SPANIER–WHITEHEAD CATEGORIES OF RESOLVING SUBCATEGORIES AND COMPARISON WITH SINGULARITY CATEGORIES

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ABSTRACT. Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$. In this paper, we investigate the affinity of the Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$ of the stable category of $\mathcal{X}$ with the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$ of $\mathcal{A}$. We construct a fully faithful triangle functor from $\mathsf{SW}(\mathcal{X})$ to $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$, and we prove that it is dense if and only if the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ of $\mathcal{A}$ is generated by $\mathcal{X}$. Applying this result to commutative rings, we obtain characterizations of the isolated singularities, the Gorenstein rings and the Cohen–Macaulay rings. Moreover, we classify the Spanier–Whitehead categories over complete intersections. Finally, we explore a method to compute the (Rouquier) dimension of the triangulated category $\mathsf{SW}(\mathcal{X})$ in terms of generation in $\mathcal{X}$.

1. Introduction

Let $\mathcal{A}$ be an abelian category with enough projective objects, and let $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ be the bounded derived category of $\mathcal{A}$. The singularity category $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$ of $\mathcal{A}$ is by definition the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ by the perfect complexes. This category measures the homological singularity of $\mathcal{A}$ in the sense that the abelian category $\mathcal{A}$ has finite global dimension if and only if its singularity category is trivial. This notion was introduced by Buchweitz [7] in the 1980s, and studied actively ever since the relation with mirror symmetry was found by Orlov [18].

On the other hand, inspired by a well-known construction in algebraic topology (see [15]), Heller [13] defined the Spanier–Whitehead category for each left triangulated category by formally inverting the suspension (see Definition 2.4), and proved that it is always a triangulated category. This is a useful tool for the study of stable categories. Let $\mathcal{X}$ be a quasi-resolving subcategory of the abelian category $\mathcal{A}$ in the sense of [16]. Then the stable category $\underline{\mathcal{X}}$ modulo projectives forms a left triangulated category with the syzygy functor being the suspention. Thus one obtains the Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$ of $\underline{\mathcal{X}}$ together with a fully faithful triangle functor

$$\theta_{\mathcal{X}} : \mathsf{SW}(\mathcal{X}) \to \mathsf{D}_{\mathsf{sg}}(\mathcal{A}),$$

sending each object $X[i]$ to the complex concentrated in degree $-i$.

Let R be a commutative noetherian ring. Let $\mathsf{GP}(R)$ be the category of finitely generated Gorenstein projective R-modules. Since $\mathsf{GP}(R)$ is a Frobenius category, the stable category $\underline{\mathsf{GP}}(R)$ is a triangulated category. By the universal property of the Spanier–Whitehead category the canonical functor $\underline{\mathsf{GP}}(R) \to \mathsf{SW}(\mathsf{GP}(R))$ is a triangle equivalence. Combining this with a fundamental result of Buchweitz and Happel [7, 12] yields that $\theta_{\mathsf{GP}(R)}$ is dense if R is Gorenstein, whose converse follows from the main theorem of [5].

Motivated by these results, we explore in this paper the affinity of the Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$ with the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$. We prove the following theorem.

Theorem 1.1 (Theorem 3.2)**.** *Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$. Then the functor $\theta_{\mathcal{X}}$ induces a triangle equivalence between*

- *the Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$, and*
- *the full triangulated subcategory of the singularity category $\mathsf{D}_{\mathsf{sg}}(\mathcal{A})$ generated by the objects in $\mathcal{X}$.*

In particular, $\theta_{\mathcal{X}}$ is dense if and only if the derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ is generated by the objects in $\mathcal{X}$.

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Applying this theorem, we can characterize some properties of commutative noetherian rings R in terms of the density of the fully faithful triangle functor $\theta_{\mathcal{X}}$ for some resolving subcategories $\mathcal{X}$ of the category $\mathsf{mod}\,R$ of finitely generated R-modules. Denote by $\mathsf{mod}_0 R$ the full subcategory of $\mathsf{mod}\,R$ consisting of modules which are locally free on the punctured spectrum of R. For a fixed R-module M, denote by ${}^{\perp}M$ the full subcategory of $\mathsf{mod}\,R$ consisting of modules X such that $\mathsf{Ext}_R^i(X, M) = 0$ for all $i > 0$. Note that the second assertion includes our motivating results stated above.

Corollary 1.2 (Corollary 3.7). *Let R be a commutative noetherian ring.*

(1) *R is an isolated singularity if and only if $\theta_{\mathsf{mod}_0 R}$ is dense.*
(2) *R is Gorenstein if and only if $\theta_{\mathsf{GP}(R)}$ is dense, if and only if $\theta_{{}^{\perp}R}$ is dense.*
(3) *R is Cohen–Macaulay if $\theta_{{}^{\perp}M}$ is dense for some finitely generated R-module M with full support. The converse also holds when R admits a canonical module.*

Theorem 1.1 says that via the functor $\theta_{\mathcal{X}}$ the category $\mathsf{SW}(\mathcal{X})$ is equivalent to a certain full triangulated subcategory of $\mathsf{D_{sg}}(\mathcal{A})$. It is natural to ask whether this subcategory is thick, namely, closed under direct summands. For a commutative noetherian ring we set $\mathsf{D_{sg}}(R) := \mathsf{D_{sg}}(\mathsf{mod}\,R)$ and denote by $\mathsf{CM}(R)$ the category of maximal Cohen–Macaulay R-modules. We prove that for resolving subcategories over complete intersections the question is affirmative, and moreover the Spanier–Whitehead categories of the stable categories are completely classified, thanks to a theorem of Stevenson [20].

Theorem 1.3 (Theorem 4.4). *Let R be a local complete intersection.*

(1) *Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$. Then the stable category $\underline{\mathcal{X} \cap \mathsf{CM}(R)}$ is a triangulated category, and the natural functors*

$$\underline{\mathcal{X} \cap \mathsf{CM}(R)} \to \mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R)) \to \mathsf{SW}(\mathcal{X})$$

are triangle equivalences. Furthermore, $\mathsf{SW}(\mathcal{X})$ is sent by $\theta_{\mathcal{X}}$ to a thick subcategory of $\mathsf{D_{sg}}(R)$.

(2) *Suppose that R is a quotient of a regular ring. Then there is a one-to-one correspondence between*
   - *the Spanier–Whitehead categories of resolving subcategories of $\mathsf{mod}\,R$, and*
   - *the specialization-closed subsets of the singular locus of the scheme $\operatorname{Proj} S$,*

   where S is the generic hypersurface of R.

Next, we are interested in measuring the "size" of the Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$ of each resolving subcategory $\mathcal{X}$ of $\mathcal{A}$. Since $\mathsf{SW}(\mathcal{X})$ is a triangulated category, one can think of its dimension in the sense of Rouquier [19]. We present a method to compute the dimension of the triangulated category $\mathsf{SW}(\mathcal{X})$ in terms of generation in $\mathcal{X}$. To be precise, we prove the following result.

Theorem 1.4 (Theorem 5.4). *Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$. Then there is an equality*

$$\dim \mathsf{SW}(\mathcal{X}) = \inf\{n \geqslant 0 \mid \exists G \in \mathcal{X} \text{ such that } \forall X \in \mathcal{X}\ \exists k \geqslant 0 \text{ with } \Omega^k X \in [G]_{n+1}\}.$$

Let R be a commutative noetherian ring, and let $\mathcal{X}$ be a full subcategory of $\mathsf{mod}\,R$. We define the *cohomology annihilator* $\operatorname{ca}(\mathcal{X})$ of $\mathcal{X}$ to be the ideal of R consisting of elements $a \in R$ such that for each $X \in \mathcal{X}$ there exists $i > 0$ with $a\,\mathsf{Ext}_R^i(X, -) = 0$. Applying the above theorem, we obtain a relationship between the dimension of the Spanier–Whitehead category of $\mathsf{mod}_0 R$ and its cohomology annihilator.

Corollary 1.5 (Corollary 5.9). *Let R be a commutative noetherian ring.*

(1) *The factor ring $R/\operatorname{ca}(\mathsf{mod}_0 R)$ is artinian if and only if the triangulated category $\mathsf{SW}(\mathsf{mod}_0 R)$ has finite dimension.*
(2) *Suppose that R is an isolated singularity. Then $R/\operatorname{ca}(\mathsf{mod}_0 R)$ is artinian if and only if $\mathsf{D_{sg}}(R)$ has finite dimension.*

The paper is organized as follows. In Section 2, we recall the definitions of a left triangulated category and its Spanier–Whitehead category. We also recall the definition of a quasi-resolving subcategory of an abelian category with enough projective objects. We observe that the stable category is left triangulated, so that the Spanier–Whitehead category is defined. In Section 3, for an abelian category $\mathcal{A}$ with enough projective objects we compare the Spanier–Whitehead category of a quasi-resolving subcategory of $\mathcal{A}$ with the singularity category of $\mathcal{A}$. In this section, we prove the general Theorem 1.1 and Corollary 1.2 as an application. In Section 4, we consider the question asking when the Spanier–Whitehead category of a quasi-resolving subcategory is closed under direct summands. We give several partial answers in this section, including Theorem 1.3. In the final Section 5, we investigate the (Rouquier) dimension of the Spanier–Whitehead category of a quasi-resolving subcategory, and prove Theorem 1.4 and Corollary 1.5.

2. Basic definitions

In this section, we give the definitions of several basic notions which we deal with in this paper. We begin with stating our conventions.

Convention 2.1. Throughout this paper, we use the following conventions. Let $\mathcal{A}$ be an abelian category with enough projective objects. All subcategories are assumed to be full and strict (i.e., closed under isomorphism). We denote by $\mathsf{proj}\,\mathcal{A}$ the subcategory of $\mathcal{A}$ consisting of projective objects, by $\mathsf{D^b}(\mathcal{A})$ the bounded derived category of $\mathcal{A}$, by $\mathsf{D^{perf}}(\mathcal{A})$ the subcategory of $\mathsf{D^b}(\mathcal{A})$ consisting of perfect complexes (i.e., bounded complexes of objects in $\mathsf{proj}\,\mathcal{A}$), and by $\mathsf{D_{sg}}(\mathcal{A})$ the singularity category of $\mathcal{A}$, that is, the Verdier quotient of $\mathsf{D^b}(\mathcal{A})$ by $\mathsf{D^{perf}}(\mathcal{A})$. Let

$$\pi : \mathsf{D^b}(\mathcal{A}) \to \mathsf{D_{sg}}(\mathcal{A})$$

be the canonical functor. For a triangulated category $\mathcal{T}$ and a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\mathsf{tria}_{\mathcal{T}}\,\mathcal{X}$ the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$. Throughout this paper, let R be a commutative noetherian ring. We denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules, and put $\mathsf{proj}\,R := \mathsf{proj}(\mathsf{mod}\,R)$, $\mathsf{D^b}(R) := \mathsf{D^b}(\mathsf{mod}\,R)$, $\mathsf{D^{perf}}(R) := \mathsf{D^{perf}}(\mathsf{mod}\,R)$ and $\mathsf{D_{sg}}(R) := \mathsf{D_{sg}}(\mathsf{mod}\,R)$.

We recall the definition of a looped category.

Definition 2.2. A *looped category* is by definition a pair $(\mathcal{C}, \Omega)$ where $\mathcal{C}$ is an additive category and $\Omega : \mathcal{C} \to \mathcal{C}$ is an additive functor, which is called the *loop functor*. Let $(\mathcal{C}, \Omega), (\mathcal{C}', \Omega')$ be two looped categories. Then a *stable functor* $(F, \delta) : (\mathcal{C}, \Omega) \to (\mathcal{C}', \Omega')$ is by definition an additive functor $F : \mathcal{C} \to \mathcal{C}'$ together with a natural isomorphism $\delta : F\Omega \xrightarrow{\cong} \Omega' F$.

Next we recall the definition of a left triangulated category.

Definition 2.3. Let $(\mathcal{C}, \Omega)$ be a looped category. Let Δ be a collection of sequences of morphisms in $\mathcal{C}$ having the form

$$\Omega X \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X,$$

which we call *left triangles*. The triple $(\mathcal{C}, \Omega, \Delta)$ is said to be a *left triangulated category*, if the following conditions are satisfied.

(LT0) Any sequence that is isomorphic to a left triangle is a left triangle. Moreover, for any object $X \in \mathcal{C}$, the sequence $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is a left triangle.
(LT1) For any morphism $f : Y \to X$ in $\mathcal{C}$, there is a left triangle $\Omega X \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$.
(LT2) For a given left triangle $\Omega X \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$, the sequence $\Omega Y \xrightarrow{-\Omega f} \Omega X \xrightarrow{h} Z \xrightarrow{g} Y$ is a left triangle.
(LT3) For any commutative diagram of the form

$$\begin{tikzcd} \Omega X \arrow[r, "h"] \arrow[d, "\Omega\gamma"] & Z \arrow[r, "g"] & Y \arrow[r, "f"] \arrow[d, "\beta"] & X \arrow[d, "\gamma"] \\ \Omega X' \arrow[r, "h'"] & Z' \arrow[r, "g'"] & Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

where the rows are left triangles, there is a morphism $\alpha : Z \to Z'$ in $\mathcal{C}$ making the completed diagram commutative.

(LT4) For two morphisms $g : Z \to Y$ and $f : Y \to X$ in $\mathcal{C}$ there exists a commutative diagram

$$\begin{tikzcd}
\Omega A \arrow[r] \arrow[d, "\Omega t"] & C \arrow[r, "b"] \arrow[d, equal] & B \arrow[r, "a"] \arrow[d] & A \arrow[d] \\
\Omega Y \arrow[r] \arrow[d, "\Omega f"] & C \arrow[r] \arrow[d, "b"] & Z \arrow[r, "g"] \arrow[d, equal] & Y \arrow[d, "f"] \\
\Omega X \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "a"] & Z \arrow[r, "fg"] \arrow[d, "g"] & X \arrow[d, equal] \\
\Omega X \arrow[r] & A \arrow[r, "t"] & Y \arrow[r, "f"] & X
\end{tikzcd}$$

such that the rows are left triangles.

Let $\mathcal{C}$ be a left triangulated category. We state the definition of the Spanier–Whitehead category of $\mathcal{C}$, which is also known as the stabilization of $\mathcal{C}$; see [3, Section 3]. We should point out that a triangulated category is indeed a left triangulated category whose loop functor is an equivalence.

Definition 2.4. Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category. The *Spanier–Whitehead category* $\mathsf{SW}(\mathcal{C})$ of $\mathcal{C}$ is defined as follows:

- The objects are the pairs (X, n) of objects X of $\mathcal{C}$ and integers n which we denote by $X[n]$.
- For any two objects $X[n]$ and $Y[m]$ of $\mathsf{SW}(\mathcal{C})$, the hom-set is defined by

$$\mathsf{Hom}_{\mathsf{SW}(\mathcal{C})}(X[n], Y[m]) = \varinjlim_{i \geqslant n,m} \mathsf{Hom}_{\mathcal{C}}(\Omega^{i-n}X, \Omega^{i-m}Y).$$

For the basic properties of Spanier-Whitehead categories, we refer the reader to [15, Chapter 1].

The Spanier–Whitehead category $\mathsf{SW}(\mathcal{C})$ is always a triangulated category. In fact, we define the functor $\tilde{\Omega} : \mathsf{SW}(\mathcal{C}) \to \mathsf{SW}(\mathcal{C})$ by $\tilde{\Omega}(X[n]) = X[n-1]$. Then $\tilde{\Omega}$ is an automorphism with inverse $\tilde{\Omega}^{-1}$ given by $\tilde{\Omega}^{-1}(X[n]) = X[n+1]$ for each $X[n] \in \mathsf{SW}(\mathcal{C})$. Moreover, we denote by $\tilde{\Delta}$ the collection of sequences $\tilde{\Omega}X[n] \to Z[l] \to Y[m] \to X[n]$ such that there exist an integer $i \geq n, m, l$ and a left triangle $\Omega(\Omega^{i-n}X) \to \Omega^{i-l}Z \to \Omega^{i-m}Y \to \Omega^{i-n}X$ in $\mathcal{C}$. The triple $(\mathsf{SW}(\mathcal{C}), \tilde{\Omega}, \tilde{\Delta})$ has the structure of a triangulated category; see [15, Chapter 1, Theorem 7] for the details. We should remark that this process is called in [3] the *stabilization* of the left triangulated category $\mathcal{C}$.

We define the functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{SW}(\mathcal{C})$ by $\eta_{\mathcal{C}}(X) = X[0]$ for $X \in \mathcal{C}$. There is a natural isomorphism $\nu_X : \Omega X[0] \to X[-1]$, which gives rise to an isomorphism $\nu : \eta_{\mathcal{C}}\Omega \xrightarrow{\cong} \tilde{\Omega}\eta_{\mathcal{C}}$. Thus we obtain a stable functor $(\eta_{\mathcal{C}}, \nu) : (\mathcal{C}, \Omega) \to (\mathsf{SW}(\mathcal{C}), \tilde{\Omega})$. The Spanier–Whitehead category is uniquely determined up to isomorphism by the universal property: any stable functor out of $(\mathcal{C}, \Omega)$ to a triangulated category has a unique factorization through $(\eta_{\mathcal{C}}, \nu)$; see [13, Proposition 1.1].

Finally, we recall the definitions of stable categories and (quasi-)resolving subcategories.

Definition 2.5. Let $\mathcal{X}$ be an additive subcategory of $\mathcal{A}$ containing $\mathsf{proj}\,\mathcal{A}$ (possibly $\mathcal{X} = \mathcal{A}$). The *stable category* $\underline{\mathcal{X}}$ of $\mathcal{X}$ is defined as follows.

- The objects of $\underline{\mathcal{X}}$ are the same as those of $\mathcal{X}$.
- The hom-set $\mathsf{Hom}_{\underline{\mathcal{X}}}(M, N)$ is the quotient $\underline{\mathsf{Hom}}_{\mathcal{A}}(M, N)$ of the abelian group $\mathsf{Hom}_{\mathcal{A}}(M, N)$ by the subgroup consisting of morphisms $M \to N$ factoring through objects in $\mathsf{proj}\,\mathcal{A}$.

Then $\underline{\mathcal{A}}$ is an additive category, and $\underline{\mathcal{X}}$ is an additive (full) subcategory of $\underline{\mathcal{A}}$.

Definition 2.6. (1) A subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *quasi-resolving* if $\mathcal{X}$ satisfies the following conditions.
  (a) $\mathcal{X}$ contains $\mathsf{proj}\,\mathcal{A}$.
  (b) $\mathcal{X}$ is closed under finite direct sums: for a finite number of objects $X_1, \dots, X_n$ in $\mathcal{X}$ the direct sum $X_1 \oplus \cdots \oplus X_n$ is in $\mathcal{X}$.
  (c) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, if M and N are in $\mathcal{X}$, then so is L.

(2) A quasi-resolving subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *resolving* if $\mathcal{X}$ satisfies the following conditions.

(a) $\mathcal{X}$ is closed under direct summands: if X is an object in $\mathcal{X}$ and Y is a direct summand of X in $\mathcal{A}$, then Y is in $\mathcal{X}$.
(b) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, if L and N are in $\mathcal{X}$, then so is M.

Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$, and let X be an object in $\mathcal{X}$. As $\mathcal{A}$ has enough projective objects, there exists an exact sequence

$$0 \to \Omega X \xrightarrow{\lambda_X} P_X \xrightarrow{\pi_X} X \to 0$$

in $\mathcal{A}$ with $P_X \in \operatorname{proj}\mathcal{A}$, where the object ΩX is called the (first) syzygy of X. The assignment $X \mapsto \Omega X$ gives rise to an endofunctor $\Omega : \underline{\mathcal{X}} \to \underline{\mathcal{X}}$ of the stable category $\underline{\mathcal{X}}$ of $\mathcal{X}$, which is called the syzygy functor. Take a morphism $f : X \to Y$ in $\mathcal{X}$ and an epimorphism $\pi_Y : P_Y \to Y$ in $\mathcal{A}$ with $P_Y \in \operatorname{proj}\mathcal{A}$. There exists a pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega Y \arrow[r, "\alpha_f"] \arrow[d, equal] & C_f \arrow[r, "\beta_f"] \arrow[d, "\gamma_f"] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega Y \arrow[r, "\lambda_Y"] & P_Y \arrow[r, "\pi_Y"] & Y \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ with exact rows. This gives the sequence $\Omega Y \xrightarrow{\alpha_f} C_f \xrightarrow{-\beta_f} X \xrightarrow{f} Y$ in $\underline{\mathcal{X}}$, and we define a left triangle in $\underline{\mathcal{X}}$ to be a sequence isomorphic to a sequence of this form. Then $\underline{\mathcal{X}}$ is a left triangulated category by [4, Theorem 2.12]. Hence the Spanier–Whitehead category of $\underline{\mathcal{X}}$ is defined; we simply set

$$\mathsf{SW}(\mathcal{X}) := \mathsf{SW}(\underline{\mathcal{X}}), \qquad \eta_{\mathcal{X}} := \eta_{\underline{\mathcal{X}}},$$

and call $\mathsf{SW}(\mathcal{X})$ the Spanier–Whitehead category of $\mathcal{X}$. Note that $\eta_{\mathcal{X}}(M) = M[0]$ for each $M \in \mathcal{X}$.

3. Comparison of Spanier–Whitehead categories and singularity categories

In this section, we compare the Spanier–Whitehead categories of resolving subcategories and the singularity category. We start by establishing a lemma to prove our main result.

Lemma 3.1. (1) *For a subcategory $\mathcal{X}$ of $\mathsf{D^b}(\mathcal{A})$ one has $\pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}) = \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X})$.*
(2) (a) *For an object $X \in \mathsf{D^b}(\mathcal{A})$ there exists an exact triangle $P \to X \to \Sigma^n M \rightsquigarrow$ in $\mathsf{D^b}(\mathcal{A})$ such that $P \in \mathsf{D^{perf}}(\mathcal{A})$, $M \in \mathcal{A}$ and $n \in \mathbb{Z}$.*
(b) *Let $M \in \mathcal{A}$ be an object and $n \geqslant 0$ an integer. Let N be the nth syzygy of M. Then there is an exact triangle $P \to M \to \Sigma^n N \rightsquigarrow$ in $\mathsf{D^b}(\mathcal{A})$ with $P \in \mathsf{D^{perf}}(\mathcal{A})$.*
(3) *The triangulated subcategory $\mathsf{D^{perf}}(\mathcal{A})$ of $\mathsf{D^b}(\mathcal{A})$ is thick, namely, it is closed under direct summands.*
(4) *Let M, N be objects of $\mathsf{D^b}(\mathcal{A})$. Then $\pi M \cong \pi N$ if and only if there exist exact triangles*

$$X \to M \to P \rightsquigarrow \quad \text{and} \quad X \to N \to Q \rightsquigarrow$$

in $\mathsf{D^b}(\mathcal{A})$ such that $P, Q \in \mathsf{D^{perf}}(\mathcal{A})$.

Proof. (1) As $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}$ contains $\mathcal{X}$, the triangulated subcategory $\pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X})$ of $\mathsf{D_{sg}}(\mathcal{A})$ contains $\pi\mathcal{X}$, and hence it contains $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X})$. The subcategory of $\mathsf{D^b}(\mathcal{A})$ consisting of objects M with $\pi M \in \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X})$ is triangulated and contains $\mathcal{X}$. Hence it contains $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}$, and therefore $\pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X})$ is contained in $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X})$. Thus the equality $\pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}) = \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X})$ holds.

(2) The assertion follows from [11, Lemma 2.4].

(3) It suffices to show that

$$\mathsf{D^{perf}}(\mathcal{A}) = \{X \in \mathsf{D^b}(\mathcal{A}) \mid \text{there exists } r \in \mathbb{Z} \text{ such that } \mathsf{Ext}^i_{\mathcal{A}}(X, A) = 0 \text{ for all } A \in \mathcal{A} \text{ and } i > r\}.$$

Put $\mathcal{C}$ to be the right-hand side. The inclusion of $\mathsf{D^{perf}}(\mathcal{A})$ in $\mathcal{C}$ is obvious. To show the opposite inclusion, let $X \in \mathcal{C}$. Taking an exact triangle $P \to X \to \Sigma^n M \rightsquigarrow$ as in (2), we see that M is in $\mathcal{C}$, which implies that there exists an integer s such that $\mathsf{Ext}^{>s}_{\mathcal{A}}(M, \mathcal{A}) = 0$. This means that M has projective dimension at most s. Hence M is isomorphic in $\mathsf{D^b}(\mathcal{A})$ to a perfect complex, that is to say, M belongs to $\mathsf{D^{perf}}(\mathcal{A})$, and so does X.

(4) The "if" part: Application of the triangle functor π gives exact triangles $\Sigma^{-1}\pi P \to \pi X \to \pi M \to \pi P$ and $\Sigma^{-1}\pi Q \to \pi X \to \pi N \to \pi Q$ in $\mathsf{D_{sg}}(\mathcal{A})$. Since πP and πQ are zero by [17, Theorem 2.1.8], we get isomorphisms $\pi M \cong \pi X \cong \pi N$. Conversely, suppose that $\pi M \cong \pi N$. Then, using [17, Remark 2.1.23],

we observe that there exist morphisms $M \xleftarrow{s} X \xrightarrow{f} N$ in $\mathsf{D^b}(\mathcal{A})$ such that cone(s) is in $\mathsf{D^{perf}}(\mathcal{A})$ and πf is an isomorphism. It follows from [17, Proposition 2.1.35] and (3) that cone(f) belongs to $\mathsf{D^{perf}}(\mathcal{A})$. ■

Now we can prove the following theorem, which is the main result of this section.

Theorem 3.2. *Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$.*

(1) *The assignment $X[n] \mapsto \Sigma^n X$, where $X \in \mathcal{X}$ and $n \in \mathbb{Z}$, gives rise to a fully faithful triangle functor $\theta_{\mathcal{X}} : \mathsf{SW}(\mathcal{X}) \to \mathsf{D_{sg}}(\mathcal{A})$. One has a sequence of additive functors*

$$\underline{\mathcal{X}} \xrightarrow{\eta_{\mathcal{X}}} \mathsf{SW}(\mathcal{X}) \xrightarrow{\theta_{\mathcal{X}}} \mathsf{D_{sg}}(\mathcal{A}),$$

and the equality $\theta_{\mathcal{X}}\eta_{\mathcal{X}}(X) = \pi X$ holds for each $X \in \mathcal{X}$.

(2) *The Spanier–Whitehead category $\mathsf{SW}(\mathcal{X})$ is identified with the triangulated subcategory of $\mathsf{D_{sg}}(\mathcal{A})$ generated by the objects in $\mathcal{X}$. More precisely, it holds that*

$$\mathsf{Im}\,\theta_{\mathcal{X}} = \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}}) = \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}\,\pi\mathcal{X}.$$

(3) *There are equivalences*

$$\mathsf{SW}(\mathcal{X}) \underset{\theta_{\mathcal{X}}}{\cong} \mathsf{D_{sg}}(\mathcal{A}) \iff \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}\,\pi\mathcal{X} = \mathsf{D_{sg}}(\mathcal{A}) \iff \mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X} = \mathsf{D^b}(\mathcal{A}).$$

Proof. (1) It follows from [3, Theorem 3.8] that the functor $\theta_{\mathcal{A}} : \mathsf{SW}(\mathcal{A}) \to \mathsf{D_{sg}}(\mathcal{A})$ is a triangle equivalence. The inclusion functor $\iota : \underline{\mathcal{X}} \to \underline{\mathcal{A}}$ induces a functor $\mathsf{SW}(\iota) : \mathsf{SW}(\mathcal{X}) \to \mathsf{SW}(\mathcal{A})$, which satisfies $\mathsf{SW}(\iota)(X[n]) = X[n]$ for $X \in \mathcal{X}$ and $n \in \mathbb{Z}$. It is clear that $\mathsf{SW}(\iota)$ is a triangle functor. The functor ι is fully faithful, and it is easy to see from this that $\mathsf{SW}(\iota)$ is also fully faithful. Note that the composition of $\mathsf{SW}(\iota)$ and $\theta_{\mathcal{A}}$ coincides with $\theta_{\mathcal{X}}$. Therefore, $\theta_{\mathcal{X}}$ is a fully faithful triangle functor. The last assertion is straightforward.

(2) Since $\theta_{\mathcal{X}}$ is a fully faithful triangle functor, $\mathsf{Im}\,\theta_{\mathcal{X}}$ is a triangulated subcategory of $\mathsf{D_{sg}}(\mathcal{A})$ containing $\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}}$. Hence

$$\mathsf{Im}\,\theta_{\mathcal{X}} \supseteq \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}}) \supseteq \mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}}.$$

Take any object $Y \in \mathsf{Im}\,\theta_{\mathcal{X}}$. Then $Y = \theta_{\mathcal{X}}(X[n]) = \Sigma^n X$ for some $X \in \mathcal{X}$ and $n \in \mathbb{Z}$. As $X = \theta_{\mathcal{X}}\eta_{\mathcal{X}}(X)$ is in $\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}}$, it is also in $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}})$. Since $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}})$ is triangulated, it contains $\Sigma^n X = Y$. Therefore we get $\mathsf{Im}\,\theta_{\mathcal{X}} = \mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}})$. As $\theta_{\mathcal{X}}\eta_{\mathcal{X}}(M) = \pi(M)$ for each object M in $\mathcal{X}$, there is an equality $\mathsf{Im}\,\theta_{\mathcal{X}}\eta_{\mathcal{X}} = \pi\mathcal{X}$.

(3) The first equivalence directly follows from (2). Let us show the second equivalence. Using Lemma 3.1(1), we see that $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X} = \mathsf{D^b}(\mathcal{A})$ implies $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X}) = \pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}) = \pi(\mathsf{D^b}(\mathcal{A})) = \mathsf{D_{sg}}(\mathcal{A})$. Conversely, assume that the equality $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X}) = \mathsf{D_{sg}}(\mathcal{A})$ holds and take any object $M \in \mathsf{D^b}(\mathcal{A})$. Then πM belongs to $\mathsf{tria}_{\mathsf{D_{sg}}(\mathcal{A})}(\pi\mathcal{X}) = \pi(\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X})$ again by Lemma 3.1(1), and we find an object $N \in \mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}$ such that $\pi M \cong \pi N$. There exist exact triangles $X \to M \to P \rightsquigarrow$ and $X \to N \to Q \rightsquigarrow$ in $\mathsf{D^b}(\mathcal{A})$ with $P, Q \in \mathsf{D^{perf}}(\mathcal{A})$ by Lemma 3.1(4). Note that $\mathsf{D^{perf}}(\mathcal{A}) = \mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}(\mathsf{proj}\,\mathcal{A})$. As $\mathcal{X}$ contains $\mathsf{proj}\,\mathcal{A}$, the object Q is in $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}$. Therefore X belongs to $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X}$, and so does M. We conclude that the equality $\mathsf{tria}_{\mathsf{D^b}(\mathcal{A})}\,\mathcal{X} = \mathsf{D^b}(\mathcal{A})$ holds. ■

Let us give the definitions of several kinds of resolving subcategories of the module category of a commutative ring to apply the above theorem to them.

Definition 3.3. (1) Let S be a subset of $\operatorname{Spec} R$. We denote by the subcategory $\mathcal{F}(S)$ (resp. $\mathcal{P}(S)$) of $\mathsf{mod}\,R$ consisting of modules which are locally free (resp. of finite projective dimension) on S.

(2) Let M be an R-module. We denote by $M_\perp$ (resp. ${}^\perp M$) the subcategory of $\mathsf{mod}\,R$ consisting of modules N satisfying $\mathsf{Tor}^R_{>0}(M, N) = 0$ (resp. $\mathsf{Ext}^{>0}_R(N, M) = 0$).

Note that all the subcategories $\mathcal{F}(S), \mathcal{P}(S), M_\perp, {}^\perp M$ of $\mathsf{mod}\,R$ are resolving.

Using Theorem 3.2 for these resolving subcategories, we can characterize the local regularity of a commutative ring and the finiteness of homological dimensions of a module.

Corollary 3.4. (1) *There are equivalences for a subset S of* $\operatorname{Spec} R$*:*

$$R \text{ is locally regular on } S \iff \theta_{\mathcal{F}(S)} \text{ is dense} \iff \theta_{\mathcal{P}(S)} \text{ is dense.}$$

(2) *Let M be a finitely generated R-module. Then*

$$M \text{ locally has finite projective (resp. injective) dimension} \iff \theta_{M_\perp} \text{ (resp. } \theta_{{}^\perp M}\text{) is dense.}$$

Proof. For a finitely generated R-module M we set

$$\operatorname{Rfd}_R M := \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}. \tag{3.4.1}$$

By virtue of [1, Theorem 1.1] this is always finite.

(1) We call the first, second and third conditions (a), (b) and (c), respectively.

(a) $\Longrightarrow$ (b): According to Theorem 3.2(3), it is enough to show that $\operatorname{tria}\mathcal{F}(S) = \mathsf{D}^\mathsf{b}(R)$. Let X be a complex in $\mathsf{D}^\mathsf{b}(R)$. Then there exists an exact triangle $P \to X \to \Sigma^n M \rightsquigarrow$ in $\mathsf{D}^\mathsf{b}(R)$ such that $P \in \mathsf{D}^{\mathsf{perf}}(R)$, $M \in \operatorname{mod} R$ and $n \in \mathbb{Z}$ by Lemma 3.1(2a). As $\mathcal{F}(S)$ contains $\operatorname{proj} R$, it is seen that $\operatorname{tria}\mathcal{F}(S)$ contains P. Set $m := \operatorname{Rfd}_R M$. For each $\mathfrak{p} \in S$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ has projective dimension at most m, and hence the mth syzygy N of the R-module M belongs to $\mathcal{F}(S)$. Therefore M belongs to $\operatorname{tria}\mathcal{F}(S)$, and so does $\Sigma^n M$. Thus X is in $\operatorname{tria}\mathcal{F}(S)$, and we obtain $\operatorname{tria}\mathcal{F}(S) = \mathsf{D}^\mathsf{b}(R)$.

(b) $\Longrightarrow$ (c): As $\mathcal{F}(S) \subseteq \mathcal{P}(S)$, we see $\theta_{\mathcal{F}(S)}$ factors through $\theta_{\mathcal{P}(S)}$. The implication follows from this.

(c) $\Longrightarrow$ (a): Theorem 3.2(3) implies $\operatorname{tria}\mathcal{P}(S) = \mathsf{D}^\mathsf{b}(R)$. Let $\mathcal{Y}$ be the subcategory of $\mathsf{D}^\mathsf{b}(R)$ consisting of complexes which are locally of finite projective dimension on S. Then $\mathcal{Y}$ is triangulated and contains $\mathcal{P}(S)$, and hence it also contains $\operatorname{tria}\mathcal{P}(S)$. Thus we obtain $\mathcal{Y} = \mathsf{D}^\mathsf{b}(R)$. For each $\mathfrak{p} \in S$ the module $R/\mathfrak{p}$ is in $\mathcal{Y}$, and the residue field $\kappa(\mathfrak{p})$ of $R_\mathfrak{p}$ has finite projective dimension as an $R_\mathfrak{p}$-module. This implies that $R_\mathfrak{p}$ is a regular local ring.

(2)(i) Suppose that $\theta_{M_\perp}$ is dense. Then it follows from Theorem 3.2(3) that $\operatorname{tria}(M_\perp) = \mathsf{D}^\mathsf{b}(R)$. Let $\mathcal{Y}$ be the subcategory of $\mathsf{D}^\mathsf{b}(R)$ consisting of complexes Y with $\operatorname{Tor}^R_{\gg 0}(M, Y) = 0$. Then $\mathcal{Y}$ is triangulated and contains $M_\perp$. Hence $\mathcal{Y} = \operatorname{tria}(M_\perp) = \mathsf{D}^\mathsf{b}(R)$. This especially says that $\operatorname{Tor}^R_{\gg 0}(M, R/\mathfrak{p}) = 0$ for each $\mathfrak{p} \in \operatorname{Spec} R$. Therefore $\operatorname{Tor}^{R_\mathfrak{p}}_{\gg 0}(M_\mathfrak{p}, \kappa(\mathfrak{p})) = 0$, which implies that $M_\mathfrak{p}$ has finite projective dimension.

(ii) Assume that M is locally of finite projective dimension. Set $m := \operatorname{Rfd}_R M$. For each $\mathfrak{p} \in \operatorname{Spec} R$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ has projective dimension at most m, and $\operatorname{Tor}^R_{>m}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for each $N \in \operatorname{mod} R$. Hence the mth syzygy of N belongs to $M_\perp$, and N is in $\operatorname{tria}(M_\perp)$. Therefore $\operatorname{mod} R$ is contained in $\operatorname{tria}(M_\perp)$, and so is $\operatorname{tria}(\operatorname{proj} R) = \mathsf{D}^{\mathsf{perf}}(R)$. Applying Lemma 3.1(2a), we observe that $\mathsf{D}^\mathsf{b}(R) = \operatorname{tria}(M_\perp)$ holds. Theorem 3.2(3) implies that $\theta_{M_\perp}$ is dense.

(iii) Assume that the functor $\theta_{{}^\perp M}$ is dense. In the argument (i), replace $M_\perp, \operatorname{Tor}^R_{\gg 0}(M, Y) = 0, \dots$ with ${}^\perp M, \operatorname{Ext}_R^{\gg 0}(Y, M) = 0, \dots$ respectively. Then it is observed that M locally has finite injective dimension.

(iv) Suppose that M is locally of finite injective dimension. Fix an R-module $N \in \operatorname{mod} R$ and set $n := \operatorname{Rfd}_R N$. Then $\operatorname{Ext}_{R_\mathfrak{p}}^{>n}(N_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ by [6, Exercise 3.1.24]. Hence $\operatorname{Ext}_R^{>n}(N, M) = 0$, which implies that the nth syzygy of N belongs to ${}^\perp M$. Making a similar argument as in the latter half of (ii), we deduce that $\mathsf{D}^\mathsf{b}(R) = \operatorname{tria}({}^\perp M)$, and $\theta_{{}^\perp M}$ is dense. ■

Remark 3.5. Combining Corollary 3.4(2) with [2, Lemma 4.5] shows that a finitely generated R-module M has finite projective dimension if and only if $\theta_{M_\perp}$ is dense.

Recall that the *punctured spectrum* of R is defined to be the set of nonmaximal prime ideals of R. We give the definitions of two resolving subcategories that are well-studied in the literature.

Definition 3.6. (1) We denote by $\operatorname{mod}_0 R$ the subcategory of $\operatorname{mod} R$ consisting of modules that are locally free on the punctured spectrum of R.

(2) We denote by $\mathsf{GP}(R)$ the subcategory of $\operatorname{mod} R$ consisting of Gorenstein projective R-modules. Here, a finitely generated R-module M is called *Gorenstein projective* (or *totally reflexive*) if M is reflexive, $\operatorname{Ext}_R^{>0}(M, R) = 0$ and $\operatorname{Ext}_R^{>0}(\operatorname{Hom}_R(M, R), R) = 0$. For the details, we refer the reader to [8].

Both $\operatorname{mod}_0 R$ and $\mathsf{GP}(R)$ are resolving subcategories of $\operatorname{mod} R$.

Recall that R is called an *isolated singularity* if the punctured spectrum of R is regular, or in other words, $R_\mathfrak{p}$ is a regular local ring for all nonmaximal prime ideals $\mathfrak{p}$ of R. Also, a finitely generated R-module M is said to have *full support* if $\operatorname{Supp} M = \operatorname{Spec} R$. Applying Corollary 3.4, we obtain characterizations of several classes of commutative rings.

Corollary 3.7. (1) *R is an isolated singularity if and only if $\theta_{\operatorname{mod}_0 R}$ is dense.*

(2) *R is a Gorenstein ring if and only if $\theta_{\mathsf{GP}(R)}$ is dense, if and only if $\theta_{{}^\perp R}$ is dense.*

(3) *R is a Cohen–Macaulay ring if $\theta_{{}^\perp M}$ is dense for some $M \in \mathsf{mod}\, R$ with full support. The converse also holds when R admits a canonical module.*

Proof. (1) Letting S be the punctured spectrum of R in Corollary 3.4(1) immediately shows the assertion.

(2) Corollary 3.4(2) shows that R is Gorenstein if and only if $\theta_{{}^\perp R}$ is dense. As ${}^\perp R$ contains $\mathsf{GP}(R)$, the functor $\theta_{{}^\perp R}$ is dense if so is $\theta_{\mathsf{GP}(R)}$. If R is Gorenstein, then $\mathsf{GP}(R) = {}^\perp R$, whence $\theta_{\mathsf{GP}(R)}$ is dense.

(3) Let M be a finitely generated R-module with full support. By Corollary 3.4(2) the functor $\theta_{{}^\perp M}$ is dense if and only if for each $\mathfrak{p} \in \operatorname{Spec} R$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is nonzero and has finite injective dimension. The latter condition implies that R is Cohen–Macaulay by [6, Remark 9.6.4(a)]. Conversely, assume that R is a Cohen–Macaulay ring with a canonical module ω. Then ω has full support and is locally of finite injective dimension (see [6, Definition 3.3.16]). Hence $\theta_{{}^\perp \omega}$ is a dense functor. ■

4. The structure of Spanier–Whitehead categories

Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$. Theorem 3.2(1) shows that $\theta_\mathcal{X} : \mathsf{SW}(\mathcal{X}) \to \mathsf{D_{sg}}(\mathcal{A})$ is a fully faithful triangle functor. In this section, passing through this functor, we regard $\mathsf{SW}(\mathcal{X})$ as a full triangulated subcategory of $\mathsf{D_{sg}}(\mathcal{A})$, and study the structure of this subcategory. More precisely, we consider the following natural question.

Question 4.1. Let $\mathcal{X}$ be a quasi-resolving subcategory of $\mathcal{A}$. When is $\mathsf{SW}(\mathcal{X})$ a thick subcategory of $\mathsf{D_{sg}}(\mathcal{A})$? In other words, when is it closed under direct summands?

The following proposition gives an answer to this question.

Proposition 4.2. *Let S be a subset of $\operatorname{Spec} R$. Then the Spanier–Whitehead categories $\mathsf{SW}(\mathcal{P}(S))$ and $\mathsf{SW}(\mathcal{F}(S))$ are thick subcategories of $\mathsf{D_{sg}}(R)$.*

Proof. (1) We prove the assertion on $\mathcal{P}(S)$. Let X, Y be objects of $\mathsf{D_{sg}}(R)$ such that $X \oplus Y$ is in $\mathsf{SW}(\mathcal{P}(S))$. Then $X \oplus Y \cong \Sigma^n Z$ in $\mathsf{D_{sg}}(R)$ for some $Z \in \mathcal{P}(S)$ and $n \in \mathbb{Z}$. Fix a prime ideal $\mathfrak{p} \in S$. We have

$$X_\mathfrak{p} \oplus Y_\mathfrak{p} \cong (X \oplus Y)_\mathfrak{p} \cong \Sigma^n Z_\mathfrak{p} \cong 0$$

in $\mathsf{D_{sg}}(R_\mathfrak{p})$, where the last isomorphism follows from the fact that $Z_\mathfrak{p}$ has finite projective dimension. Hence $X_\mathfrak{p} \cong 0 \cong Y_\mathfrak{p}$ in $\mathsf{D_{sg}}(R_\mathfrak{p})$, which particularly says that $X_\mathfrak{p}$ has finite projective dimension over $R_\mathfrak{p}$. Therefore X is locally of finite projective dimension on S. Applying Lemma 3.1(2a), we see that $X \cong \Sigma^m M$ for some $M \in \mathsf{mod}\, R$ and $m \in \mathbb{Z}$. It is observed that the R-module M locally has finite projective dimension on S, that is to say, $M \in \mathcal{P}(S)$. Consequently, $X \cong \Sigma^m M$ belongs to $\mathsf{SW}(\mathcal{P}(S))$. We conclude that $\mathsf{SW}(\mathcal{P}(S))$ is closed under direct summands.

(2) We prove the assertion on $\mathcal{F}(S)$. Let X, Y be objects of $\mathsf{D_{sg}}(R)$ such that $X \oplus Y$ is in $\mathsf{SW}(\mathcal{F}(S))$. Then $X \oplus Y \cong \Sigma^n Z$ in $\mathsf{D_{sg}}(R)$ for some $Z \in \mathcal{F}(S)$ and $n \in \mathbb{Z}$. Note that $\mathcal{F}(S)$ is contained in $\mathcal{P}(S)$. Similarly as in (1), we have $X \cong \Sigma^m M$ for some $M \in \mathsf{mod}\, R$ and $m \in \mathbb{Z}$ and M is locally of finite projective dimension on S. Setting $r = \operatorname{Rfd}_R M$, we observe that the rth syzygy N of M is locally free on S, namely, $N \in \mathcal{F}(S)$. Lemma 3.1(2b) gives an isomorphism $M \cong \Sigma^r N$, which is in $\mathsf{SW}(\mathcal{F}(S))$. It follows that $X \cong \Sigma^m M$ is also in $\mathsf{SW}(\mathcal{F}(S))$. Thus $\mathsf{SW}(\mathcal{F}(S))$ is closed under direct summands. ■

A finitely generated R-module M is called *maximal Cohen–Macaulay* if $\operatorname{depth} M_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. (Note that $\operatorname{depth} 0 = \infty$, and hence 0 is maximal Cohen–Macaulay.) We denote by $\mathsf{CM}(R)$ the subcategory of $\mathsf{mod}\, R$ consisting of maximal Cohen–Macaulay R-modules. We investigate the Spanier–Whitehead categories of a quasi-resolving subcategory of finitely generated modules and its restriction to maximal Cohen–Macaulay modules.

Proposition 4.3. *Let R be a Cohen–Macaulay ring. Let $\mathcal{X}$ be a quasi-resolving (resp. resolving) subcategory of $\mathsf{mod}\, R$. Then the restriction $\mathcal{X} \cap \mathsf{CM}(R)$ is also a quasi-resolving (resp. resolving) subcategory of $\mathsf{mod}\, R$, and there is an equality*

$$\mathsf{SW}(\mathcal{X}) = \mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R)).$$

In particular, it holds that $\mathsf{SW}(\mathsf{mod}\, R) = \mathsf{SW}(\mathsf{CM}(R))$.

Proof. The first assertion is obvious. Let us prove the second assertion. The inclusion $\mathcal{X} \cap \mathsf{CM}(R) \subseteq \mathcal{X}$ implies the inclusion $\mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R)) \subseteq \mathsf{SW}(\mathcal{X})$. To show the opposite inclusion, pick any object $X[n] \in \mathsf{SW}(\mathcal{X})$. Choose an integer $r \geqslant 0$ such that the rth syzygy C of X is maximal Cohen–Macaulay.

(For example, if we take $r = \operatorname{Rfd}_R M$, which is finite by (3.4.1), then the depth lemma implies that $\operatorname{depth}_{R_\mathfrak{p}} C_\mathfrak{p} \geqslant \operatorname{depth} R_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, that is, C is maximal Cohen–Macaulay.) Note that C belongs to $\mathcal{X} \cap \mathsf{CM}(R)$. By the definition of $\mathsf{SW}(\mathcal{X})$, we have $X[n] \cong C[n+r]$. Hence $X[n]$ belongs to $\mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R))$. Therefore the equality $\mathsf{SW}(\mathcal{X}) = \mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R))$ holds. The third assertion follows by letting $\mathcal{X} = \mathsf{mod}\, R$ in the second assertion. ∎

Now we can state and prove the main result of this section, which classifies the Spanier–Whitehead categories over a complete intersection.

Theorem 4.4. *Let R be a local complete intersection.*

(1) *Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\, R$. Then one has*
$$\mathsf{SW}(\mathcal{X}) = \underline{\mathcal{X} \cap \mathsf{CM}(R)},$$
which is a thick subcategory of $\mathsf{D_{sg}}(R)$.

(2) *Suppose that R is a quotient of a regular ring. Then there is a one-to-one correspondence between*
- *the Spanier–Whitehead categories of resolving subcategories of $\mathsf{mod}\, R$, and*
- *the specialization-closed subsets of the singular locus of the scheme $\operatorname{Proj} S$,*

where S is the generic hypersurface of R.

Proof. First of all, recall that the stable category $\underline{\mathsf{CM}(R)}$ is triangulated, and identified with $\mathsf{D_{sg}}(R)$; we refer the reader to [7, Theorem 4.4.1] for the details.

(1) Proposition 4.3 implies that $\mathsf{SW}(\mathcal{X}) = \mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R))$. It follows from [9, Corollary 4.16] that $\underline{\mathcal{X} \cap \mathsf{CM}(R)}$ is a thick subcategory of $\underline{\mathsf{CM}(R)}$, and in particular it is triangulated. Hence $\mathsf{SW}(\mathcal{X} \cap \mathsf{CM}(R)) = \underline{\mathcal{X} \cap \mathsf{CM}(R)}$ by [3, Corollary 3.3(1)].

(2) There is a one-to-one correspondence between the thick subcategories of $\underline{\mathsf{CM}(R)}$ and the specialization-closed subsets of the singular locus of $\operatorname{Proj} S$; see [20, Remark 10.7]. For a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, the stable category $\underline{\mathcal{X} \cap \mathsf{CM}(R)}$ is a thick subcategory of $\underline{\mathsf{CM}(R)}$ by (1), while each thick subcategory $\mathcal{C}$ of $\underline{\mathsf{CM}(R)}$ has the form $\underline{\mathcal{X}}$ for some resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ contained in $\mathsf{CM}(R)$. Now the assertion follows. ∎

5. The Rouquier dimension of the Spanier–Whitehead category

In this section, we explore a method to compute the (Rouquier) dimension of the Spanier–Whitehead category of a resolving subcategory. Using this result, a relationship between the dimension of $\mathsf{D_{sg}}(R)$ and its cohomology annihilator is obtained. First of all, we recall some definitions involved with generation in triangulated and abelian categories, which are given by Rouquier and Dao–Takahashi. For further details including basic properties, we refer the reader to their papers [10, 19].

Definition 5.1. (1) Let $\mathcal{T}$ be a triangulated category.

(i) For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ that is closed under finite direct sums, direct summands and shifts, i.e.,
$$\langle \mathcal{X} \rangle = \mathsf{add}_\mathcal{T}\{\Sigma^i X \mid i \in \mathbb{Z},\ X \in \mathcal{X}\}.$$
When $\mathcal{X}$ consists of a single object X, we simply denote it by $\langle X \rangle$.

(ii) For subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects M which fit into an exact triangle $X \to M \to Y \to \Sigma X$ in $\mathcal{T}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(iii) For a subcategory $\mathcal{C}$ of $\mathcal{T}$ we set:
$$\langle \mathcal{C} \rangle_r = \begin{cases} \{0\} & (r = 0), \\ \langle \mathcal{C} \rangle & (r = 1), \\ \langle \langle \mathcal{C} \rangle_{r-1} * \langle \mathcal{C} \rangle \rangle & (r \geqslant 2). \end{cases}$$
If $\mathcal{C}$ consists of a single object C, then we simply denote it by $\langle C \rangle_r$.

(iv) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We define the *(Rouquier) dimension* of $\mathcal{X}$, denoted by $\dim \mathcal{X}$ (or $\dim_\mathcal{T} \mathcal{X}$), as the infimum of the integers $n \geqslant 0$ such that $\mathcal{X} = \langle G \rangle_{n+1}$ for some $G \in \mathcal{X}$.

(2) Let $\mathcal{A}$ be an abelian category with enough projective objects.

(i) For a subcategory $\mathcal{X}$ of $\mathcal{A}$ we denote by $[\mathcal{X}]$ the smallest subcategory of $\mathcal{A}$ containing $\mathsf{proj}\,\mathcal{A}$ and $\mathcal{X}$ that is closed under finite direct sums, direct summands and syzygies, i.e.,

$$[\mathcal{X}] = \mathsf{add}_{\mathcal{A}}(\mathsf{proj}\,\mathcal{A} \cup \{\Omega^i X \mid i \geqslant 0,\ X \in \mathcal{X}\}).$$

When $\mathcal{X}$ consists of a single object X, we simply denote it by $[X]$.

(ii) For subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{A}$ we denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\mathcal{A}$ consisting of objects M which fit into an exact sequence $0 \to X \to M \to Y \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(iii) For a subcategory $\mathcal{C}$ of $\mathcal{A}$ we set:

$$[\mathcal{C}]_r = \begin{cases} \{0\} & (r = 0), \\ [\mathcal{C}] & (r = 1), \\ [[\mathcal{C}]_{r-1} \circ [\mathcal{C}]] & (r \geqslant 2). \end{cases}$$

If $\mathcal{C}$ consists of a single object C, then we simply denote it by $[C]_r$.

(iv) Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We define the *(Dao–Takahashi) dimension* of $\mathcal{X}$, denoted by $\dim \mathcal{X}$ (or $\dim_{\mathcal{A}} \mathcal{X}$), as the infimum of the integers $n \geqslant 0$ such that $\mathcal{X} = [G]_{n+1}$ for some $G \in \mathcal{X}$.

We simply write X to denote $X[0] \in \mathsf{SW}(\mathcal{X})$.

Lemma 5.2. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$.*

(1) *Let $X, Y \in \mathcal{X}$ and $n \geqslant 0$. If $X \in [Y]_n$ in $\mathcal{A}$, then $X \in \langle Y \rangle_n$ in $\mathsf{SW}(\mathcal{X})$.*

(2) *Let $X, Y \in \mathcal{X}$ and $m, n \in \mathbb{Z}$. Then:*

$$X[m] \cong Y[n] \text{ in } \mathsf{SW}(\mathcal{X}) \iff \Omega^{k-m} X \cong \Omega^{k-n} Y \text{ in } \underline{\mathcal{A}} \text{ for some } k \geqslant m, n.$$

(3) *Let $X, Y \in \mathcal{X}$ and $m, n \in \mathbb{Z}$. Then $X[m]$ is a direct summand of $Y[n]$ in $\mathsf{SW}(\mathcal{X})$ if and only if $\Omega^{k-m} X$ is a direct summand of $\Omega^{k-n} Y$ in $\underline{\mathcal{A}}$ for some $k \geqslant m, n$.*

(4) *An exact triangle $X[a] \to Y[b] \to Z[c] \rightsquigarrow$ in $\mathsf{SW}(\mathcal{X})$ with $X, Y, Z \in \mathcal{X}$ and $a, b, c \in \mathbb{Z}$ induces a short exact sequence*

$$0 \to \Omega^{d-a} X \to \Omega^{d-b} Y \to \Omega^{d-c} Z \to 0$$

in $\mathcal{A}$ with $d \geqslant a, b, c$, up to projective summands. Conversely, this short exact sequence induces an exact triangle as above.

Proof. (1) We show the assertion by induction on n. Both $[Y]_0$ and $\langle Y \rangle_0$ are zero, which settles the case $n = 0$. Pick any object $A \in \mathcal{A}$. If A belongs to $[Y] = \mathsf{add}(\mathsf{proj}\,\mathcal{A} \cup \{\Omega^i Y \mid i \geqslant 0\})$, then in the category $\mathsf{SW}(\mathcal{X})$ we have $A \in \mathsf{add}\{\Omega^i Y \mid i \geqslant 0\} \subseteq \langle Y \rangle$. This settles the case $n = 1$.

Now let $n \geqslant 2$. Then there exists an exact sequence $0 \to L \xrightarrow{g} M \xrightarrow{f} N \to 0$ in $\mathcal{A}$ with $L \in [Y]_{n-1}$ and $N \in [Y]$ such that X is a direct summand of M in $\mathcal{A}$. Since $\mathcal{X}$ is resolving and contains Y, we see that L, N are in $\mathcal{X}$, and so is M. Thus the above exact sequence induces an exact triangle $L \to M \to N \rightsquigarrow$ in $\mathsf{SW}(\mathcal{X})$. Indeed, there exists a pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d, "g"] & \\
0 \arrow[r] & \Omega N \arrow[r, "\alpha_f"] \arrow[d, equal] & C_f \arrow[r, "\beta_f"] \arrow[d, "\gamma_f"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega Y \arrow[r, "\lambda_N"] & P_N \arrow[r, "\pi_N"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in $\mathcal{X}$, which induces a left triangle $\Omega N \to C_f \to M \to N$ in $\underline{\mathcal{X}}$. Since $P_N \in \operatorname{proj} \mathcal{A}$, we have $C_f \cong P_N \oplus L$ and so $C_f \cong L$ in $\underline{\mathcal{X}}$. The induction hypothesis and basis imply that $L \in \langle Y \rangle_{n-1}$ and $N \in \langle Y \rangle$, while X is a direct summand of M in $\mathsf{SW}(\mathcal{X})$. It follows that $X \in \langle Y \rangle_n$.

(2) The assertion is a direct consequence of [3, Corollary 3.3(3)]; see also [15, Page 7, Line 5-6].

(3) The "if" part: There is an object $Z \in \underline{\mathcal{A}}$ such that $\Omega^{k-m} X \oplus Z \cong \Omega^{k-n} Y$ in $\underline{\mathcal{A}}$. Since $\mathcal{X}$ is resolving and contains Y, we see that $\mathcal{X}$ contains Z as well. Hence we obtain isomorphisms

$$X[m-k] \oplus Z \cong \Omega^{k-m} X \oplus Z \cong \Omega^{k-n} Y \cong Y[n-k]$$

in $\mathsf{SW}(\mathcal{X})$, which implies $X[m] \oplus Z[k] \cong Y[n]$ in $\mathsf{SW}(\mathcal{X})$.

The "only if" part: There is an isomorphism $X[m] \oplus Z[h] \cong Y[n]$ in $\mathsf{SW}(\mathcal{X})$, where $Z \in \mathcal{X}$ and $h \in \mathbb{Z}$. The object $X[m] \oplus Z[h]$ is isomorphic to $(\Omega^{s-m} X \oplus \Omega^{s-h} Z)[s]$, where $s = \max\{m, h\}$. It follows from (2) that there exists an integer $l \geqslant s, n$ such that $\Omega^{l-s}(\Omega^{s-m} X \oplus \Omega^{s-h} Z) \cong \Omega^{l-n} Y$ in $\underline{\mathcal{A}}$. Hence $\Omega^{l-m} X \oplus \Omega^{l-h} Z \cong \Omega^{l-n} Y$.

(4) The last statement is evident. Let us show the first statement. By the definition of Spanier–Whitehead category, the exact triangle $X[a] \to Y[b] \to Z[c] \rightsquigarrow$ is induced from a left triangle

$$\Omega^{d-c+1} Z \to \Omega^{d-a} X \to \Omega^{d-b} Y \to \Omega^{d-c} Z \tag{5.2.1}$$

in the left triangulated category $\underline{\mathcal{X}}$, where $d \geqslant \max\{a, b, c\}$ is an integer. By the definition of left triangles in $\underline{\mathcal{X}}$, there exists a pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega(\Omega^{d-c} Z) \arrow[r, "\gamma"] \arrow[d, equal] & W \arrow[r, "\beta"] \arrow[d, "\theta"] & \Omega^{d-b} Y \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \Omega(\Omega^{d-c} Z) \arrow[r, "\lambda"'] & P \arrow[r, "\pi"'] & \Omega^{d-c} Z \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ with $P \in \operatorname{proj} \mathcal{A}$ such that the sequence $\Omega(\Omega^{d-c} Z) \xrightarrow{\gamma} W \xrightarrow{\beta} \Omega^{d-b} Y \xrightarrow{\alpha} \Omega^{d-c} Z$ is isomorphic to the sequence (5.2.1) in $\underline{\mathcal{X}}$. Thus $W \cong \Omega^{d-a} X$ in $\underline{\mathcal{X}}$, that is, W is isomorphic in $\mathcal{A}$ to $\Omega^{d-a} X$ up to projective summands. On the other hand we have the short exact sequence

$$0 \to W \xrightarrow{\binom{-\beta}{\theta}} \Omega^{d-b} Y \oplus P \xrightarrow{(\alpha, \pi)} \Omega^{d-c} Z \to 0$$

in $\mathcal{A}$, which gives a short exact sequence as in the assertion. ■

Proposition 5.3. *Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$. Then the following conditions are equivalent for any $G \in \mathcal{X}$ and $n > 0$.*

(1) *One has $\mathsf{SW}(\mathcal{X}) = \langle G \rangle_n$.*

(2) *For each $X \in \mathcal{X}$ there exists an integer $k \geqslant 0$ such that $\Omega^k X \in [G]_n$.*

Proof. (2) $\Longrightarrow$ (1): Pick any object $M \in \mathsf{SW}(\mathcal{X})$. Then $M = X[m]$ for some $X \in \mathcal{X}$ and $m \in \mathbb{Z}$. By assumption we have $\Omega^k X \in [G]_n$ for some $k \geqslant 0$, and $\Omega^k X \in \mathcal{X}$. Lemma 5.2(1) implies that $\Omega^k X \in \langle G \rangle_n$. Hence $X = \Omega^k X[k]$ belongs to $\langle G \rangle_n$, and so does $M = X[m]$. This shows that $\mathsf{SW}(\mathcal{X}) = \langle G \rangle_n$.

(1) $\Longrightarrow$ (2): We use induction on n. When $n = 1$, we observe that X is a direct summand in $\mathsf{SW}(\mathcal{X})$ of $C := (G[a_1])^{\oplus b_1} \oplus \cdots \oplus (G[a_r])^{\oplus b_r}$ for some $a_1, \dots, a_r \in \mathbb{Z}$ and $b_1, \dots, b_r \geqslant 1$. Put $a = \max\{a_1, \dots, a_r\}$ and $D = (\Omega^{a-a_1} G)^{\oplus b_1} \oplus \cdots \oplus (\Omega^{a-a_r} G)^{\oplus b_r}$. Then X is a direct summand of $D[a]$, and D belongs to $[G]$. Lemma 5.2(3) implies that $\Omega^s X \in [G]$ for some $s \geqslant 0$.

Now let us consider the case $n \geqslant 2$. There exists an exact triangle $A[a] \to B[b] \to C[c] \rightsquigarrow$ in $\mathsf{SW}(\mathcal{X})$ with $A, B, C \in \mathcal{X}$, $a, b, c \in \mathbb{Z}$, $A[a] \in \langle G \rangle_{n-1}$ and $C[c] \in \langle G \rangle$ such that X is a direct summand of $B[b]$ in $\mathsf{SW}(\mathcal{X})$. By Lemma 5.2(4) there are an integer $d \geqslant \max\{a, b, c\}$ and an exact sequence $0 \to \Omega^{d-a} A \to \Omega^{d-b} B \to \Omega^{d-c} C \to 0$ in $\mathcal{A}$ up to projective summands. We have $A \in \langle G \rangle_{n-1}$ and $C \in \langle G \rangle$, and the induction hypothesis and basis imply that $\Omega^p A \in [G]_{n-1}$ and $\Omega^q C \in [G]$ for some $p, q \geqslant 0$, while $\Omega^r X$ is a direct summand in $\underline{\mathcal{A}}$ of $\Omega^{r-b} B$ for some $r \geqslant b$ by Lemma 5.2(3). Putting $s = \max\{p, q, r\}$, we obtain an exact sequence

$$0 \to \Omega^{s+d-a} A \to \Omega^{s+d-b} B \to \Omega^{s+d-c} C \to 0$$

in $\mathcal{A}$ (up to projective summand) with $\Omega^{s+d-a} A \in [G]_{n-1}$ and $\Omega^{s+d-c} C \in [G]$ such that $\Omega^{s+d} X$ is a direct summand of $\Omega^{s+d-b} B$. Therefore $\Omega^{s+d} X$ belongs to $[G]_n$. ■

Now we can prove the following theorem. This result gives rise to a method to compute the Rouquier dimension of the Spanier–Whitehead category of each resolving subcategory $\mathcal{X}$ in terms of generation in $\mathcal{X}$, which yields a relationship between Rouquier and Dao–Takahashi dimensions.

Theorem 5.4. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{A}$. Then there is an equality*

$$\dim \mathsf{SW}(\mathcal{X}) = \inf\{n \geqslant 0 \mid \exists\, G \in \mathcal{X} \text{ such that } \forall X \in \mathcal{X}\ \exists\, k \geqslant 0 \text{ with } \Omega^k X \in [G]_{n+1}\}.$$

In particular, one has an inequality

$$\dim \mathsf{SW}(\mathcal{X}) \leqslant \dim \mathcal{X}. \tag{5.4.1}$$

Proof. We only give a proof of the first assertion; the second assertion immediately follows from it. Put s, t to be the left and right sides of the equality in the theorem (note that $s, t \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$). Suppose that s is finite. Then $\mathsf{SW}(\mathcal{X}) = \langle C \rangle_{s+1}$ for some $C \in \mathsf{SW}(\mathcal{X})$. Writing $C = G[c]$ with $G \in \mathcal{X}$ and $c \in \mathbb{Z}$, we have $\mathsf{SW}(\mathcal{X}) = \langle G \rangle_{s+1}$. Proposition 5.3 implies that for each $X \in \mathcal{X}$ there exists an integer $k \geqslant 0$ such that $\Omega^k X \in [G]_{s+1}$. This shows that $s \geqslant t$. Conversely, assume that t is finite. Then there exists an object $G \in \mathcal{X}$ such that for any $X \in \mathcal{X}$ there is $k \geqslant 0$ with $\Omega^k X \in [G]_{t+1}$. We have $\mathsf{SW}(\mathcal{X}) = \langle G \rangle_{t+1}$ by Proposition 5.3, which shows $s \leqslant t$. Now we conclude $s = t$, and the first assertion follows. ■

The equality in (5.4.1) is sometimes strict. Here is an example.

Example 5.5. Let R be an equicharacteristic excellent local ring of Krull dimension d. By virtue of [14, Theorem 5.3], the derived category $\mathsf{D}^{\mathsf{b}}(R)$ has finite dimension, and so does the singularity category $\mathsf{D}_{\mathsf{sg}}(R)$. As $\mathsf{D}_{\mathsf{sg}}(R)$ and $\mathsf{SW}(\operatorname{mod} R)$ are triangle equivalent by Theorem 3.2(3), $\mathsf{SW}(\operatorname{mod} R)$ has finite dimension, as well. On the other hand, it follows from [9, Theorem 4.4] that $\dim(\operatorname{mod} R) = \infty$. Thus

$$\dim \mathsf{SW}(\operatorname{mod} R) < \dim(\operatorname{mod} R).$$

As an application of Theorems 5.4 and 3.2(3) we obtain the following result.

Corollary 5.6. *Assume that one of the following conditions holds.*

- *R is a Cohen–Macaulay ring and $\mathcal{X} = \mathsf{CM}(R)$.*
- *R is an isolated singularity and $\mathcal{X} = \operatorname{mod}_0 R$.*

Then one has $\dim \mathsf{D}_{\mathsf{sg}}(R) = \inf\{n \geqslant 0 \mid \exists\, G \in \mathcal{X} \text{ such that } \forall X \in \mathcal{X}\ \exists\, k \geqslant 0 \text{ with } \Omega^k X \in [G]_{n+1}\}$.

To give our next result, we introduce some notation.

Definition 5.7. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(1) We define the *cohomology annihilator* of $\mathcal{X}$ by

$$\operatorname{ca}(\mathcal{X}) = \bigcap_{X \in \mathcal{X}} \bigcup_{i>0} \operatorname{ann}_R \mathsf{Ext}^i_R(X, -) = \bigcap_{X \in \mathcal{X}} \bigcup_{i>0} \bigcap_{M \in \operatorname{mod} R} \operatorname{ann}_R \mathsf{Ext}^i_R(X, M),$$

which is an ideal of R.

(2) The *infinite projective dimension locus* of $\mathcal{X}$, denoted by $\operatorname{IPD}(\mathcal{X})$, is by definition the set of prime ideals $\mathfrak{p}$ of R such that $X_\mathfrak{p}$ has infinite projective dimension over $R_\mathfrak{p}$ for some $X \in \mathcal{X}$. Note that $\operatorname{IPD}(\mathcal{X})$ is a specialization-closed subset of $\operatorname{Spec} R$, and contained in the singular locus $\operatorname{Sing} R$ of R.

Recall that for a specialization-closed subset W of $\operatorname{Spec} R$ there is an equality

$$\sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in W\} = \sup\{n \geqslant 0 \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } W\},$$

and this value is defined as the *(Krull) dimension* of W, which is denoted by $\dim W$. For an ideal I of R, we denote by $\mu(I)$ the minimal number of generators of I. For an artinian ring A, we denote by $\ell\ell(A)$ the *Loewy length* of A, that is, the minimum of integers $n \geqslant 0$ such that $(\operatorname{rad} A)^n = 0$.

Now we can state another application of Theorem 5.4.

Corollary 5.8. *Let R be a commutative noetherian ring. Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$.

(1) *If* $\dim \mathsf{SW}(\mathcal{X}) < \infty$, *then* $\dim R/\operatorname{ca}(\mathcal{X}) \leqslant \dim \operatorname{IPD}(\mathcal{X})$, *and hence* $\dim R/\operatorname{ca}(\mathcal{X}) \leqslant \dim \operatorname{Sing} R$.

(2) *Suppose that the ring $A := R/\operatorname{ca}(\mathcal{X})$ is artinian and the module $A/\operatorname{rad} A$ belongs to $\mathcal{X}$. Then*

$$\dim \mathsf{SW}(\mathcal{X}) < \ell\ell(R/\operatorname{ca}(\mathcal{X})) \cdot (\mu(\operatorname{ca}(\mathcal{X})) + 1) < \infty.$$

Proof. (1) There is nothing to show when $\dim \mathrm{IPD}(\mathcal{X}) = \infty$, so assume $\dim \mathrm{IPD}(\mathcal{X}) =: t < \infty$. Put $\dim \mathsf{SW}(\mathcal{X}) =: n < \infty$. Theorem 5.4 implies that there exists $G \in \mathcal{X}$ such that for all $X \in \mathcal{X}$ there is $k \geqslant 0$ with $\Omega^k X \in [G]_{n+1}$. Put $r := \mathrm{Rfd}_R\, G < \infty$; see (3.4.1). Setting $I = \mathrm{ann}_R \mathsf{Ext}_R^{r+1}(G, \Omega^{r+1} G)$, we have $I \mathsf{Ext}_R^{>r}(G, -) = 0$ by [14, Lemma 2.14]. Induction on n shows $I^{n+1} \mathsf{Ext}_R^{>r}(C, -) = 0$ for all $C \in [G]_{n+1}$, and hence $I^{n+1} \mathsf{Ext}_R^{>(r+k)}(X, -) = 0$. Therefore I^{n+1} is contained in $\mathrm{ca}(\mathcal{X})$, and $\dim R/\mathrm{ca}(\mathcal{X}) \leqslant \dim R/I^{n+1} = \dim R/I$. Note that $\dim \mathrm{IPD}(G) \leqslant t$. Take any prime ideal $\mathfrak{p}$ of R such that $\dim R/\mathfrak{p} > t$. Then the $R_\mathfrak{p}$-module $G_\mathfrak{p}$ has finite projective dimension, and it is at most r by the Auslander–Buchsbaum formula. Hence $\mathfrak{p}$ does not contain I. This shows that $\dim R/I \leqslant t$, and we are done.

(2) Choose a system of generators $\boldsymbol{x} = x_1, \dots, x_n$ of $\mathrm{ca}(\mathcal{X})$ with $n = \mu(\mathrm{ca}(\mathcal{X}))$. Fix $1 \leqslant k \leqslant n$. For each $X \in \mathcal{X}$ there is $i_k > 0$ such that $x_k \mathsf{Ext}_R^{i_k}(X, -) = 0$, and $x_k \mathsf{Ext}_R^{\geqslant i_k}(X, -) = 0$ by [14, Lemma 2.14]. Putting $t = \max_{1 \leqslant k \leqslant n} i_k$ and $Y = \Omega^{t-1} X$, we have $x_k \mathsf{Ext}_R^{>0}(Y, -) = 0$. It follows from [21, Corollary 3.2(2)] that Y belongs to $[\bigoplus_{j=0}^n \mathsf{H}_j(\boldsymbol{x}, Y)]_{n+1}$ in $\mathsf{mod}\, R$, where $\mathsf{H}_j(\boldsymbol{x}, Y)$ stands for the Koszul homology. Each $\mathsf{H}_j(\boldsymbol{x}, Y)$ is a module over the artinian ring $A = R/(\boldsymbol{x})$, so it belongs to $[G]_h$ in $\mathsf{mod}\, R$, where $G := A/\mathrm{rad}\, A$ and $h := \ell\ell(A)$. Therefore $Y \in [G]_{(n+1)h}$. Applying Theorem 5.4, we are done. ■

Corollaries 5.8 and 3.7 immediately yield the following result.

Corollary 5.9. *The ring $R/\mathrm{ca}(\mathsf{mod}_0 R)$ is artinian if and only if $\mathsf{SW}(\mathsf{mod}_0 R)$ has finite dimension. When R is an isolated singularity, $R/\mathrm{ca}(\mathsf{mod}_0 R)$ is artinian if and only if $\mathsf{D_{sg}}(R)$ has finite dimension.*

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