UPPER COHEN-MACAULAY DIMENSION

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Abstract. In this paper, we define a homological invariant for finitely generated modules over a commutative noetherian local ring, which we call upper Cohen-Macaulay dimension. This invariant is quite similar to Cohen-Macaulay dimension that has been introduced by Gerko. Also we define a homological invariant with respect to a local homomorphism of local rings. This invariant links upper Cohen-Macaulay dimension with Gorenstein dimension.

1. Introduction

Throughout the present paper, all rings are assumed to be commutative noetherian rings, and all modules are assumed to be finitely generated modules.

Let R be a local ring with residue class field k. Projective dimension pd_R is one of the most classical homological dimensions. Complete intersection dimension (abbr. CI-dimension) CI-$\dim_R$ was introduced by Avramov, Gasharov, and Peeva [4]. Gorenstein dimension (abbr. G-dimension) G-$\dim_R$ was defined by Auslander [1], and was developed by Auslander and Bridger [2]. Cohen-Macaulay dimension (abbr. CM-dimension) CM-$\dim_R$ was introduced by Gerko [11].

Every one of these dimensions is a homological invariant for R-modules which characterizes a certain property of local rings and satisfies a certain equality. Let i_R be a numerical invariant for R-modules, i.e. $i_R(M) \in \mathbb{N} \cup \{\infty\}$ for an R-module M, and let $\mathcal{P}$ be a property of local rings. The following conditions hold for the pairs $(\mathcal{P}, i_R) =$ (regular, pd_R), (complete intersection, CI-$\dim_R$), (Gorenstein, G-$\dim_R$), and (Cohen-Macaulay, CM-$\dim_R$).

(a) The following conditions are equivalent.
   i) R satisfies the property $\mathcal{P}$.
   ii) $i_R(M) < \infty$ for any R-module M.
   iii) $i_R(k) < \infty$.

(b) Let M be a non-zero R-module with $i_R(M) < \infty$. Then

$$i_R(M) = \operatorname{depth} R - \operatorname{depth}_R M.$$

In this paper, modifying the definition of CM-dimension, we will define a new homological invariant for R-modules which we will call upper Cohen-Macaulay dimension (abbr. CM*-dimension) and will denote by CM*-$\dim_R$. This invariant interpolates between CM-dimension and G-dimension: let M be an R-module. Then

$$\text{CM-}\dim_R M \leq \text{CM}^*\text{-}\dim_R M \leq \text{G-}\dim_R M.$$

The equalities hold to the left of any finite dimension.

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CM*-dimension is quite similar to CM-dimension: it has many properties analogous to those of CM-dimension. For example, the above two conditions (a), (b) also hold for the pair ($\mathcal{P}$, i_R)=(Cohen-Macaulay, CM*-dim$_R$).

Let $\phi : S \to R$ be a local homomorphism of local rings. The main purpose of this paper is to provide a new homological invariant for R-modules with respect to the homomorphism ϕ, which we call upper Cohen-Macaulay dimension relative to ϕ and denote by CM*-dim$_\phi$. We define it by using the idea of G-factorizations.

In Section 2, we will make a list of properties of CM*-dimension. In our sense, it will be *absolute* CM*-dimension.

In Section 3, which is the main section of this paper, we will make the precise definition of *relative* CM*-dimension CM*-dim$_\phi$, and will study the properties of this dimension. Namely, we shall prove the following:

(A) The following conditions are equivalent.
  i) R is Cohen-Macaulay and S is Gorenstein.
  ii) CM*-dim$_\phi M < \infty$ for any R-module M.
  iii) CM*-dim$_\phi k < \infty$.

(B) Let M be a non-zero R-module with CM*-dim$_\phi M < \infty$. Then
$$\text{CM}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(C) i) Suppose that ϕ is faithfully flat. Let M be an R-module. Then
$$\text{CM}^*\text{-dim}_R M \leq \text{CM}^*\text{-dim}_\phi M \leq \text{G-dim}_R M.$$
  The equalities hold to the left of any finite dimension.
  ii) If S is the prime field of R and ϕ is the natural embedding, then
$$\text{CM}^*\text{-dim}_\phi M = \text{CM}^*\text{-dim}_R M$$
  for any R-module M.
  iii) If S is equal to R and ϕ is the identity map, then
$$\text{CM}^*\text{-dim}_\phi M = \text{G-dim}_R M$$
  for any R-module M.

The results (A), (B) are analogues of the conditions (a), (b). The result (C) says that relative CM*-dimension connects absolute CM*-dimension with G-dimension; relative CM*-dimension coincides with absolute CM*-dimension (resp. G-dimension) as a numerical invariant for R-modules if S is the "smallest" (resp. "largest") subring of R.

2. Preliminary

Throughout this section, $(R, \mathfrak{m}, k)$ is always a local ring. We begin with recalling the definition of Gorenstein dimension (abbr. G-dimension). Denote by $\Omega_R^n M$ the nth syzygy module of an R-module M.

Definition 2.1. Let M be an R-module.

(1) If the following conditions hold, then we say that M has *G-dimension zero*, and write G-dim$_R M = 0$.
  i) The natural homomorphism $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R)$ is an isomorphism.
  ii) $\operatorname{Ext}_R^i(M, R) = 0$ for every $i > 0$.
  iii) $\operatorname{Ext}_R^i(\operatorname{Hom}_R(M, R), R) = 0$ for every $i > 0$.

(2) If $\Omega_R^n M$ has G-dimension zero for a non-negative integer n, then we say that M has *G-dimension at most* n, and write G-$\dim_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write G-$\dim_R M = \infty$.

(3) If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has *G-dimension* n, and write G-$\dim_R M = n$.

For the properties of G-dimension, we refer to [2], [6], [13], and [15].

Now we recall the definition of Cohen-Macaulay dimension (abbr. CM-dimension) which has been introduced by Gerko.

Definition 2.2. [11, Definition 3.1, 3.2]

(1) An R-module M is called *G-perfect* if G-$\dim_R M = \mathrm{grade}_R M$.

(2) A local homomorphism $\phi : S \to R$ of local rings is called a *G-deformation* if ϕ is surjective and R is G-perfect as an S-module.

(3) A diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings is called a *G-quasideformation* of R if α is faithfully flat and ϕ is a G-deformation.

(4) For an R-module M, the *Cohen-Macaulay dimension* of M is defined as follows:

$$\text{CM-}\dim_R M = \inf \left\{ \begin{array}{r|l} \text{G-}\dim_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is a} \\ -\text{G-}\dim_S R' & \text{G-quasideformation of } R \end{array} \right\}.$$

Modifying the above definition, we make the following definition.

Definition 2.3. (1) We call a diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings an *upper G-quasideformation* of R if it is a G-quasideformation and the closed fiber of α is regular.

(2) For an R-module M, we define the *upper Cohen-Macaulay dimension* (abbr. CM*-dimension) of M as follows:

$$\text{CM}^*\text{-}\dim_R M = \inf \left\{ \begin{array}{r|l} \text{G-}\dim_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is an upper} \\ -\text{G-}\dim_S R' & \text{G-quasideformation of } R \end{array} \right\}.$$

Comparing the definition of CM*-dimension with that of CM-dimension, one easily see that

$$\text{CM-}\dim_R M \leq \text{CM}^*\text{-}\dim_R M$$

for any R-module M; the equality holds if CM*-$\dim_R M < \infty$. CM*-dimension shares a lot of properties with CM-dimension. We shall exhibit a list of them in the rest of this section. We will omit the proofs of them because they can be proved quite similarly to the corresponding results of CM-dimension.

Theorem 2.4. [11, Theorem 3.9] *The following conditions are equivalent.*

i) *R is Cohen-Macaulay.*

ii) CM*-$\dim_R M < \infty$ *for any R-module M.*

iii) CM*-$\dim_R k < \infty$.

The CM*-dimension satisfies the equality analogous to the Auslander-Buchsbaum formula:

Theorem 2.5. [11, Theorem 3.8] *Let M be a non-zero R-module. If* CM*-$\dim_R M < \infty$, *then*

$$\text{CM}^*\text{-}\dim_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Christensen defines a *semi-dualizing module* in his paper [7], which Gerko and Golod call a *suitable module* in [11] and [12]. Developing this concept a little, we make the following definition as a matter of convenience.

Definition 2.6. Let M and C be R-modules. We call C a *semi-dualizing module for M* if it satisfies the following conditions.

i) The natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.
ii) $\mathrm{Ext}^i_R(C, C) = 0$ for any $i > 0$.
iii) The natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism.
iv) $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for any $i > 0$.

It is worth noting that an R-module M has G-dimension zero if and only if R is a semi-dualizing module for M.

Refering to [8, Proposition 1.1], one can easily show that semi-dualizing modules enjoy the following properties.

Proposition 2.7. *Let C be a semi-dualizing R-module for some R-module. Then,*

(1) *C is faithful. In particular,* $\dim_R C = \dim R$.
(2) *A sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ in R is R-regular if and only if it is C-regular. In particular,* $\mathrm{depth}_R C = \mathrm{depth}\, R$.

It is possible to describe CM*-dimension in terms of a semi-dualizing module:

Theorem 2.8. [11, Theorem 3.7] *The following conditions are equivalent for an R-module M and a non-negative integer n.*

i) $\mathrm{CM}^*\text{-}\mathrm{dim}_R M \leq n$.
ii) *There exist a faithfully flat homomorphism $R \to R'$ of local rings whose closed fiber is regular, and an R'-module C such that C is a semi-dualizing module for $\Omega^n_R M \otimes_R R'$ as an R'-module.*

In particular, $\mathrm{CM}^*\text{-}\mathrm{dim}_R M \geq 0$ *for any R-module M.*

Corollary 2.9. *For an R-module M, we have*

$$\mathrm{CM}^*\text{-}\mathrm{dim}_R M \leq \mathrm{G}\text{-}\mathrm{dim}_R M.$$

The equality holds if $\mathrm{G}\text{-}\mathrm{dim}_R M < \infty$.

We end off this section by making a remark on G-dimension for later use:

Theorem 2.10. [15, Theorem 2.7] *For an R-module M,* $\mathrm{G}\text{-}\mathrm{dim}_R M < \infty$ *if and only if the natural morphism $M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$ is an isomorphism in the derived category of the category of R-modules.*

3. Relative CM*-dimension

In this section, we observe CM*-dimension from a relative point of view. Throughout the section, ϕ always denotes a local homomorphism from a local ring $(S, \mathfrak{n}, \ell)$ to a local ring $(R, \mathfrak{m}, k)$.

We consider a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R \end{array}$$

of local homomorphisms of local rings, which we call a *G-factorization* of ϕ if β is a faithfully flat homomorphism and $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is an upper G-quasideformation of R. Using the idea of G-factorization, we make the following definition.

Definition 3.1. Let M be an R-module. We define the *upper Cohen-Macaulay dimension* of M *relative* to ϕ, denoted by $\mathrm{CM}^*\text{-}\dim_\phi M$, as follows:

$$\mathrm{CM}^*\text{-}\dim_\phi M = \inf \left\{ \begin{array}{r|l} \text{G-}\dim_{S'}(M \otimes_R R') & S \to S' \to R' \leftarrow R \\ -\text{G-}\dim_{S'} R' & \text{is a G-factorization of } \phi \end{array} \right\}.$$

In the rest of this paper, the dimensions $\mathrm{CM}^*\text{-}\dim_R$ and $\mathrm{CM}^*\text{-}\dim_\phi$ will be often called *absolute* CM^*-dimension and *relative* CM^*-dimension, respectively.

We use the convention that the infimum of the empty set is ∞. It is natural to ask whether ϕ always has a G-factorization. The following example says that this is not true in general.

Example 3.2. Suppose that $R = \ell$ is the residue class field of S, and ϕ is the natural surjection from S to ℓ. Furthermore, suppose that S is not Gorenstein. Then ϕ does not have a G-factorization. (Hence we have $\mathrm{CM}^*\text{-}\dim_\phi M = \infty$ for any R-module M.)

Indeed, assume that ϕ has a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$. Then, since the closed fiber of α is regular, R' is a regular local ring. Let $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ be a regular system of parameters of R'. Since $\text{G-}\dim_{S'} R' = \mathrm{grade}_{S'} R' < \infty$ and $\boldsymbol{x}$ is an R'-regular sequence, we see that $\text{G-}\dim_{S'} R'/(\boldsymbol{x}) < \infty$. Note that $R'/(\boldsymbol{x})$ is isomorphic to the residue class field of S'. Therefore S' is a Gorenstein local ring, and hence so is S because β is faithfully flat. This contradicts our assumption.

From the above example, we see that ϕ does not necessarily have a G-factorization in a general setting. However it seems that ϕ has a G-factorization at least when S is Gorenstein. We can prove it if we furthermore assume that S contains a field. To do this, we prepare a couple of lemmas.

Lemma 3.3. *Let $\phi : S \to R$ be a local homomorphism of complete local rings which have the same coefficient field k. Put $S' = S \widehat{\otimes}_k R$, and define $\lambda : S \to S'$ by $\lambda(b) = b \widehat{\otimes} 1$, $\varepsilon : S' \to R$ by $\varepsilon(b \widehat{\otimes} a) = \phi(b)a$. Suppose that S is Gorenstein. Then $S \xrightarrow{\lambda} S' \xrightarrow{\varepsilon} R \xleftarrow{\mathrm{id}} R$ is a G-factorization of ϕ.*

Proof. Take a minimal system of generators $y_1, y_2, \cdots, y_s$ of the maximal ideal of S. Put $J = \mathrm{Ker}\ \varepsilon$ and $dy_i = y_i \widehat{\otimes} 1 - 1 \widehat{\otimes} \phi(y_i) \in S'$ for each $1 \leq i \leq s$.

Claim 1. $J = (dy_1, dy_2, \cdots, dy_s)S'$.

Indeed, put $J_0 = (dy_1, dy_2, \cdots, dy_s)$. Take an element $z = b \widehat{\otimes} a$ in J, and let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have $b \widehat{\otimes} 1 = \sum b_{i_1 i_2 \cdots i_s} (y_1 \widehat{\otimes} 1)^{i_1} (y_2 \widehat{\otimes} 1)^{i_2} \cdots (y_s \widehat{\otimes} 1)^{i_s} \equiv \sum b_{i_1 i_2 \cdots i_s} (1 \widehat{\otimes} \phi(y_1))^{i_1} (1 \widehat{\otimes} \phi(y_2))^{i_2} \cdots (1 \widehat{\otimes} \phi(y_s))^{i_s} = 1 \widehat{\otimes} \phi(b)$ modulo J_0. It follows that $z \equiv 1 \widehat{\otimes} \phi(b) a$ modulo J_0. Since $\phi(b)a = \varepsilon(b \widehat{\otimes} a) = 0$, we have $z \equiv 0$ modulo J_0. Hence $z \in J_0$, and we see that $J = J_0$.

Claim 2. If S is regular, then the sequence $dy_1, dy_2, \cdots, dy_s$ is an S'-regular sequence.

In fact, since S is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i)$ for

$1 \leq i \leq s$. Note that there is an automorphism on S' which sends Y_i to dy_i. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is S'-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form a regular sequence on S'.

Now, let $T = k[[Y_1, Y_2, \cdots, Y_s]]$ be a formal power series ring and consider S to be a T-algebra in the natural way. Put $T' = T\widehat{\otimes}_k R$. Since the rings S, T are Gorenstein, we have $\mathbf{R}\mathrm{Hom}_T(S, T) \cong S[-e]$, where $e = \dim T - \dim S$. Note that T' is faithfully flat over T. Hence $\mathbf{R}\mathrm{Hom}_{T'}(S', T') \cong S'[-e]$. On the other hand, since T is regular, it follows from the claims that the sequence $Y_1 - \phi(y_1), Y_2 - \phi(y_2), \cdots, Y_s - \phi(y_s)$ in T' is a T'-regular sequence. Hence we see that $\mathbf{R}\mathrm{Hom}_{T'}(R, T') \cong R[-s]$. Therefore we have $\mathbf{R}\mathrm{Hom}_{S'}(R, S') \cong \mathbf{R}\mathrm{Hom}_{S'}(R, \mathbf{R}\mathrm{Hom}_{T'}(S', T')[e]) \cong \mathbf{R}\mathrm{Hom}_{T'}(R, T')[e] \cong R[e - s]$. Thus it follows that $\text{G-dim}_{S'} R = \mathrm{grade}_{S'} R = s - e < \infty$. $\square$

To show the existence of G-factorizations, we need the following type of factorizations, which are called *Cohen factorizations.*

Lemma 3.4. [3, Theorem 1.1] *Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism of local rings, and $\alpha : R \to \widehat{R}$ be the natural embedding into the $\mathfrak{m}$-adic completion. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & \widehat{R} \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R \end{array}$$

such that S' is a local ring, β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is a surjective homomorphism.

Now we can prove the following theorem.

Theorem 3.5. *Let S be a Gorenstein local ring containing a field. Then any local homomorphism $\phi : S \to R$ of local rings has a G-factorization.*

Proof. Replacing R and S with their completions respectively, we may assume that R and S are complete. By Lemma 3.4, ϕ has a Cohen factorization

$$\begin{tikzcd} & S' \arrow[dr, "\phi'"] & \\ S \arrow[ur, "\beta"] \arrow[rr, "\phi"'] & & R, \end{tikzcd}$$

where β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is surjective. Hence S' is also Gorenstein. Thus, replacing S with S', we may assume that ϕ is surjective. In particular, R and S have the same coefficient field. Then it follows from Lemma 3.3 that ϕ has a G-factorization. $\square$

Conjecture 3.6. If S is an arbitrary Gorenstein local ring which may not contain a field, then every local homomorphism $\phi : S \to R$ has a G-factorization.

In the following theorem, we compare relative CM*-dimension with absolute CM*-dimension.

Theorem 3.7. *Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism as before.*

(1) *For any R-module M, we have*

$$\mathrm{CM}^*\text{-}\dim_\phi M \geq \mathrm{CM}^*\text{-}\dim_R M.$$

In particular, $\mathrm{CM}^*\text{-}\dim_\phi M \geq 0$.

(2) *If S is regular and ϕ is faithfully flat, then*

$$\mathrm{CM}^*\text{-}\dim_\phi M = \mathrm{CM}^*\text{-}\dim_R M$$

for any R-module M.

Proof. (1) If $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a G-factorization of ϕ, then $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is an upper G-quasideformation of R. Hence, comparing Definition 3.1 with Definition 2.3, we have the required inequality.

(2) It is enough to show that if $\mathrm{CM}^*\text{-}\dim_R M = n < \infty$ then $\mathrm{CM}^*\text{-}\dim_\phi M \leq n$. Theorem 2.8 says that there exist a faithfully flat homomorphism $\alpha : R \to R'$ of local rings with regular closed fiber, and a semi-dualizing R'-module C for $N := \Omega^n_{R'}(M \otimes_R R')$. Let $S' = R' \ltimes C$ be the trivial extension of R' by C. Let $\beta : S \to S'$ be the composite map of ϕ, α, and the natural inclusion $R' \to S'$, and let $\phi' : S' \to R'$ be the natural surjection.

Claim 1. β is faithfully flat.

In fact, let $\boldsymbol{y} = y_1, y_2, \cdots, y_n$ be a regular system of parameters of S. Since ϕ and α are faithfully flat, $\boldsymbol{y}$ is an R'-regular sequence, and hence is a C-regular sequence by Proposition 2.7.2. Note that the Koszul complex $\mathrm{K}_\bullet(\boldsymbol{y}, S)$ is an S-free resolution of $S/(\boldsymbol{y}) = S/\mathfrak{n}$. Since $\mathrm{K}_\bullet(\boldsymbol{y}, C) \cong \mathrm{K}_\bullet(\boldsymbol{y}, S) \otimes_S C$ and $\boldsymbol{y}$ is a C-regular sequence, we have $\mathrm{Tor}_1^S(S/\mathfrak{n}, C) \cong \mathrm{H}_1(\boldsymbol{y}, C) = 0$. It follows from the local criteria of flatness that C is flat over S. Since R' is also flat over S, so is S'. Therefore β is a flat local homomorphism, and hence is faithfully flat.

Claim 2. $\mathrm{G}\text{-}\dim_{S'} R' = 0$ and $\mathrm{G}\text{-}\dim_{S'}(M \otimes_R R') = n$.

Indeed, note that $\mathbf{R}\mathrm{Hom}_{R'}(S', C) \cong S'$. Hence we have $\mathbf{R}\mathrm{Hom}_{S'}(R', S') \cong C$. Therefore we see that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S'), S') &\cong \mathbf{R}\mathrm{Hom}_{S'}(C, \mathbf{R}\mathrm{Hom}_{R'}(S', C)) \\ &\cong \mathbf{R}\mathrm{Hom}_{R'}(C, C) \\ &\cong R'\end{aligned}$$

because C is a semi-dualizing R'-module. It follows from Theorem 2.10 that $\mathrm{G}\text{-}\dim_{S'} R' < \infty$. Thus, we have $\mathrm{G}\text{-}\dim_{S'} R' = \operatorname{depth} S' - \operatorname{depth} R' = 0$. On the other hand, since C is a semi-dualizing module for N as an R'-module, it is easy to see that $\mathbf{R}\mathrm{Hom}_{R'}(N, C) \cong \mathrm{Hom}_{R'}(N, C)$ and

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(N, S'), S') &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{R'}(N, C), C) \\ &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathrm{Hom}_{R'}(N, C), C) \\ &\cong \mathrm{Hom}_{R'}(\mathrm{Hom}_{R'}(N, C), C) \\ &\cong N.\end{aligned}$$

Applying Theorem 2.10 again, we see that $\mathrm{G}\text{-}\dim_{S'} N < \infty$. In the above we have shown that $\mathrm{G}\text{-}\dim_{S'} R' < \infty$. Hence $\mathrm{G}\text{-}\dim_{S'} F < \infty$ for any free R'-module F. Therefore we have $\mathrm{G}\text{-}\dim_{S'}(M \otimes_R R') < \infty$. Thus, we see that $\mathrm{G}\text{-}\dim_{S'}(M \otimes_R R') = \operatorname{depth} S' - \operatorname{depth}(M \otimes_R R') = \operatorname{depth} R - \operatorname{depth} M = \mathrm{CM}^*\text{-}\dim_R M = n$.

The above claims imply that $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a G-factorization of ϕ, and we have $\mathrm{CM}^*\text{-}\dim_\phi M \leq \mathrm{G}\text{-}\dim_{S'}(M \otimes_R R') - \mathrm{G}\text{-}\dim_{S'} R' = n$ as desired. □

Let us consider the case that R contains a field K (e.g. K is the prime field of R). The second assertion of the above proposition especially says that if $S = K$ and $\phi : K \to R$ is the natural inclusion then $\text{CM}^*\text{-dim}_\phi M = \text{CM}^*\text{-dim}_R M$ for all R-module M. In other words, CM*-dimension relative to the map giving R the structure of a K-algebra, is absolute CM*-dimension. This leads us to the following conjecture.

Conjecture 3.8. If S is the prime local ring of R and ϕ is the natural inclusion, then relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ coincides with absolute CM*-dimension $\text{CM}^*\text{-dim}_R$.

Our next goal is to give some properties of relative CM*-dimension, which are similar to those of absolute CM*-dimension. First of all, relative CM*-dimension also satisfies the Auslander-Buchsbaum-type equality.

Theorem 3.9. *Let M be a non-zero R-module. If $\text{CM}^*\text{-dim}_\phi M < \infty$, then*

$$\text{CM}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Hence we especially have $\text{CM}^*\text{-dim}_\phi M = \text{CM}^*\text{-dim}_R M$.

Proof. Since $\text{CM}^*\text{-dim}_\phi M < \infty$, there exists a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of ϕ such that $\text{CM}^*\text{-dim}_\phi M = \text{G-dim}_{S'}(M \otimes_R R') - \text{G-dim}_{S'} R' < \infty$. Hence we have

$$\begin{aligned}
\text{CM}^*\text{-dim}_\phi M &= \text{G-dim}_{S'}(M \otimes_R R') - \text{G-dim}_{S'} R' \\
&= (\operatorname{depth} S' - \operatorname{depth}_{S'}(M \otimes_R R')) \\
&\qquad - (\operatorname{depth} S' - \operatorname{depth}_{S'} R') \\
&= \operatorname{depth}_{S'} R' - \operatorname{depth}_{S'}(M \otimes_R R').
\end{aligned}$$

Since ϕ' is surjective and α, β are faithfully flat, we obtain two equalities

$$\begin{cases} \operatorname{depth}_{S'} R' = \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R', \\ \operatorname{depth}_{S'}(M \otimes_R R') = \operatorname{depth}_R M + \operatorname{depth} R'/\mathfrak{m}R'. \end{cases}$$

Therefore we see that $\text{CM}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$ as desired. $\Box$

Corollary 3.10. *Suppose that S is a Gorenstein local ring containing a field. Then*

$$\text{CM}^*\text{-dim}_\phi F = 0$$

for any free R-module F.

Proof. Theorem 3.5 says that ϕ has a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$. Note that $\text{G-dim}_{S'}(F \otimes_R R') = \text{G-dim}_{S'} R' < \infty$. Hence we have $\text{CM}^*\text{-dim}_\phi F < \infty$. The assertion follows from the above theorem. $\Box$

Theorem 2.4 says that absolute CM*-dimension $\text{CM}^*\text{-dim}_R$ characterizes Cohen-Macaulayness of R. As an analogous result for relative CM*-dimension, we have the following.

Theorem 3.11. *The following conditions are equivalent for a local homomorphism $\phi : (S, \mathfrak{n}, l) \to (R, \mathfrak{m}, k)$.*

i) *R is Cohen-Macaulay and S is Gorenstein.*
ii) *$\text{CM}^*\text{-dim}_\phi M < \infty$ for any R-module M.*
iii) *$\text{CM}^*\text{-dim}_\phi k < \infty$.*

Proof. i) $\Rightarrow$ ii): By Lemma 3.4, there is a Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ. Since the closed fiber of β is regular, S' is also Gorenstein. Hence we have $\mathbf{R}\mathrm{Hom}_{S'}(\widehat{R}, S') \cong \mathrm{K}_{\widehat{R}}[-e]$, where $\mathrm{K}_{\widehat{R}}$ is the canonical module of $\widehat{R}$ and $e = \dim S' - \dim \widehat{R}$. Note that $\text{G-dim}_{S'}\widehat{R} < \infty$ because S' is Gorenstein. Therefore we easily see that $\text{G-dim}_{S'}\widehat{R} = \mathrm{grade}_{S'}\widehat{R} = e$. Thus the Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ is also a G-factorization of ϕ. The Gorensteinness of S' implies that $\text{G-dim}_{S'}(M \otimes_R \widehat{R}) < \infty$ for any R-module M. The assertion follows from this.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): Theorem 3.7.1 implies that $\text{CM}^*\text{-dim}_R k < \infty$. Hence R is Cohen-Macaulay by virtue of Theorem 2.4. On the other hand, since $\text{CM}^*\text{-dim}_\phi k < \infty$, ϕ has a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\text{G-dim}_{S'}(k \otimes_R R') < \infty$. Note that the closed fiber $A := k \otimes_R R' \cong R'/\mathfrak{m}R'$ of α is regular. Let $\boldsymbol{x} = x_1, x_2, \cdots x_n$ be a regular system of parameters of A. Since $\text{G-dim}_{S'}A < \infty$ and $\boldsymbol{x}$ is an A-regular sequence, we have $\text{G-dim}_{S'}A/(\boldsymbol{x}) < \infty$. Hence S' is Gorenstein because $A/(\boldsymbol{x})$ is isomorphic to the residue class field of S'. It follows from the flatness of β that S is also Gorenstein. □

In the rest of this section, we consider the relationship between relative CM*-dimension and G-dimension. Let us consider the case that ϕ is faithfully flat. Then $S \xrightarrow{\phi} R \xrightarrow{\mathrm{id}} R \xleftarrow{\mathrm{id}} R$ is a G-factorization of ϕ. Hence, if the G-dimension of an R-module M is finite, then the CM*-dimension of M relative to ϕ is also finite. Since both relative CM*-dimension and G-dimension satisfy the Auslander-Buchsbaum-type equalities, we have the following result that slightly generalizes Corollary 2.9.

Proposition 3.12. *Suppose that ϕ is faithfully flat. Then we have*

$$\text{CM}^*\text{-dim}_\phi M \leq \text{G-dim}_R M$$

for any R-module M. The equality holds if $\text{G-dim}_R M < \infty$.

Remark 3.13. Generally speaking, there is no inequality relation between relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ and G-dimension G-dim_R:

(1) If R is Gorenstein and S is not Gorenstein, then we have $\text{CM}^*\text{-dim}_\phi k = \infty$ and $\text{G-dim}_R k < \infty$. Hence $\text{CM}^*\text{-dim}_\phi k > \text{G-dim}_R k$.

(2) If R is not Gorenstein but Cohen-Macaulay and S is Gorenstein, then we have $\text{CM}^*\text{-dim}_\phi k < \infty$ and $\text{G-dim}_R k = \infty$. Hence $\text{CM}^*\text{-dim}_\phi k < \text{G-dim}_R k$.

(Both follow immediately from Theorem 3.11.)

As we have remarked after Theorem 3.7, relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ coincides with absolute CM*-dimension $\text{CM}^*\text{-dim}_R$ if S is the prime field of R (or maybe the prime local ring of R), in other words, S is the "smallest" local subring of R. In contrast with this, if S is the "largest" local subring of R, i.e. $S = R$, then relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ coincides with G-dimension G-dim_R.

Theorem 3.14. *If $S = R$ and ϕ is the identity map of R, then*

$$\text{CM}^*\text{-dim}_\phi M = \text{G-dim}_R M$$

for any R-module M.

Proof. By Proposition 3.12, we have only to prove that if $\text{CM}^*\text{-dim}_\phi M = m < \infty$ then $\text{G-dim}_R M = m$. There exists a G-factorization $R \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of $\phi = \mathrm{id}_R$ such that $\text{G-dim}_{S'}(M \otimes_R R') - \text{G-dim}_{S'}R' = m$.

Claim 1. $\mathbf{R}\mathrm{Hom}_{S'\otimes_R k}(R'\otimes_R k, S'\otimes_R k) \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S') \otimes^{\mathbf{L}}_R k$

In fact, let $F_\bullet$ be an S'-free resolution of R'. Since R' and S' are faithfully flat over R, it is easy to see that $F_\bullet \otimes_R k$ is an $(S'\otimes_R k)$-free resolution of $R'\otimes_R k$. Note that $\mathrm{Hom}_{S'}(F_\bullet, S')$ is a complex of free S'-modules, and hence is a complex of flat R-modules. Therefore we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{S'}(R', S') \otimes^{\mathbf{L}}_R k &\cong \mathrm{Hom}_{S'}(F_\bullet, S') \otimes_R k \\ &\cong \mathrm{Hom}_{S'\otimes_R k}(F_\bullet \otimes_R k, S'\otimes_R k) \\ &\cong \mathbf{R}\mathrm{Hom}_{S'\otimes_R k}(R'\otimes_R k, S'\otimes_R k).\end{aligned}$$

Claim 2. $S'\otimes_R k$ is Gorenstein.

Indeed, putting $g = \text{G-dim}_{S'}R' = \mathrm{grade}_{S'}R'$ and $N = \mathrm{Ext}^g_{S'}(R', S')$, we have $N \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$. Then it follows from Claim 1 that

$$(*) \qquad \mathbf{R}\mathrm{Hom}_{S'\otimes_R k}(R'\otimes_R k, S'\otimes_R k) \cong (N \otimes^{\mathbf{L}}_R k)[-g].$$

In particular, we have $\mathrm{Ext}^n_{S'\otimes_R k}(R'\otimes_R k, S'\otimes_R k) \cong \mathrm{Tor}^R_{g-n}(N, k) = 0$ for all $n > g$. Now taking a regular system of parameters $\boldsymbol{x} = x_1, x_2, \cdots, x_r$ of $A := R'\otimes_R k$, we have $\mathrm{Ext}^n_{S'\otimes_R k}(A/(\boldsymbol{x}), S'\otimes_R k) = 0$ for all $n > g + r$. Since $A/(\boldsymbol{x})$ is isomorphic to the residue class field of $S'\otimes_R k$, the self injective dimension of $S'\otimes_R k$ is not bigger than $g + r$. Therefore $S'\otimes_R k$ is Gorenstein.

Claim 3. $R' \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$

Note that, since $R'\otimes_R k$ is regular, the canonical module of $R'\otimes_R k$ is isomorphic to $R'\otimes_R k$. Thus, it follows from $(*)$ and Claim 2 that $N\otimes^{\mathbf{L}}_R k \cong \mathbf{R}\mathrm{Hom}_{S'\otimes_R k}(R'\otimes_R k, S'\otimes_R k)[g] \cong R'\otimes_R k$, hence $N\otimes_R k \cong R'\otimes_R k$. Therefore we have $N\otimes_{R'} k' \cong k'$, where k' is the residue class field of R'. In other words, $N \cong R'/I$ for some ideal I of R'. On the other hand, since $\text{G-dim}_{S'}R' < \infty$, we have

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{R'}(N, N) &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S')[g], \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]) \\ &\cong \mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S'), S') \\ &\cong R'\end{aligned}$$

In particular, N is a semi-dualizing R'-module for R'. Hence by Proposition 2.7.1, we see that $I = 0$, i.e. $R' \cong N \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$.

Now we can prove that $\text{G-dim}_R M = m$. Since R' is R-flat and $\text{G-dim}_{S'}(M\otimes_R R') < \infty$, we see that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)\otimes_R R' &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{R'}(M\otimes_R R', R'), R') \\ &\cong \mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(M\otimes_R R', S'), S') \\ &\cong M\otimes_R R'\end{aligned}$$

by Claim 3. It follows from the faithful flatness of $\alpha : R \to R'$ that $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R) \cong M$, and hence $\text{G-dim}_R M < \infty$. Note that Claim 3 implies $\mathbf{R}\mathrm{Hom}_{R'}(M\otimes_R R', R') \cong \mathbf{R}\mathrm{Hom}_{S'}(M\otimes_R R', S')[g]$. Therefore we have

$$\begin{aligned}\text{G-dim}_R M &= \text{G-dim}_{R'}(M\otimes_R R') \\ &= \text{G-dim}_{S'}(M\otimes_R R') - g \\ &= m\end{aligned}$$

as desired. □

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