ex 3.8.

$$\operatorname{depth} A = \dim A/(x) = \dim A/(y) = 1$$
$$\operatorname{depth} B = \dim B/(x,y) = 0 < 1 = \dim B/(x)$$

Cor 3.9. $(R, \mathfrak{m}, k)$ local $\quad M \neq 0$ a f.g R-mod

$$\operatorname{depth}_R M \leqslant \dim R$$

Proof. As $M \neq 0$, we find $\mathfrak{p} \in \operatorname{Ass} M$ by Cor 1.8

By Prop 3.7, $\operatorname{depth}_R M \leqslant \dim R/\mathfrak{p} \leqslant \dim R$ ■

Def 3.10. $\mathfrak{p} \in \operatorname{Spec} R$ is a <u>minimal prime</u> if there is no $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{q} \subsetneq \mathfrak{p}$. Set

$$\operatorname{Min} R = \{\text{Minimal primes of } R\}$$

Cor 3.11. $\quad \operatorname{Min} R \subseteq \operatorname{Ass} R$

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. Define

$$R_{\mathfrak{p}} := \{\tfrac{a}{s} \mid a \in R,\ s \in R - \mathfrak{p}\},$$

where

$$\tfrac{a}{s} = \tfrac{b}{t} \iff \exists\, u \in R - \mathfrak{p} \text{ s.t } u(ta - sb) = 0$$

$\rightsquigarrow$ $R_{\mathfrak{p}}$ is a commutative Noetherian ring by

$$\tfrac{a}{s} + \tfrac{b}{t} := \tfrac{ta+sb}{st}, \qquad \tfrac{a}{s} \cdot \tfrac{b}{t} := \tfrac{ab}{st}$$

Set $\mathfrak{p}R_{\mathfrak{p}} := \{\frac{a}{s} \mid a \in \mathfrak{p},\ s \in R - \mathfrak{p}\} \subseteq R_{\mathfrak{p}} \quad \rightsquigarrow \quad (R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ local

$$\begin{aligned} \mathfrak{p} \in \operatorname{Min} R \quad &\Rightarrow \quad \nexists\, P \in \operatorname{Spec} R \ \text{ s.t } \ P \subseteq \mathfrak{p}R_{\mathfrak{p}} \\ &\Rightarrow \quad \operatorname{Spec} R_{\mathfrak{p}} = \{\mathfrak{p}R_{\mathfrak{p}}\} \\ &\Rightarrow \quad \dim R_{\mathfrak{p}} = 0 \text{ by Cor 3.6} \end{aligned}$$

By Cor 20,

$$0 \leqslant \operatorname{depth}_{R_{\mathfrak{p}}} R_{\mathfrak{p}} \leqslant \dim R_{\mathfrak{p}} = 0 \quad \therefore \operatorname{depth}_{R_{\mathfrak{p}}} R_{\mathfrak{p}} = 0$$

By Prop 2.6, $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} R_{\mathfrak{p}} \quad \therefore \exists\, \frac{a}{s} \in R_{\mathfrak{p}}$ s.t $\mathfrak{p}R_{\mathfrak{p}} = \operatorname{ann}_{R_{\mathfrak{p}}}(\frac{a}{s})$

Write $\mathfrak{p} = (b_1, \dots, b_n)$

$$\begin{aligned} \tfrac{b_i}{1} \in \mathfrak{p}R_{\mathfrak{p}} = \operatorname{ann}_{R_{\mathfrak{p}}}(\tfrac{a}{s}) \ &\Rightarrow \ \tfrac{b_i a}{s} = \tfrac{0}{1} \\ &\Rightarrow \ \exists\, t_i \in R - \mathfrak{p} \text{ s.t } t_i b_i a = 0 \end{aligned}$$

$t := t_1 \cdots t_n \in R - \mathfrak{p}$

<u>Claim</u> $\quad \mathfrak{p} = \operatorname{ann}_R(ta)$ ($\therefore \mathfrak{p} \in \operatorname{Ass}_R R$)

($\subseteq$) $b_i(ta) = t_1 \cdots t_{i-1} t_{i+1} \cdots t_n (t_i b_i a) = 0$

($\supseteq$) $c \in \operatorname{ann}(ta) \Rightarrow 0 = c(ta) = t(ca)$

$$\Rightarrow \tfrac{ca}{s} = \tfrac{0}{1} \Rightarrow \tfrac{c}{1} \in \operatorname{ann}(\tfrac{a}{s}) = \mathfrak{p}R_{\mathfrak{p}} \Rightarrow c \in \mathfrak{p}$$ ■

Ex 3.12.

$$\operatorname{Min} A = \{(x), (y)\} = \operatorname{Ass} A$$
$$\operatorname{Min} B = \{(x)\} \subsetneq \{(x), (x, y)\} = \operatorname{Ass} B$$

By Cor 3.11 and Thm 1.12 :

Cor 3.13. $\# \operatorname{Min} R < \infty$

Prop 3.14. $(R, \mathfrak{m}, k)$ local $x \in \mathfrak{m}$
(1) $\dim R/(x) \geqslant \dim R - 1$
(2) "=" holds if x is a NZD on R

Proof. $n := \dim R/(x)$
(1) By Fact 3.4, $\exists\, \overline{y_1}, \ldots, \overline{y_n} \in \mathfrak{m}/(x)$ s.t

$$(R/(x))/(\overline{y_1}, \ldots, \overline{y_n}) = R/(x, y_1, \ldots, y_n)$$

is Artinian. By Fact 3.4 again, $\dim R \leqslant n + 1$
(2) $\exists\, \mathfrak{p}_0, \ldots, \mathfrak{p}_n \in \operatorname{Spec} R$ s.t

$$(x) \subseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

Suppose $\mathfrak{p}_0 \in \operatorname{Min} R$. Then $\mathfrak{p}_0 \in \operatorname{Ass} R$ by Cor 3.11
Cor 1.10 says all elements in $\mathfrak{p}_0$ are ZDs on R, contradicting that x is a NZD on R
$\therefore \mathfrak{p}_0 \notin \operatorname{Min} R \quad \therefore \exists\, \mathfrak{q} \in \operatorname{Spec} R$ s.t $\mathfrak{q} \subsetneq \mathfrak{p}_0$
The chain

$$\mathfrak{q} \subsetneq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

in $\operatorname{Spec} R$ implies $\dim R \geqslant n + 1$ ■

Ex 3.15.
(1) $\begin{cases} x - y \text{ a NZD on } A \\ \dim A/(x-y) = \dim k[[X]]/(X^2) = 0 = \dim A - 1 \end{cases}$
(2) $\begin{cases} x \text{ a ZD on } A \\ \dim A/(x) = \dim k[[Y]] = 1 > 0 = \dim A - 1 \end{cases}$

4. Cohen–Macaulay rings and Gorenstein rings

Throughout this section, let $(R, \mathfrak{m}, k)$ be local.

Def 4.1. (1) A f.g R-mod M is <u>(maximal) Cohen–Macaulay</u> (CM) if ($M = 0$ or) $\operatorname{depth} M = \dim R$
(2) R is <u>Cohen–Macaulay</u> (CM) if it is CM as an R-module, i.e., $\operatorname{depth} R = \dim R$

Rem 4.2. Any f.g free R-mod is CM if R is CM

Ex 4.3.
(1) A is a CM local ring
(2) B is not a CM local ring
(3) $A/(x), A/(y)$ are CM A-modules

Def 4.4. R is <u>Gorenstein</u> if

$$\mathrm{Ext}^i_R(k,R) \cong \begin{cases} k & (i = \dim R), \\ 0 & (i \neq \dim R). \end{cases}$$

Prop 4.5. A Gor local ring is CM

Prop 4.6. Let $x \in \mathfrak{m}$ be a NZD on R
(1) R is CM $\iff$ $R/(x)$ is CM
(2) R is Gor $\iff$ $R/(x)$ is Gor

Proof. (1) By Props 2.12(2) and 3.14(2),

$$\mathrm{depth}_{R/(x)} R/(x) = \mathrm{depth}_R R - 1$$
$$\dim R/(x) = \dim R - 1$$
$$\therefore\ \mathrm{depth}_R R = \dim R \ \Leftrightarrow\ \mathrm{depth}_{R/(x)} R/(x) = \dim R/(x)$$

(2) By Lem 2.11 and Prop 3.14(2),

$$\mathrm{Ext}^i_{R/(x)}(k, R/(x)) \cong \mathrm{Ext}^{i+1}_R(k,R)$$
$$\dim R/(x) = \dim R - 1$$
$$\therefore\quad \mathrm{Ext}^i_R(k,R) \cong \begin{cases} k & (i = \dim R) \\ 0 & (i \neq \dim R) \end{cases} \ \Leftrightarrow\ \mathrm{Ext}^i_{R/(x)}(k, R/(x)) \cong \begin{cases} k & (i = \dim R/(x)) \\ 0 & (i \neq \dim R/(x)) \end{cases}$$

■

Ex 4.7. $x - y$ a NZD on A

$$A/(x-y) \cong k[[X]]/(X^2)$$

is Gor $\quad\therefore A$ is Gor by Prop 4.6

Thm 4.8 (Characterization of CM modules over Gor local rings)**.** R: Gor $\quad M$: a f.g R-mod $\quad$ TFAE
(1) M is CM
(2) The following hold, where $(-)^* = \mathrm{Hom}_R(-,R)$
  (i) $M \cong M^{**}$
  (ii) $\mathrm{Ext}^i_R(M,R) = 0$ $(\forall\, i > 0)$
  (iii) $\mathrm{Ext}^i_R(M^*,R) = 0$ $(\forall\, i > 0)$

Sketch. (Details: Appendix B)
Induction on $d = \dim R$
(2) $\Rightarrow$ (1):
<u>$d = 0$</u> Easy
<u>$d > 0$</u> $\exists\, x \in \mathfrak{m}$ a NZD on $R \rightsquigarrow$ a NZD on M, M^*

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

$\rightsquigarrow M/xM$ satisfies (i)–(iii)
Ind hyp $\Rightarrow M/xM$ a CM $R/(x)$-mod $\Rightarrow M$ a CM R-mod
(1) $\Rightarrow$ (2):

$\underline{d=0}$ Ind on $\ell = \ell(M)$

$\underline{\ell=1}$ $M \cong k$, $\mathrm{Ext}^i_R(k,R) \cong \begin{cases} k & (i=0) \\ 0 & (i>0) \end{cases} \rightsquigarrow$ done

$\underline{\ell>1}$ $\exists\, 0 \to N \to M \to k \to 0, \quad \ell(N) = \ell - 1$

Ind hyp to $N \rightsquigarrow$ done

$\underline{d>0}$ $\exists\, x \in \mathfrak{m}$ a NZD on $R, M \rightsquigarrow$ a NZD on M^*

M/xM a CM $R/(x)$-mod

Ind hyp $\Rightarrow$ M/xM satisfies (i)–(iii) $\rightsquigarrow$ M satisfies (i)–(iii) ■

Ex 4.9. $A/(x)$ is a CM A-module

$$\begin{cases} A/(x) \cong (A/(x))^* \cong (A/(x))^{**} \\ \mathrm{Ext}^i_R(A/(x), A) = 0 \ (\forall\, i > 0) \end{cases}$$

Cor 4.10. R: Gor $\quad M$: a CM R-mod

(1) M^* is a CM R-mod

(2) $\exists$ exact seq of f.g R-mod

$$0 \to M \to F \to N \to 0$$

s.t F: free, N: CM

Proof. (1) Immediate from Thm 4.8

(2) $\exists\, 0 \to K \to P \to M^* \to 0$: exact s.t P: f.g free

$$\rightsquigarrow \quad 0 \to M^{**} \to P^* \to K^* \to \mathrm{Ext}^1(M^*, R) \quad \text{exact}$$

By Thm 4.8, $M^{**} \cong M$ and $\mathrm{Ext}^1(M^*, R) = 0$

$$\therefore\ 0 \to M \to P^* \to K^* \to 0 \quad \text{exact}$$

By Prop 2.4,

$$\operatorname{depth} K \geqslant \inf\{\operatorname{depth} P, \operatorname{depth} M^* + 1\} = \dim R$$

as $\operatorname{depth} P = \dim R = \operatorname{depth} M^*$

$\therefore K$ is CM $\quad \therefore K^*$ is CM by (1) $\qquad F := P^*,\ N := K^*$ ■

Ex 4.11.

- $A/(x), A/(y)$ are CM
- $(A/(x))^*$ is CM as $(A/(x))^* \cong A/(x)$
- $\exists\, 0 \to A/(x) \to A \to A/(y) \to 0$

Appendix A. Proof of Theorem 1.12

To prove Theorem 1.12, we establish three lemmas.

Lemma A.1. *Let M be a finitely generated R-module. Then there exists a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

of submodules of M such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$.

Proof. We have $M_0 := 0 \subseteq M$. If $M = 0$, then taking $n = 0$, we are done. Hence let us assume $M \neq 0$. By Corollary 1.8 we find an associated prime $\mathfrak{p}_1$ of M. Proposition 1.6 implies that there is an injective homomorphism $f_1 : R/\mathfrak{p}_1 \to M$. Set $M_1 = \operatorname{Im} f_1 \subseteq M$. If $M_1 = M$, then taking $n = 1$, we are done. So we assume $M_1 \neq M$. Then by Corollary 1.8 we find an associated prime $\mathfrak{p}_2$ of M/M_1, and by Proposition 1.6 there is an injection $f_2 : R/\mathfrak{p}_2 \to M/M_1$. Writing $\operatorname{Im} f_2 = M_2/M_1$, we get $M_1 \subseteq M_2 \subseteq M$. If $M_2 = M$, then taking $n = 2$, we are done. Iterating this procedure, we obtain an ascending chain

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subseteq M$$

of submodules of M such that for each i one has $M_i/M_{i-1} = \operatorname{Im} f_i \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$. Since M is Noetherian, this procedure ends in finitely many steps. This means that there exists an integer $n \geqslant 0$ such that $M_n = M$. ■

Lemma A.2. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be an exact sequence of R-modules. Then*

$$\operatorname{Ass} M \subseteq \operatorname{Ass} L \cup \operatorname{Ass} N.$$

Proof. Let $\mathfrak{p}$ be an associated prime of M. Then $\mathfrak{p} = \operatorname{ann}(x)$ for some $x \in M$.

If $Rx \cap \operatorname{Im} f = 0$, then it holds that

$$R/\mathfrak{p} \cong Rx = Rx/(Rx \cap \operatorname{Im} f) \cong (Rx + \operatorname{Im} f)/\operatorname{Im} f \subseteq \operatorname{Coker} f \cong N,$$

which shows that $\mathfrak{p}$ is an associated prime of N by Proposition 1.6.

Let us consider the case where $Rx \cap \operatorname{Im} f \neq 0$. In this case, there exist elements $a \in R$ and $y \in L$ such that $ax = f(y) \neq 0$. We claim that $\mathfrak{p} = \operatorname{ann}(y)$. Indeed, pick any element $b \in \mathfrak{p}$. Then $bx = 0$, and we have

$$f(by) = bf(y) = b(ax) = a(bx) = a \cdot 0 = 0.$$

The injectivity of the map f implies $by = 0$, whence $b \in \operatorname{ann}(y)$. Conversely, pick any element $c \in \operatorname{ann}(y)$. Then $cy = 0$, and we have

$$(ac)x = c(ax) = cf(y) = f(cy) = f(0) = 0,$$

whence $ac \in \mathfrak{p}$. Since $ax \neq 0$, the element a is not in the prime ideal $\mathfrak{p}$. Hence c must be in $\mathfrak{p}$. Now the claim follows, and $\mathfrak{p}$ is an associated prime of L.

Consequently, $\mathfrak{p}$ belongs to the union $\operatorname{Ass} L \cup \operatorname{Ass} N$, which shows the lemma. ■

Lemma A.3. *For each prime ideal $\mathfrak{p}$ of R one has*

$$\operatorname{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}.$$

Proof. We have $\mathfrak{p} = \operatorname{ann}(\overline{1}) \in \operatorname{Ass}_R(R/\mathfrak{p})$. Take any $\mathfrak{q} \in \operatorname{Ass}_R(R/\mathfrak{p})$. Then $\mathfrak{q} = \operatorname{ann}(\overline{x})$ for some $\overline{x} \in R/\mathfrak{p}$. It suffices to show $\mathfrak{q} = \mathfrak{p}$. Fix an element $a \in R$. If $a \in \mathfrak{p}$, then $a \cdot \overline{x} = \overline{ax} = \overline{0}$ as $ax \in \mathfrak{p}$. Hence $a \in \mathfrak{q}$, and we get $\mathfrak{p} \subseteq \mathfrak{q}$. Conversely, if $a \in \mathfrak{q}$, then $a \cdot \overline{x} = \overline{0}$, which implies $ax \in \mathfrak{p}$. Since $\mathfrak{q}$ is a prime ideal, $\mathfrak{q} \neq R$. Hence $\overline{x} \neq \overline{0}$, that is, $x \notin \mathfrak{p}$. Therefore $a \in \mathfrak{p}$, which shows $\mathfrak{q} \subseteq \mathfrak{p}$. ■

Remark A.4. (1) The filtration in Lemma A.1 can be regarded as a generalization of a composition series of a module of finite length.

(2) The conclusion of Lemma A.2 can be improved as $\operatorname{Ass} L \subseteq \operatorname{Ass} M \subseteq \operatorname{Ass} L \cup \operatorname{Ass} N$.

(3) Lemma A.3 especially says that the only associated prime of a Noetherian domain is the zero ideal.

Now we can give a proof of Theorem 1.12.

Proof of Theorem 1.12. Lemma A.1 yields a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

of submodules of M such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec} R$. For each $1 \leqslant i \leqslant n$ there is an exact sequence

$$0 \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to 0.$$

Applying Lemmas A.2 and A.3, we get

$$\operatorname{Ass} M_i \subseteq \operatorname{Ass} M_{i-1} \cup \operatorname{Ass} R/\mathfrak{p}_i = \operatorname{Ass} M_{i-1} \cup \{\mathfrak{p}_i\}.$$

Thus we obtain

$$\begin{aligned} \operatorname{Ass} M = \operatorname{Ass} M_n &\subseteq \operatorname{Ass} M_{n-1} \cup \{\mathfrak{p}_n\} \\ &\subseteq \operatorname{Ass} M_{n-2} \cup \{\mathfrak{p}_{n-1}, \mathfrak{p}_n\} \\ &\subseteq \cdots \\ &\subseteq \operatorname{Ass} M_0 \cup \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}, \end{aligned}$$

where the last equality holds since $\operatorname{Ass} M_0 = \emptyset$ by Corollary 1.8. Therefore $\operatorname{Ass} M$ is a finite set. ■

Appendix B. Proof of Theorem 4.8

To prove Theorem 4.8, we establish three lemmas.

Lemma B.1. *Let $(R, \mathfrak{m}, k)$ be an Artinian local ring. For each finitely generated R-module M, there exists a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

of submodules of M such that $M_i/M_{i-1} \cong k$ for all $1 \leqslant i \leqslant n$.

Proof. This is a direct consequence of Lemma A.1 and Corollary 3.6. ■

Lemma B.2. *Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of R-modules and R-homomorphisms with exact rows. If α and γ are isomorphisms, then so is β.

Proof. This is easy to show by diagram chasing. ■

Lemma B.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $f : M \to N$ be a homomorphism of finitely generated R-modules. Let $x \in \mathfrak{m}$ be an nonzerodivisor on N. If the induced homomorphism $\overline{f} : M/xM \to N/xN$ is an isomorphism, then so is f.*

Proof. There is an exact sequence

$$0 \to K \xrightarrow{\theta} M \xrightarrow{f} N \xrightarrow{\pi} C \to 0,$$

where $K = \operatorname{Ker} f$, $C = \operatorname{Coker} f$, and θ, π are natural maps. Applying $- \otimes_R R/(x)$ to this, we get an exact sequence

$$M/xM \xrightarrow{\overline{f}} N/xN \xrightarrow{\overline{\pi}} C/xC \to 0.$$

Since $\overline{f}$ is an isomorphism, we have $C/xC = 0$. Proposition 3.2 implies $C = 0$. Thus there is a short exact sequence

$$0 \to K \xrightarrow{\theta} M \xrightarrow{f} N \to 0.$$

We claim that the induced map $\overline{\theta} : K/xK \to M/xM$ is injective. Indeed, let $\overline{z} \in K/xK$ be an element such that $\overline{\theta}(\overline{z}) = \overline{0}$. Then $\overline{z} = \overline{0}$ in M/xM. This means $z \in xM$, and $z = xw$ for some $w \in M$. As $z \in K = \operatorname{Ker} f$, there are equalities

$$0 = f(z) = f(xw) = xf(w)$$

in N. Since x is a nonzerodivisor on N, we have $f(w) = 0$, which says $w \in K$. Therefore $z = xw \in xK$, and $\overline{z} = \overline{0}$ in K/xK. The claim now follows.

By this claim, the induced sequence

$$0 \to K/xK \xrightarrow{\overline{\theta}} M/xM \xrightarrow{\overline{f}} N/xN \to 0$$

is exact, and we get $K/xK = 0$ as $\overline{f}$ is an isomorphism. Using Proposition 3.2 again, we obtain $K = 0$. Consequently, f is an isomorphism. ■

Remark B.4. (1) Lemma B.1 is included in the Jordan–Hölder theorem for modules of finite length; see [1, Chapter 6].

(2) Lemma B.2 is a special case of the five lemma; see [5, Proposition 2.72].

Now we can give a proof of Theorem 4.8.

Proof of Theorem 4.8. Put $d := \dim R = \operatorname{depth} R$. We may assume $M \neq 0$.

(2) $\Rightarrow$ (1): We use induction on d. When $d = 0$, we have $0 \leqslant \operatorname{depth} M \leqslant d = 0$ by Corollary 3.9, whence $\operatorname{depth} M = d$ and M is Cohen–Macaulay. Let $d \geqslant 1$. We find a nonzerodivisor $x \in \mathfrak{m}$ on R by Proposition 2.9. There is an exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0.$$

Applying $\operatorname{Hom}_R(M, -)$ and $\operatorname{Hom}_R(M^*, -)$ to this and using the isomorphism $M \cong M^{**}$, we obtain exact sequences

$$0 \to M^* \xrightarrow{x} M^* \quad \text{and} \quad 0 \to M \xrightarrow{x} M,$$

which say that x is a nonzerodivisor on M and M^*. Hence there are exact sequences

$$0 \to M \xrightarrow{x} M \to M/xM \to 0, \tag{B.4.1}$$

$$0 \to M^* \xrightarrow{x} M^* \to M^*/xM^* \to 0. \tag{B.4.2}$$

Applying $(-)^*$ to (B.4.1) makes an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M/xM)^* & \longrightarrow & M^* & \xrightarrow{x} & M^* \\ & \longrightarrow & \mathrm{Ext}^1_R(M/xM,R) & \longrightarrow & \mathrm{Ext}^1_R(M,R) & \xrightarrow{x} & \mathrm{Ext}^1_R(M,R) \\ & \longrightarrow & \mathrm{Ext}^2_R(M/xM,R) & \longrightarrow & \mathrm{Ext}^2_R(M,R) & \xrightarrow{x} & \mathrm{Ext}^2_R(M,R) \\ & \longrightarrow & \cdots & & & & \end{array}$$

Since $\mathrm{Ext}^{>0}_R(M,R)=0$ and x is a nonzerodivisor on M^*, we see that

$$\mathrm{Ext}^i_{R/(x)}(M/xM,R/(x)) \cong \mathrm{Ext}^{i+1}_R(M/xM,R) \cong \begin{cases} M^*/xM^* & (i=0), \\ 0 & (i\neq 0), \end{cases}$$

where the first isomorphism follows from Lemma 2.11. An analogous argument using (B.4.2) shows

$$\mathrm{Ext}^i_{R/(x)}(M^*/xM^*,R/(x)) \cong \mathrm{Ext}^{i+1}_R(M^*/xM^*,R) \cong \begin{cases} M/xM & (i=0), \\ 0 & (i\neq 0). \end{cases}$$

In particular, we have

$$M/xM \cong \mathrm{Hom}_{R/(x)}(M^*/xM^*,R/(x)) \cong \mathrm{Hom}_{R/(x)}(\mathrm{Hom}_{R/(x)}(M/xM,R/(x)),R/(x)).$$

Proposition 3.14(2) implies $\dim R/(x)=d-1$, so by the induction hypothesis M/xM is a Cohen–Macaulay $R/(x)$-module. Hence $\mathrm{depth}_{R/(x)} M/xM = \dim R/(x)$, and we have

$$\mathrm{depth}_R M - 1 = \mathrm{depth}_{R/(x)} M/xM = \dim R/(x) = d-1,$$

where the first equality follows from Proposition 2.12(2). Therefore $\mathrm{depth}_R M = d$, that is, M is Cohen–Macaulay.

(1) $\Rightarrow$ (2): Again we use induction on d. Let us consider the case $d=0$. (In this case, note that every finitely generated R-module is Cohen–Macaulay by the proof of (2) $\Rightarrow$ (1) for $d=0$.) As R is Gorenstein, we have

$$\mathrm{Ext}^i_R(k,R) \cong \begin{cases} k & (i=0), \\ 0 & (i\neq 0). \end{cases}$$

Since R is Artinian by Corollary 3.6, it follows from Lemma B.1 that there is a filtration

$$0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M$$

of submodules of M such that $M_i/M_{i-1}\cong k$ for each $1\leqslant i\leqslant n$.

Let us deduce (2) by induction on n. When $n=1$, the module M is isomorphic to k. Since $k\cong \mathrm{Ext}^0_R(k,R)=\mathrm{Hom}_R(k,R)=k^*$, we have

$$k\cong k^{**}, \quad \mathrm{Ext}^i_R(k^*,R)\cong \mathrm{Ext}^i_R(k,R)=0 \text{ for all } i>0. \tag{B.4.3}$$

Therefore (2) holds when $n=1$. Next, let $n\geqslant 2$. Setting $N=M_{n-1}$, we have an exact sequence

$$0\to N\to M\to k\to 0.$$

We can apply the hypothesis of induction (on n) to N to see that

$$N\cong N^{**}, \quad \mathrm{Ext}^{>0}_R(N,R)=0=\mathrm{Ext}^{>0}_R(N^*,R). \tag{B.4.4}$$

Applying $(-)^*$ to the above short exact sequence gives rise to an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & k^* & \longrightarrow & M^* & \longrightarrow & N^* \\ & \longrightarrow & \mathrm{Ext}^1_R(k,R) & \longrightarrow & \mathrm{Ext}^1_R(M,R) & \longrightarrow & \mathrm{Ext}^1_R(N,R) \\ & \longrightarrow & \mathrm{Ext}^2_R(k,R) & \longrightarrow & \mathrm{Ext}^2_R(M,R) & \longrightarrow & \mathrm{Ext}^2_R(N,R) \\ & \longrightarrow & \cdots & & & & \end{array}$$

This exact sequence together with (B.4.3) and (B.4.4) shows that there is an exact sequence

$$0 \to k^* \to M^* \to N^* \to 0,$$

and $\mathrm{Ext}^{>0}_R(M,R) = 0$. Applying $(-)^*$ to this short exact sequence, we get another exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N^{**} & \longrightarrow & M^{**} & \longrightarrow & k^{**} \\ & \longrightarrow & \mathrm{Ext}^1_R(N^*,R) & \longrightarrow & \mathrm{Ext}^1_R(M^*,R) & \longrightarrow & \mathrm{Ext}^1_R(k^*,R) \\ & \longrightarrow & \mathrm{Ext}^2_R(N^*,R) & \longrightarrow & \mathrm{Ext}^2_R(M^*,R) & \longrightarrow & \mathrm{Ext}^2_R(k^*,R) \\ & \longrightarrow & \cdots & & & & \end{array}$$

From this exact sequence together with (B.4.3) and (B.4.4) again, we obtain an exact sequence

$$0 \to N^{**} \to M^{**} \to k^{**} \to 0,$$

and $\mathrm{Ext}^{>0}_R(M^*,R) = 0$. There is a commutative diagram with natural vertical maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N^{**} & \longrightarrow & M^{**} & \longrightarrow & k^{**} & \longrightarrow & 0 \end{array}$$

By (B.4.3), (B.4.4) again, the left and right vertical maps are isomorphisms. It follows from Lemma B.2 that $M \cong M^{**}$. Thus the proof of the case $d = 0$ is completed.

Next, let us consider the case $d \geqslant 1$. Since $\operatorname{depth} R = \operatorname{depth} M = d \geqslant 1$, the maximal ideal $\mathfrak{m}$ is not an associated prime of R or M by Proposition 2.6. Hence $\mathfrak{m}$ is not contained in any prime ideal belonging to $\operatorname{Ass} R \cup \operatorname{Ass} M$, and we can choose an element $x \in \mathfrak{m}$ such that

$$x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass} R \cup \operatorname{Ass} M} \mathfrak{p}$$

by Theorem 1.12 and Lemma 2.8. Corollary 1.10 says that x is a nonzerodivisor on R and M. Thus $R/(x)$ is Gorenstein by Proposition 4.6(2), and

$$\operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M - 1 = d - 1 = \dim R/(x)$$

by Propositions 2.12(2) and 3.14(2). Hence M/xM is a Cohen–Macaulay $R/(x)$-module. The induction hypothesis implies:

$$\begin{cases} M/xM \cong \mathrm{Hom}_{R/(x)}(\mathrm{Hom}_{R/(x)}(M/xM, R/(x)), R/(x)), \\ \mathrm{Ext}^i_{R/(x)}(M/xM, R/(x)) = 0 \text{ for all } i \neq 0, \\ \mathrm{Ext}^i_{R/(x)}(\mathrm{Hom}_{R/(x)}(M/xM, R/(x)), R/(x)) = 0 \text{ for all } i \neq 0. \end{cases}$$

Using Lemma 2.11, we get

$$\begin{cases} M/xM \cong \operatorname{Ext}^1_R(\operatorname{Ext}^1_R(M/xM, R), R), \\ \operatorname{Ext}^{i+1}_R(M/xM, R) = 0 \text{ for all } i \neq 0, \\ \operatorname{Ext}^{i+1}_R(\operatorname{Ext}^1_R(M/xM, R), R) = 0 \text{ for all } i \neq 0. \end{cases}$$

From the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ we get an exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M/xM)^* & \longrightarrow & M^* & \xrightarrow{x} & M^* \\ & \longrightarrow & \operatorname{Ext}^1_R(M/xM, R) & \longrightarrow & \operatorname{Ext}^1_R(M, R) & \xrightarrow{x} & \operatorname{Ext}^1_R(M, R) \\ & \longrightarrow & \operatorname{Ext}^2_R(M/xM, R) & \longrightarrow & \operatorname{Ext}^2_R(M, R) & \xrightarrow{x} & \operatorname{Ext}^2_R(M, R) \\ & \longrightarrow & \cdots & & & & \end{array}$$

We see from this that x is a nonzerodivisor on M^*, and that $\operatorname{Ext}^{>0}_R(M, R) = 0$ by Proposition 3.2, whence $\operatorname{Ext}^1_R(M/xM, R) \cong M^*/xM^*$. Therefore

$$M/xM \cong \operatorname{Ext}^1_R(M^*/xM^*, R), \qquad \operatorname{Ext}^{\neq 1}_R(M^*/xM^*, R) = 0.$$

A similar argument for the exact sequence $0 \to M^* \xrightarrow{x} M^* \to M^*/xM^* \to 0$ shows that x is a nonzerodivisor on M^{**}, that $\operatorname{Ext}^{>0}_R(M^*, R) = 0$ and that $\operatorname{Ext}^1_R(M^*/xM^*, R) \cong M^{**}/xM^{**}$. We obtain an isomorphism $M/xM \cong M^{**}/xM^{**}$, and it follows from Lemma B.3 that $M \cong M^{**}$.

Now the proof of the theorem is completed. ■

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