WHEN IS THERE A NONTRIVIAL EXTENSION-CLOSED SUBCATEGORY?

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Abstract. Let $R$ be a commutative Noetherian local ring, and denote by $\text{mod } R$ the category of finitely generated $R$-modules. In this paper, we consider when $\text{mod } R$ has a nontrivial extension-closed subcategory. We prove that this is the case if there are part of a minimal system of generators $x, y$ of the maximal ideal with $xy = 0$, and that it holds if $R$ is a stretched Artinian Gorenstein local ring which is not a hypersurface.

Introduction

Let $R$ be a commutative Noetherian local ring with maximal ideal $m$. Denote by $\text{mod } R$ the category of finitely generated $R$-modules. An extension-closed subcategory of $\text{mod } R$ is by definition a nonempty strict full subcategory of $\text{mod } R$ closed under direct summands and extensions. The zero $R$-module, the finitely generated free $R$-modules and all the finitely generated $R$-modules form extension-closed subcategories of $\text{mod } R$, respectively. We call these three subcategories trivial extension-closed subcategories of $\text{mod } R$.

In this paper, we consider when there are only trivial extension-closed subcategories and when a nontrivial one exists. In the case where $R$ is an Artinian hypersurface, all the extension-closed subcategories of $\text{mod } R$ are trivial. Our conjecture is that the converse also holds true.

Conjecture. The following are equivalent.
(1) $R$ is an Artinian hypersurface.
(2) $\text{mod } R$ has only trivial extension-closed subcategories.

Both conditions in this conjecture imply that $R$ is an Artinian Gorenstein local ring. The conjecture holds if $R$ is a complete intersection.

The main result of this paper is the following theorem.

Theorem. Let $x, y$ be part of a minimal system of generators of $m$ with $xy = 0$. Then $R/m$ does not belong to the smallest extension-closed subcategory of $\text{mod } R$ containing $R/((x))$, and hence it is a nontrivial extension-closed subcategory.

Let $R$ be an Artinian local ring of length $l$ with embedding dimension $e$. Recall that $R$ is said to be stretched if $m^{l-e} \neq 0$. An Artinian Gorenstein local ring which is not a field and the cube of whose maximal ideal is zero is an example of a stretched Artinian Gorenstein local ring. The above theorem yields the following corollary, which guarantees that our conjecture holds when $R$ is a stretched Artinian Gorenstein local ring.

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Corollary. Let $R$ be a stretched Artinian Gorenstein local ring. Then the following are equivalent.

(1) $R$ is an Artinian hypersurface.
(2) $	ext{mod } R$ has only trivial extension-closed subcategories.

Convention

1. Throughout the rest of this paper, we assume that all rings are commutative Noetherian local rings, and that all modules are finitely generated. Let $R$ be a commutative Noetherian local ring. We denote by $m$ the maximal ideal of $R$, by $k$ the residue field of $R$ and by $	ext{mod } R$ the category of finitely generated $R$-modules.

2. Let $C$ be a category. In this paper, by a subcategory of $C$, we always mean a nonempty strict full subcategory of $C$. (Recall that a subcategory $X$ of $C$ is said to be strict if every object of $C$ that is isomorphic in $C$ to some object of $X$ belongs to $X$.) By the subcategory of $C$ consisting of objects $\{M_\lambda\}_{\lambda \in \Lambda}$, we always mean the smallest strict full subcategory of $C$ to which $M_\lambda$ belongs for all $\lambda \in \Lambda$. Note that this coincides with the full subcategory of $C$ consisting of all objects $X \in C$ such that $X \cong M_\lambda$ for some $\lambda \in \Lambda$.

3. We will often omit a letter indicating the base ring if there is no fear of confusion.

1. Some observations

We begin with recalling the precise definition of an extension-closed subcategory of $\text{mod } R$.

Definition 1.1. Let $\mathcal{X}$ be a subcategory of $\text{mod } R$. We say that $\mathcal{X}$ is extension-closed if $\mathcal{X}$ satisfies the following two conditions.

(1) $\mathcal{X}$ is closed under direct summands: if $M$ is an $R$-module in $\mathcal{X}$ and $N$ is a direct summand of $M$, then $N$ is also in $\mathcal{X}$.
(2) $\mathcal{X}$ is closed under extensions: for every exact sequence $0 \to L \to M \to N \to 0$ of $R$-modules, if $L$ and $N$ are in $\mathcal{X}$, then $M$ is also in $\mathcal{X}$.

For an $R$-module $X$, we denote by $\text{add}_R X$ the additive closure of $X$, namely, the smallest subcategory of $\text{mod } R$ containing $X$ which is closed under finite direct sums and direct summands. This is nothing but the subcategory of $\text{mod } R$ consisting of all direct summands of finite direct sums of copies of $X$. Note that the additive closure $\text{add}_R R$ of $R$ is the same as the subcategory of $\text{mod } R$ consisting of all free $R$-modules.

We call the subcategory of $\text{mod } R$ consisting of the zero $R$-module the zero subcategory of $\text{mod } R$, and denote it by $\mathbf{0}$. Clearly,

$\mathbf{0}, \text{add } R, \text{mod } R$

are all extension-closed subcategories of $\text{mod } R$. We call these three subcategories trivial extension-closed subcategories of $\text{mod } R$.

Definition 1.2. We say that $\text{mod } R$ has only trivial extension-closed subcategories if all the extension-closed subcategories of $\text{mod } R$ are $\mathbf{0}$, $\text{add } R$ and $\text{mod } R$. If there exists an extension-closed subcategory of $\text{mod } R$ other than these three, then we say that $\text{mod } R$ has a nontrivial extension-closed subcategory.
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Over an Artinian hypersurface, there exists no nontrivial extension-closed subcategory.

**Proposition 1.3.** If $R$ is an Artinian hypersurface, then $\text{mod } R$ has only trivial extension-closed subcategories.

**Proof.** This is proved in [6, Proposition 5.6]. For the convenience of the reader, we give here a proof. There exist a discrete valuation ring $S$ with maximal ideal $(x)$ and a positive integer $n$ such that $R$ is isomorphic to $S/(x^n)$. Applying to $S$ the structure theorem for finitely generated modules over a principal ideal domain, we have

$$\text{mod } R = \text{add}_R (R \oplus R/(x) \oplus R/(x^2) \oplus \cdots \oplus R/(x^{n-1})).$$

Let $\mathcal{X}$ be an extension-closed subcategory of $\text{mod } R$. Suppose that $\mathcal{X}$ is neither $0$ nor $\text{add } R$. Then $\mathcal{X}$ contains $R/(x^l)$ for some $1 \leq l \leq n - 1$. For each integer $1 \leq i \leq n - 1$ there exists an exact sequence

$$0 \rightarrow R/(x^i) \xrightarrow{f} R/(x^{i-1}) \oplus R/(x^{i+1}) \xrightarrow{g} R/(x^i) \rightarrow 0$$

of $R$-modules, where $x^0 := 1$, $f([a]) = \left( \frac{a}{x} \right)$ and $g(\left[ \frac{a}{x} \right]) = ax - b$. Hence $\mathcal{X}$ contains both $R/(x^{i-1})$ and $R/(x^{i+1})$. An inductive argument implies that $\mathcal{X}$ contains $R/(x), R/(x^2), \ldots, R/(x^{n-1}), R/(x^n) = R$. Therefore $\mathcal{X}$ coincides with $\text{mod } R$. □

We conjecture that the converse of Proposition 1.3 also holds. The main purpose of this paper is to study this conjecture.

**Conjecture 1.4.** If $\text{mod } R$ has only trivial extension-closed subcategories, then $R$ is an Artinian hypersurface.

One can show that the assumption of Conjecture 1.4 implies that $R$ is Artinian and Gorenstein.

**Proposition 1.5.** If $\text{mod } R$ has only trivial extension-closed subcategories, then $R$ is an Artinian Gorenstein ring.

**Proof.** First, let $\mathcal{X}$ be the subcategory of $\text{mod } R$ consisting of all $R$-modules of finite length. Clearly, $\mathcal{X}$ is an extension-closed subcategory of $\text{mod } R$. Using the fact that $\mathcal{X}$ contains $k$ and our assumption, we easily deduce that $\mathcal{X}$ coincides with $\text{mod } R$, which implies that $R$ is Artinian.

Next, let $\mathcal{Y}$ be the subcategory of $\text{mod } R$ consisting of all injective $R$-modules. It is obvious that $\mathcal{Y}$ is extension-closed, and the injective hull of $k$ belongs to $\mathcal{Y}$. Our assumption implies that $\mathcal{Y}$ is equal to $\text{add } R$, and we see that $R$ is Gorenstein. □

In the proposition below, we give a sufficient condition for $\text{mod } R$ to have a nontrivial extension-closed subcategory. This sufficient condition is a little complicated, but by using this, we will obtain some explicit sufficient conditions.

**Proposition 1.6.** Let $S \rightarrow R$ be a homomorphism of local rings. Assume that there exist $R$-modules $M, N$ such that:

- $M$ is $S$-flat and not $R$-free,
- $N$ is not $S$-flat.

Then $\text{mod } R$ has a nontrivial extension-closed subcategory.
Proof. Let $\mathcal{X}$ be the subcategory of $\text{mod } R$ consisting of all $S$-flat $R$-modules. It is easy to see that $\mathcal{X}$ is an extension-closed subcategory of $\text{mod } R$. The existence of $M$ and $N$ shows that $\mathcal{X}$ does not coincide with any of 0, $\text{add } R$, $\text{mod } R$. \hfill \Box

The following result is a direct consequence of Proposition 1.6.

**Corollary 1.7.** Suppose that there exists a local subring $S \subseteq R$ which is not a field and an ideal $I \subseteq R$ such that the composition $S \to R \to R/I$ is an isomorphism. Then $\text{mod } R$ has a nontrivial extension-closed subcategory.

**Proof.** Apply Proposition 1.6 to $M = R/I$ and $N = k$. \hfill \Box

The next three results, which give explicit sufficient conditions for $\text{mod } R$ to have a nontrivial extension-closed subcategory, are all deduced from Corollary 1.7.

**Corollary 1.8.** Let $S$ be a local ring which is not a field and $N$ a nonzero $S$-module. Let $R = S \ltimes N$ be the idealization of $N$ over $S$. Then $\text{mod } R$ has a nontrivial extension-closed subcategory.

**Proof.** Setting $I = \{ (0, n) \in R \mid n \in N \}$, we see that the composite map $S \to R \to R/I$ of natural homomorphisms is an isomorphism. Corollary 1.7 yields the conclusion. \hfill \Box

**Corollary 1.9.** Let $S, T$ be complete local rings which are not fields and have the same coefficient field $k$. Let $R = S \hat{\otimes}_k T$ be the complete tensor product of $S$ and $T$ over $k$. Then $\text{mod } R$ has a nontrivial extension-closed subcategory.

**Proof.** We can write $S \cong k[[x_1, \ldots, x_n]]/(f_1, \ldots, f_a)$ and $T \cong k[[y_1, \ldots, y_m]]/(g_1, \ldots, g_b)$, where $n, m \geq 1$, $f_1, \ldots, f_a \in (x_1, \ldots, x_n)^2$ and $g_1, \ldots, g_b \in (y_1, \ldots, y_m)^2$. Then $R$ is isomorphic to the ring $k[[x_1, \ldots, x_n, y_1, \ldots, y_m]]/(f_1, \ldots, f_a, g_1, \ldots, g_b)$. The composition $S \to R \to R/(y_1, \ldots, y_m)R$ of natural maps is an isomorphism, and we can use Corollary 1.7. \hfill \Box

The following result is due to Shiro Goto.

**Corollary 1.10.** Let $R = k[[X_1, \ldots, X_n, Y]]/\mathfrak{a}$ be a residue ring of a formal power series ring over a field $k$ with $n \geq 1$. Assume that $Y^{l+1} \in \mathfrak{a} \subseteq (X_1, \ldots, X_n, Y)^{l+1}$ holds for some $l \geq 1$. Then $\text{mod } R$ has a nontrivial extension-closed subcategory.

**Proof.** Let $x_1, \ldots, x_n, y \in R$ be the residue classes of $X_1, \ldots, X_n, Y$. Let $k[[y]]$ be the $k$-subalgebra of $R$ generated by $y$. Since $Y^{l+1} = 0$, we have a surjective ring homomorphism $\phi : k[[t]]/(t^{l+1}) \to k[[y]]$ given by $\phi(f(t)) = f(y)$ for $f(t) \in k[[t]]$, where $t$ is an indeterminate over $k$. Thus we obtain a ring homomorphism $\psi : k[[t]]/(t^{l+1}) \xrightarrow{\phi} k[[y]] \subseteq R \to R/(x_1, \ldots, x_n) + \mathfrak{m}^{l+1} = k[[Y]]/(Y^{l+1})$. We see that $\psi$ is an isomorphism. Hence $\phi$ is injective, and therefore it is an isomorphism. Applying Corollary 1.7 to $S = k[[y]]$ and $I = (x_1, \ldots, x_n) + \mathfrak{m}^{l+1}$, we get the conclusion. \hfill \Box

Using Corollaries 1.8 and 1.9, let us construct examples of a ring $R$ such that $\text{mod } R$ has a nontrivial extension-closed subcategory.
Example 1.11. Let $k$ be a field.

(1) Consider the ring
$$R = k[[x, y, z, w]]/(x^2, xy, xz - yw, xw, y^2, yz, z^2, zw, w^2).$$
This is an Artinian Gorenstein local ring. Putting $S = k[[x, y]]/(x^2, xy, y^2)$, we observe that $R$ is isomorphic to the idealization $S \ltimes E_S(k)$, where $E_S(k)$ denotes the injective hull of the $S$-module $k$. Hence it follows from Corollary 1.8 that $\text{mod } R$ has a nontrivial extension-closed subcategory.

In fact, for instance, let $X$ be the subcategory of $\text{mod } R$ consisting of all $R$-modules $X$ satisfying $\text{Tor}_1^R(R/(x), X) = 0$. It is clear that $X$ is extension-closed. We have an exact sequence
$$0 \to R/(x, y, w) \overset{f}{\to} R \to R/(x) \to 0,$$
where $f(1) = x$. Making the tensor product over $R$ of this exact sequence with $R/(z)$, we get an exact sequence
$$0 \to \text{Tor}_1^R(R/(x), R/(z)) \overset{g}{\to} k \to R/(z) \to R/(x, z) \to 0,$$
where $g(1) = \pi$. We see that $\text{Tor}_1^R(R/(x), R/(z)) = 0$, namely, $R/(z)$ belongs to $X$. Since $R/(x)$ is not a free $R$-module, $k$ does not belong to $X$. Thus $X$ is an extension-closed subcategory of $\text{mod } R$ which is different from any of 0, $\text{add } R$, $\text{mod } R$.

(2) Let
$$R = k[[x, y]]/(x^n, y^m)$$
with $n, m \geq 2$. This is an Artinian complete intersection. Since we have an isomorphism $R \cong k[[x]]/(x^n) \otimes_k k[[y]]/(y^m)$ of rings, $\text{mod } R$ has a nontrivial extension-closed subcategory by Corollary 1.9.

Indeed, for example, the subcategory of $\text{mod } R$ consisting of all $R$-modules $X$ with $\text{Tor}_1^R(R/(x), X) = 0$ is extension-closed, and does not coincide with any of 0, $\text{add } R$, $\text{mod } R$ because it contains $R/(y)$ and does not contain $k$.

Now, we verify that Conjecture 1.4 holds for a ring admitting a module with bounded Betti numbers.

Proposition 1.12. Suppose that $\text{mod } R$ has only trivial extension-closed subcategories. If there exists a nonfree $R$-module $M$ whose Betti numbers are bounded, then $R$ is an Artinian hypersurface.

Proof. That the local ring $R$ is Artinian follows from Proposition 1.5. Let $X$ be the subcategory of $\text{mod } R$ consisting of all $R$-modules whose Betti numbers are bounded. Then it is easy to see that $X$ is extension-closed. Since the nonfree $R$-module $M$ belongs to $X$, our assumption implies that $X$ coincides with $\text{mod } R$. In particular, the module $k$ is in $X$, which forces $R$ to be a hypersurface (cf. [7] or [1, Remarks 8.1.1(3)]). \hfill $\square$

Using [3, Theorem 3.2], we observe that such a module $M$ as in Proposition 1.12 exists when there exists an $R$-complex of finite complete intersection dimension and of infinite projective dimension. (See [2] for the details of complete intersection dimension.) Thus we obtain:
Corollary 1.13. Assume that there exists an \( R \)-complex of finite complete intersection dimension and of infinite projective dimension. If \( \text{mod} \, R \) has only trivial extension-closed subcategories, then \( R \) is an Artinian hypersurface.

Since over a complete intersection local ring every module has finite complete intersection dimension, Corollary 1.13 and Proposition 1.5 guarantee that Conjecture 1.4 holds true in the case where the local ring \( R \) is a complete intersection. Combining this with Proposition 1.3, we get the following result.

Corollary 1.14. If \( R \) is a complete intersection, then the following are equivalent.

1. \( R \) is an Artinian hypersurface.
2. \( \text{mod} \, R \) has only trivial extension-closed subcategories.

2. Main results

In this section, we conduct a closer investigation of the condition that \( \text{mod} \, R \) has a nontrivial extension-closed subcategory. Establishing a certain assumption on the ring \( R \), we shall construct an explicit nontrivial extension-closed subcategory. For this purpose, we begin with introducing a notion of a subcategory constructed from a single module.

Definition 2.1. Let \( X \) be a nonzero \( R \)-module. We define the subcategory \( \text{filt}^n_R X \) of \( \text{mod} \, R \) inductively as follows.

1. Let \( \text{filt}^1_R X \) be the subcategory consisting of \( X \).
2. For \( n \geq 2 \), let \( \text{filt}^n_R X \) be the subcategory consisting of all \( R \)-modules \( M \) such that there are exact sequences

\[
0 \to Y \to M \to X \to 0
\]

of \( R \)-modules with \( Y \in \text{filt}^{n-1}_R X \).

We denote by \( \text{filt}_R X \) the subcategory of \( \text{mod} \, R \) consisting of all \( R \)-modules \( M \) such that \( M \in \text{filt}^n_R X \) for some \( n \geq 1 \).

Here is a result concerning the structure of \( \text{filt}^n_R X \). Its name comes from its property stated in the first assertion.

Proposition 2.2. Let \( X \) be a nonzero \( R \)-module.

1. An \( R \)-module \( M \) belongs to \( \text{filt}^n_R X \) if and only if there exists a filtration

\[
0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n = M
\]

of \( R \)-submodules of \( M \) with \( M_i/M_{i-1} \cong X \) for all \( 1 \leq i \leq n \).

2. If \( \text{filt}^p_R X \) intersects \( \text{filt}^q_R X \), then \( p = q \).

Proof. (1) This can be proved by induction on \( n \).

(2) It is seen from the definition that if an \( R \)-module \( M \) belongs to \( \text{filt}^n_R X \), then we have \( e(M) = n \cdot e(X) \), where \( e(-) \) denotes the multiplicity. The assertion immediately follows from this.

Corollary 2.3. Let \( X \) be a nonzero \( R \)-module.
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Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be an exact sequence of \( R \)-modules. If \( L \) is in \( \text{filt}_R^n X \) and \( N \) is in \( \text{filt}_R^{p+q} X \), then \( M \) is in \( \text{filt}_R^{p+q} X \).

The subcategory \( \text{filt}_R X \) of \( \text{mod} R \) is closed under extensions.

Proof. (1) Using Proposition 2.2(1), we can prove the assertion.

(2) This assertion follows from (1). □

For an \( R \)-module \( X \), we denote by \( \text{ext}_R X \) the extension closure of \( X \), that is, the smallest extension-closed subcategory of \( \text{mod} R \) containing \( X \). One can describe \( \text{ext}_R X \) by using \( \text{filt}_R X \).

**Proposition 2.4.** Let \( X \) be a nonzero \( R \)-module. Then \( \text{ext}_R X \) coincides with the subcategory of \( \text{mod} R \) consisting of all direct summands of modules in \( \text{filt}_R X \).

Proof. Let \( \mathcal{X} \) be the subcategory of \( \text{mod} R \) consisting of all direct summands of modules in \( \text{filt}_R X \). It suffices to prove the following two statements.

(1) \( \mathcal{X} \) is an extension-closed subcategory of \( \text{mod} R \) containing \( X \).

(2) If \( \mathcal{X}' \) is an extension-closed subcategory of \( \text{mod} R \) containing \( X \), then \( \mathcal{X}' \) contains \( \mathcal{X} \).

As to (1): Obviously, \( \mathcal{X} \) contains \( X \) and is closed under direct summands. Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be an exact sequence of \( R \)-modules with \( L, N \in \mathcal{X} \). Then we have isomorphisms \( L \oplus L' \cong Y \) and \( N \oplus N' \cong Z \) for some \( L', N' \in \text{mod} R \) and \( Y, Z \in \text{filt} X \). Taking the direct sum of the above exact sequence with the exact sequences \( 0 \rightarrow L' \rightarrow L' \rightarrow 0 \rightarrow 0 \) and \( 0 \rightarrow 0 \rightarrow N' \rightarrow N' \rightarrow 0 \), we get an exact sequence

\[
0 \rightarrow Y \rightarrow L' \oplus M \oplus N' \rightarrow Z \rightarrow 0.
\]

Since \( Y, Z \) are in \( \text{filt} X \), so is \( L' \oplus M \oplus N' \), and hence \( M \) belongs to \( \mathcal{X} \). Thus \( \mathcal{X} \) is closed under extensions.

As to (2): Since \( \mathcal{X}' \) is closed under direct summands, we have only to prove that \( \mathcal{X}' \) contains \( \text{filt} X \), equivalently, that \( \mathcal{X}' \) contains \( \text{filt}^n X \) for every \( n \geq 1 \). This can easily be shown by induction on \( n \). □

Let \( x \) be an element of \( R \). To understand the subcategory \( \text{ext}_R(R/(x)) \), we investigate the structure of each module in \( \text{filt}_R^n(R/(x)) \) for \( n \geq 1 \).

**Proposition 2.5.** Let \( x \in R \) and \( n \geq 1 \). Let \( M \) be an \( R \)-module in \( \text{filt}_R^n(R/(x)) \). Then there exists an exact sequence

\[
\begin{pmatrix}
  x & c_{1,2} & \cdots & c_{1,n} \\
  0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & c_{n-1,n} \\
  0 & \cdots & 0 & x
\end{pmatrix}
\]

of \( R \)-modules with each \( c_{i,j} \) being in \( R \) such that

\[
\begin{pmatrix}
  c_{1,j} \\
  \vdots \\
  c_{j-1,j}
\end{pmatrix}
\]

is in \( \text{span}_R(x) \).
for all $2 \leq j \leq n$.

Proof. We prove the proposition by induction on $n$. When $n = 1$, we have $M \cong R/(x)$, and there is an exact sequence $R \xrightarrow{x} R \rightarrow M \rightarrow 0$. Let $n \geq 2$. We have an exact sequence $0 \rightarrow Y \rightarrow M \rightarrow R/(x) \rightarrow 0$ of $R$-modules with $Y \in \text{filt}^{n-1}(R/(x))$. The induction hypothesis shows that there is an exact sequence $R^{n-1} \xrightarrow{A} R^{n-1} \rightarrow Y \rightarrow 0$ with $A = \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-2,n-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$ such that $\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \\ 0 \end{pmatrix} (0 : x) \subseteq \text{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$ for all $2 \leq j \leq n-1$. We have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & R^{n-1} & \xrightarrow{(A \begin{pmatrix} 1 \\ 0 \end{pmatrix})} & R^{n-1} \oplus R & \xrightarrow{(0 \ 1)} & R & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R^{n-1} & \xrightarrow{(0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix})} & R^{n-1} \oplus R & \xrightarrow{(0 \ 1)} & R & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Y & \rightarrow & M & \rightarrow & R/(x) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

with exact rows and columns. The induced map $g : (0 : x) \rightarrow Y$ is the zero map by the snake lemma. By diagram chasing, we see that $g(r) = f(Br)$ holds for each $r \in (0 : x)$. Hence we have $f(Br) = 0$ for all $r \in (0 : x)$, whence $Br$ is in the image of the map $A : R^{n-1} \rightarrow R^{n-1}$. Writing $B = \begin{pmatrix} c_{1,n} \\ \vdots \\ c_{n-1,n} \end{pmatrix}$, we obtain an inclusion relation $\begin{pmatrix} c_{1,n} \\ \vdots \\ c_{n-1,n} \end{pmatrix} (0 : x) \subseteq \text{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-2,n-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$. Consequently, we have $\begin{pmatrix} c_{1,j} \\ \vdots \\ c_{j-1,j} \end{pmatrix} (0 : x) \subseteq \text{Im} \begin{pmatrix} x & c_{1,2} & \cdots & c_{1,j-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{j-2,j-1} \\ 0 & \cdots & 0 & x \end{pmatrix}$.
for all $2 \leq j \leq n$. The middle column of the above diagram gives an exact sequence

$$
\begin{pmatrix}
  x & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\
  0 & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \cdots \\
  0 & \cdots & 0 & x & c_{n-1,n} \\
  0 & \cdots & 0 & 0 & x
\end{pmatrix}
$$

$$R^n \to R^n \to R^n \to M \to 0.$$

Thus the proof of the proposition is completed.

Now we can prove the following result concerning the structure of $\text{ext}_R(R/(x))$, which is the main result of this paper.

**Theorem 2.6.** Let $x, y$ be part of a minimal system of generators of $m$ with $xy = 0$. Then $k$ does not belong to $\text{ext}_R(R/(x))$.

**Proof.** Let $e$ be the embedding dimension of $R$. We have $e \geq 2$, and write $m = (x, y, z_3, \ldots, z_e)$. Let us assume that $k$ belongs to $\text{ext}_R(R/(x))$. We want to derive a contradiction. By Proposition 2.4, the module $k$ is isomorphic to a direct summand of a module $M \in \text{filt}_R(R/(x))$. We have an isomorphism $M \cong k \oplus N$ for some $R$-module $N$, and $M$ belongs to $\text{filt}_R^n(R/(x))$ for some $n \geq 1$. Proposition 2.5 gives an exact sequence

$$(2.6.1) \quad R^n \to R^n \to M \to 0$$

of $R$-modules such that

$$\left( \begin{array}{c}
  c_{1,j} \\
  \vdots \\
  c_{j-1,j}
\end{array} \right) (0 : x) \subseteq \text{Im} \left( \begin{array}{c}
  x & c_{1,2} & \cdots & c_{1,j-1} \\
  0 & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \ddots & \cdots \\
  \vdots & \ddots & \ddots & \cdots \\
  0 & \cdots & 0 & x
\end{array} \right)$$

for all $2 \leq j \leq n$. Since $y$ is in $(0 : x)$, there are elements $d_{1,j}, \ldots, d_{j-1,j} \in R$ such that

$$\left( \begin{array}{c}
  c_{1,j} y \\
  \vdots \\
  c_{j-1,j} y
\end{array} \right) = \left( \begin{array}{c}
  x & c_{1,2} & \cdots & c_{1,j-1} \\
  0 & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \ddots & \cdots \\
  \vdots & \ddots & \ddots & \cdots \\
  0 & \cdots & 0 & x
\end{array} \right) \left( \begin{array}{c}
  d_{1,j} \\
  \vdots \\
  d_{j-1,j}
\end{array} \right).$$

Hence the equality

$$c_{i,j} y = xd_{i,j} + c_{i,i+1} d_{i+1,j} + \cdots + c_{i,j-1} d_{j-1,j}$$

holds for $2 \leq j \leq n$ and $1 \leq i \leq j - 1$.

We claim that the elements $c_{i,j}, d_{i,j}$ belong to $m$ for all $2 \leq j \leq n$ and $1 \leq i \leq j - 1$. Indeed, the hypothesis of induction on $j$ implies that $c_{i,j}$ is in $m$ for $i + 1 \leq l \leq j - 1$, and the assumption of descending induction on $i$ shows that $d_{i,j}$ is in $m$ for $i + 1 \leq l \leq j - 1$. Hence we have $c_{i,j} y - xd_{i,j} \in m^2$, which gives an equality

$$\overline{c_{i,j}} \cdot \overline{y} - \overline{x} \cdot \overline{d_{i,j}} = \overline{0}$$

in $m/m^2$. Since $x, y$ are part of a $k$-basis of $m/m^2$, we have $\overline{c_{i,j}} = \overline{d_{i,j}} = \overline{0}$ in $k$. Therefore, $c_{i,j}, d_{i,j}$ belong to $m$, as desired.
By elementary column operations, the matrix
\[
\begin{pmatrix}
x & c_1,2 & \cdots & c_1,n \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & c_{n-1},n \\
0 & \cdots & 0 & x
\end{pmatrix}
\]
can be transformed into a matrix
\[
\begin{pmatrix}
x & b_1,2 & \cdots & b_1,n \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & b_{n-1},n \\
0 & \cdots & 0 & x
\end{pmatrix}
\]
such that each \(b_{i,j}\) is an element of the ideal \(I = (y, z_3, \ldots, z_e)\). We have an exact sequence
\[
\begin{array}{c}
R^n \\ R^n \\
\text{ann.} \\
M \\
0
\end{array} \rightarrow \\
\begin{array}{c}
R^n \\
R^n \\
\text{ann.} \\
M/IM \\
0
\end{array}
\]
and applying \(- \otimes_R R/I\) to this, we get an exact sequence
\[
\begin{array}{c}
(R/I)^n \\
(R/I)^n \\
(R/I)^n \\
M/IM \\
0
\end{array} \rightarrow \\
\begin{array}{c}
(R/I)^n \\
(R/I)^n \\
(R/I)^n \\
M/IM \\
0
\end{array}
\]
Hence we have an isomorphism \(M/IM \cong (R/I + (x))^n = k^n\). Since \(M/IM \cong k \oplus N/IN\), we see that \(N/IN\) is isomorphic to \(k^{n-1}\), and get an equality
\[
\beta_1^R(N/IN) = (n-1)\beta_1^R(k)
\]
of Betti numbers. There is an exact sequence \(R^\beta_1^R(N) \rightarrow R^\beta_1^R(N) \rightarrow N \rightarrow 0\) of \(R\)-modules, and tensoring \(R/I\) with this gives an exact sequence \((R/I)^\beta_1^R(N) \rightarrow (R/I)^\beta_1^R(N) \rightarrow N/IN \rightarrow 0\) of \(R/I\)-modules. It follows from this that
\[
\beta_1^R(N/IN) \leq \beta_1^R(N).
\]
The isomorphism \(M \cong k \oplus N\) shows
\[
\beta_1^R(M) = \beta_1^R(k) + \beta_1^R(N) = e + \beta_1^R(N).
\]
The existence of the exact sequence (2.6.1) implies
\[
\beta_1^R(M) \leq n.
\]
Since \(m/I = x(R/I)\) and \(x \notin I\), we have
\[
\beta_1^R(k) = 1.
\]
Using the (in)equalities (2.6.2)–(2.6.6), we obtain
\[
n - 1 = (n - 1)\beta_1^R(k) = \beta_1^R(N/IN) \leq \beta_1^R(N) = \beta_1^R(M) - e \leq n - e,
\]
whence \(e \leq 1\). This is a desired contradiction; this contradiction completes the proof of the theorem. \(\square\)

Let \(R\) be an Artinian local ring. Then, using the fact that every \(R\)-module \(M\) is annihilated by the ideal \(m^{\ell(M)}\), we can check that the equality \(m^{\ell(R) - \text{edim}R + 1} = 0\) holds. (Here, \(\ell(M)\) and \(\text{edim} R\) denote the length of \(M\) and the embedding dimension of \(R\),...
respectively.) Recall that $R$ is called *stretched* if $m^i \neq 0$ for all $i < \ell(R) - \text{edim } R + 1$, or equivalently, if $m^{\ell(R) - \text{edim } R + 1} \neq 0$.

**Example 2.7.** (1) Every Artinian Gorenstein local ring $R$ with $m^3 = 0$ that is not a field is stretched.

(2) Let $k$ be a field, and let $R = k[[x, y, z]]/(xy, xz, yz, x^3 - y^2, x^3 - z^2)$ be a residue ring of a formal power series ring over $k$. Then $R$ is an Artinian Gorenstein local ring. Since $\ell(R) = 6$, $\text{edim } R = 3$ and $m^3 = (x^3) \neq 0$, the ring $R$ is stretched.

Now we have a sufficient condition for $\text{mod } R$ to have a nontrivial extension-closed subcategory.

**Corollary 2.8.** Let $R$ be a stretched Artinian Gorenstein local ring with $\text{edim } R \geq 2$. Then $\text{mod } R$ has a nontrivial extension-closed subcategory.

**Proof.** If $\text{edim } R < \ell(R) - 2$, then by [5, Theorem 1.1] there exist elements $x, y \in R$ with $xy = 0$ which form part of minimal system of generators of $m$, and Theorem 2.6 shows that $\text{ext}_R(R/(x))$ is a nontrivial extension-closed subcategory of $\text{mod } R$.

Let $\text{edim } R \geq \ell(R) - 2$. Then we have $m^3 = 0$. Take an element $x \in m \setminus m^2$. First, assume that $(0 : x)$ is not contained in $(x) + m^2$. Then there exists an element $y \in (0 : x)$ which does not belong to $(x) + m^2$, and we see that $\overline{x}, \overline{y}$ form part of a $k$-basis of $m/m^2$. Hence $x, y$ are part of a minimal system of generators of $m$ with $xy = 0$, and the assertion follows from Theorem 2.6.

Next, assume that $(0 : x)$ is contained in $(x) + m^2$. Then we have

$$ (x) = (0 : (0 : x)) \supseteq (0 : (x) + m^2) = (0 : x) \cap (0 : m^2) = (0 : x). $$

Here, the equality (a) follows from the double annihilator property (cf. [4, Exercise 3.2.15]), and (b) from the inclusion $(0 : m^2) \supseteq m$. Suppose that $(0 : x) \neq (x)$. Then we have $xm \subseteq m^2 \subseteq (0 : x) \subseteq (x)$ and $\ell_R((x)/xm) = 1$, which imply $xm = m^2 = (0 : x)$. Hence $m \subseteq (0 : m^2) = (0 : (0 : x)) = (x)$, which contradicts the assumption that $\text{edim } R \geq 2$. Thus the equality $(0 : x) = (x)$ holds, and there exists an exact sequence

$$ \cdots \to R \xrightarrow{x} R \xrightarrow{y} R \to R/(x) \to 0 $$

of $R$-modules. This implies that $R/(x)$ belongs to the subcategory $\mathcal{X}$ of $\text{mod } R$ consisting of all $R$-modules with bounded Betti numbers, which is extension-closed. Hence $\mathcal{X}$ is neither $0$ nor $\text{add } R$, and we also have $\mathcal{X} \neq \text{mod } R$ because $R$ is not a hypersurface by the assumption that $\text{edim } R \geq 2$ again. Therefore $\mathcal{X}$ is a nontrivial extension-closed subcategory of $\text{mod } R$.\[\square\]

We can guarantee that our Conjecture 1.4 holds true for a stretched Artinian Gorenstein local ring. The following result follows from Proposition 1.3 and Corollary 2.8.

**Corollary 2.9.** Let $R$ be a stretched Artinian Gorenstein local ring. Then the following are equivalent.

(1) $R$ is an Artinian hypersurface.
mod $R$ has only trivial extension-closed subcategories.

We end this paper by posing a question.

**Question 2.10.** An extension-closed subcategory of mod $R$ is called *resolving* if it contains $R$ and is closed under syzygies. Does the assumption of Theorem 2.6 imply that $k$ does not belong to the smallest resolving subcategory of mod $R$ containing $R/(x)$?

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**References**


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