MODULES IN RESOLVING SUBCATEGORIES WHICH ARE FREE ON THE PUNCTURED SPECTRUM

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Abstract. Let $R$ be a commutative noetherian local ring, and let $\mathcal{X}$ be a resolving subcategory of the category of finitely generated $R$-modules. In this paper, we study modules in $\mathcal{X}$ by relating them to modules in $\mathcal{X}$ which are free on the punctured spectrum of $R$. We do this by investigating nonfree loci and establishing an analogue of the notion of a level in a triangulated category which has been introduced by Avramov, Buchweitz, Iyengar and Miller. As an application, we prove a result on the dimension of the nonfree locus of a resolving subcategory having only countably many nonisomorphic indecomposable modules in it, which is a generalization of a theorem of Huneke and Leuschke.

1. Introduction

In the 1960s, Auslander and Bridger [3] introduced the notion of a resolving subcategory of an abelian category with enough projectives. They proved that in the category of finitely generated modules over a left and right noetherian ring, the full subcategory consisting of all modules of Gorenstein dimension zero, which are now also called totally reflexive modules, is resolving.

Let $R$ be a commutative noetherian ring, and let $\text{mod}\, R$ denote the category of finitely generated $R$-modules. A lot of important full subcategories of $\text{mod}\, R$ are known to be resolving. As trivial examples, $\text{mod}\, R$ itself and the full subcategory $\text{proj}\, R$ of $\text{mod}\, R$ consisting of all projective modules are resolving. If $R$ is a Cohen-Macaulay local ring, then the full subcategory $\text{CM}(R)$ of $\text{mod}\, R$ consisting of all maximal Cohen-Macaulay $R$-modules is resolving; see Example 2.4 for details and other examples of a resolving subcategory.

Let $R$ be a local ring, and let $\mathcal{X}$ be a resolving subcategory of $\text{mod}\, R$. In the present paper, we study modules in $\mathcal{X}$ by relating them to modules in $\mathcal{X}$ which are free on the punctured spectrum of $R$. A key role is played by the nonfree loci of $R$-modules and subcategories of $\text{mod}\, R$, which are certain closed and specialization-closed subsets of $\text{Spec}\, R$, respectively.

To be more precise, from each module $X \in \mathcal{X}$ we construct another module $X' \in \mathcal{X}$ which is free on the punctured spectrum of $R$, and count the (minimum) number of steps required to construct $X'$ from $X$. We denote the number by $\text{step}(X, X')$. (The precise definition will be given in Definition 5.1.) This invariant is an analogue of a level in a

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triangulated category which has been introduced by Avramov, Buchweitz, Iyengar and Miller [6].

We denote by $NF(X)$ and $NF(X')$ the nonfree loci of an $R$-module $X$ and a subcategory $X$ of $\text{mod} R$, respectively (cf. Definition 2.7). The main result of this paper is the following, which will be proved in Corollary 5.6.

**Theorem A.** Let $R$ be a commutative noetherian local ring, and let $X$ be a resolving subcategory of $\text{mod} R$. Then for every nonfree $R$-module $X \in X$, there exists a nonfree $R$-module $X' \in X$ satisfying the following two conditions:

1. $\text{step}(X, X') \leq 2 \dim NF(X)$, and
2. $X'$ is free on the punctured spectrum of $R$.

As an application, we consider how many nonisomorphic indecomposable modules are in $X$. We will prove the following result in Corollary 6.9.

**Theorem B.** Let $R$ be a commutative noetherian local ring which is either complete or has uncountable residue field. Let $X$ be a resolving subcategory of $\text{mod} R$ in which there are only countably many nonisomorphic indecomposable $R$-modules. Then $\dim NF(X) \leq 1$.


**Convention.** Throughout this paper, let $R$ be a commutative noetherian ring. All $R$-modules considered in this paper are assumed to be finitely generated. We denote by $\text{mod} R$ the category of finitely generated $R$-modules. By a subcategory of $\text{mod} R$, we always mean a full subcategory of $\text{mod} R$ which is closed under isomorphisms. We freely use basic definitions and results in commutative algebra which are stated in [7].

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**2. Foundations**

In this section, we define the resolving closures and the nonfree loci of an $R$-module and a subcategory of $\text{mod} R$, and study their basic properties. We begin with recalling the definition of a syzygy.

**Definition 2.1.** (1) Let $n$ be a nonnegative integer, and let $M$ be an $R$-module. If there exists an exact sequence

$$0 \rightarrow N \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

of $R$-modules where $P_i$ is a projective $R$-module for every $0 \leq i \leq n - 1$, then we call $N$ the $n$th syzygy of $M$, and denote it by $\Omega^n M$. Note that the $n$th syzygy of a given $R$-module is not uniquely determined; it is uniquely determined up to projective summand. We simply write $\Omega^1 M = \Omega M$. 

(2) In the case where $R$ is local, the $R$-module $M$ admits a minimal free resolution

$$
\cdots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \rightarrow 0.
$$

Then we define the $n$th syzygy of $M$ as the image of $\partial_n$, and denote it by $\Omega^n M$. The $n$th syzygy of a given $R$-module is uniquely determined up to isomorphism since so is a minimal free resolution. Whenever $R$ is local, we define the $n$th syzygy of an $R$-module by using its minimal free resolution.

Next let us recall the definition of a resolving subcategory.

**Definition 2.2.** A subcategory $\mathcal{X}$ of mod $R$ is called resolving if $\mathcal{X}$ satisfies the following conditions.

1. $\mathcal{X}$ contains all projective $R$-modules.
2. $\mathcal{X}$ is closed under direct summands: if $M$ is in $\mathcal{X}$ and $N$ is a direct summand of $M$, then $N$ is also in $\mathcal{X}$.
3. $\mathcal{X}$ is closed under extensions: for any exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ in mod $R$, if $L$ and $N$ are in $\mathcal{X}$, then so is $M$.
4. $\mathcal{X}$ is closed under kernels of epimorphisms: for any exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ in mod $R$, if $M$ and $N$ are in $\mathcal{X}$, then so is $L$.

A resolving subcategory is a subcategory such that any two “minimal” resolutions of a module by modules in it have the same length; see [3, Lemma (3.12)].

The closedness under kernels of epimorphisms can be replaced with a weaker condition of the closedness under syzygies.

**Remark 2.3.** [15, Lemma 3.2] A subcategory $\mathcal{X}$ of mod $R$ is resolving if and only if $\mathcal{X}$ satisfies the following conditions.

1. $\mathcal{X}$ contains all projective $R$-modules.
2. $\mathcal{X}$ is closed under direct summands.
3. $\mathcal{X}$ is closed under extensions.
4. $\mathcal{X}$ is closed under syzygies: if $M$ is in $\mathcal{X}$, then so is $\Omega M$.

A lot of important subcategories of mod $R$ are known to be resolving. Here, let us make a list of examples.

**Example 2.4.** (1) It is trivial that the subcategory mod $R$ of mod $R$ is resolving.

(2) It is obvious that the subcategory proj $R$ of mod $R$ consisting of all projective $R$-modules is resolving.

(3) Let $I$ be an ideal of $R$. Then the subcategory of mod $R$ consisting of all $R$-modules $M$ with $\text{grade}(I, M) \geq \text{grade}(I, R)$ is resolving. This can be shown by using the equality $\text{grade}(I, M) = \inf\{i \in \mathbb{Z} \mid \text{Ext}^i_R(R/I, M) \neq 0\}$.

(4) Let $R$ be a Cohen-Macaulay local ring. Then, letting $I$ be the maximal ideal of $R$ in (3), we see that the subcategory CM($R$) of mod $R$ consisting of all maximal Cohen-Macaulay $R$-modules is resolving.

(5) An $R$-module $C$ is called semidualizing if the natural homomorphism $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}^i_R(C, C) = 0$ for every $i > 0$. An $R$-module $M$ is called totally $C$-reflexive, where $C$ is a semidualizing $R$-module, if
the natural homomorphism $M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism and $\text{Ext}^i_R(M, C) = \text{Ext}^i_R(\text{Hom}_R(M, C), C) = 0$ for every $i > 0$. The subcategory $\mathcal{G}(R)$ of mod $R$ consisting of all totally $C$-reflexive $R$-modules is resolving by [1, Theorem 2.1].

(6) A totally $R$-reflexive $R$-module is simply called totally reflexive. The subcategory $\mathcal{G}(R)$ of mod $R$ consisting of all totally reflexive $R$-modules is resolving by (5); see also [3, (3.11)].

(7) Let $n$ be a nonnegative integer, and let $K$ be an $R$-module (which is not necessarily finitely generated). Then the subcategory of mod $R$ consisting of all $R$-modules $M$ with $\text{Tor}^i_R(M, K) = 0$ for $i > n$ (respectively, $i \gg 0$) and the subcategory of mod $R$ consisting of all $R$-modules $M$ with $\text{Ext}^i_R(M, K) = 0$ for $i > n$ (respectively, $i \gg 0$) are both resolving.

(8) Let $R$ be a local ring. We say that an $R$-module $M$ is bounded if there is an integer $s$ such that $\beta^B_i(M) \leq s$ for all $i \geq 0$, where $\beta^B_i(M)$ denotes the $i$th Betti number of $M$. The subcategory of mod $R$ consisting of all bounded $R$-modules is resolving. This can be shown by using the equality $\beta^B_i(M) = \dim_k \text{Tor}^i_R(M, k)$, where $k$ is the residue field of $R$.

(9) Let $R$ be local. We say that an $R$-module $M$ has complexity $c$ if $c$ is the least nonnegative integer $d$ such that there exists a real number $r$ satisfying the inequality $\beta^B_i(M) \leq ri^{d-1}$ for $i \gg 0$. The subcategory of mod $R$ consisting of all $R$-modules having finite complexity is resolving by [4, Proposition 4.2.4].

(10) Let $R$ be local. We say that an $R$-module $M$ has lower complete intersection zero if $M$ is totally reflexive and has finite complexity. The subcategory of mod $R$ consisting of all $R$-modules of lower complete intersection dimension zero is resolving by (6) and (9); see also [5, Lemma 6.3.1].

Now we define the resolving closures of a subcategory of mod $R$ and an $R$-module.

**Definition 2.5.** For a subcategory $\mathcal{X}$ of mod $R$, we denote by $\text{res} \mathcal{X}$ (or $\text{res}_R \mathcal{X}$ when there is some fear of confusion) the resolving subcategory of mod $R$ generated by $\mathcal{X}$, namely, the smallest resolving subcategory of mod $R$ containing $\mathcal{X}$. If $\mathcal{X}$ consists of a single module $X$, then we simply write $\text{res} X$ (or $\text{res}_R X$).

**Remark 2.6.** (1) Let $\{\mathcal{X}_\lambda\}_{\lambda \in \Lambda}$ be a family of resolving subcategories of mod $R$. Then the intersection $\bigcap_{\lambda \in \Lambda} \mathcal{X}_\lambda$ is also a resolving subcategory of mod $R$. Therefore, for every subcategory $\mathcal{X}$ of mod $R$, the smallest resolving subcategory of mod $R$ containing $\mathcal{X}$ exists.

(2) Let $\mathcal{X}, \mathcal{Y}$ be subcategories of mod $R$. If $\mathcal{X} \subseteq \mathcal{Y}$, then $\text{res} \mathcal{X} \subseteq \text{res} \mathcal{Y}$.

(3) A subcategory $\mathcal{X}$ of mod $R$ is resolving if and only if $\mathcal{X} = \text{res} \mathcal{X}$. In particular, $\text{res} \mathcal{X} = \text{res}(\text{res} \mathcal{X})$ for every subcategory $\mathcal{X}$ of mod $R$.

Next we recall the definition of the nonfree locus of an $R$-module and define the nonfree locus of a subcategory of mod $R$.

**Definition 2.7.** (1) We denote by $\text{NF}(X)$ (or $\text{NF}_R(X)$) the nonfree locus of an $R$-module $X$, namely, the set of prime ideals $p$ of $R$ such that the $R_p$-module $X_p$ is nonfree.
We define the nonfree locus of a subcategory \( X \) of \( \text{mod}\, R \) as the union of \( \text{NF}(X) \) where \( X \) runs through all (nonisomorphic) \( R \)-modules in \( X \), and denote it by \( \text{NF}(X) \) (or \( \text{NF}_R(X) \)).

**Remark 2.8.** (1) For a subcategory \( X \) of \( \text{mod}\, R \), one has \( \text{NF}(X) = \emptyset \) if and only if \( X \) is contained in \( \text{proj}\, R \). In particular, one has \( \text{NF}(X) = \emptyset \) for an \( R \)-module \( X \) if and only if \( X \) is projective.

(2) Let \( R \) be a local ring with maximal ideal \( \mathfrak{m} \). Then an \( R \)-module \( X \) is nonfree if and only if \( \mathfrak{m} \) is in \( \text{NF}(X) \).

(3) Let \( X, Y \) be subcategories of \( \text{mod}\, R \). If \( X \subseteq Y \), then \( \text{NF}(X) \subseteq \text{NF}(Y) \).

**Example 2.9.** Let \( R \) be a Cohen-Macaulay local ring. Then the nonfree locus \( \text{NF}(\text{CM}(R)) \) coincides with the singular locus \( \text{Sing}\, R \) of \( R \).

In fact, for every prime ideal \( \mathfrak{p} \) in \( \text{NF}(\text{CM}(R)) \) there exists a maximal Cohen-Macaulay \( R \)-module \( X \) such that the \( R_\mathfrak{p} \)-module \( X_\mathfrak{p} \) is not free. Hence \( X_\mathfrak{p} \) is a nonfree maximal Cohen-Macaulay \( R_\mathfrak{p} \)-module, which implies that the local ring \( R_\mathfrak{p} \) is singular. On the other hand, each prime ideal \( \mathfrak{p} \) in \( \text{Sing}\, R \) belongs to the nonfree locus of the maximal Cohen-Macaulay \( R \)-module \( \Omega^d(R/\mathfrak{p}) \), where \( d = \dim R \).

The nonfree locus of a module can be described as the support of an Ext module.

**Proposition 2.10.** Let \( \sigma : 0 \to Y \to P \to X \to 0 \) be an exact sequence of \( R \)-modules such that \( P \) is projective. Then one has \( \text{NF}(X) = \text{Supp} \text{Ext}^1_R(X, Y) \). Hence \( \text{NF}(X) = \text{Supp} \text{Ext}^1_R(X, \Omega X) \).

**Proof.** For a prime ideal \( \mathfrak{p} \) in \( \text{Supp} \text{Ext}^1_R(X, Y) \), the module \( \text{Ext}^1_R(X_\mathfrak{p}, Y_\mathfrak{p}) \) is nonzero. In particular, \( X_\mathfrak{p} \) is a nonfree \( R_\mathfrak{p} \)-module, and hence \( \mathfrak{p} \) is in \( \text{NF}(X) \). Conversely, let \( \mathfrak{p} \) be a prime ideal in \( \text{NF}(X) \). Localizing \( \sigma \) at \( \mathfrak{p} \), we obtain an exact sequence \( \sigma_\mathfrak{p} : 0 \to Y_\mathfrak{p} \to P_\mathfrak{p} \to X_\mathfrak{p} \to 0 \) of \( R_\mathfrak{p} \)-modules. Since \( X_\mathfrak{p} \) is not free, this exact sequence \( \sigma_\mathfrak{p} \) does not split, hence this defines a nonzero element of the module \( \text{Ext}^1_{R_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) \). Thus \( \text{Ext}^1_R(X, Y)_\mathfrak{p} \) is nonzero, that is, \( \mathfrak{p} \) is in \( \text{Supp} \text{Ext}^1_R(X, Y) \).

Recall that a subset \( Z \) of \( \text{Spec}\, R \) is called specialization-closed provided that if \( \mathfrak{p} \in Z \) and \( \mathfrak{q} \in \text{Spec}\, R \) with \( \mathfrak{p} \subseteq \mathfrak{q} \) then \( \mathfrak{q} \in Z \). Note that every closed subset of \( \text{Spec}\, R \) is specialization-closed.

**Corollary 2.11.** (1) The nonfree locus of an \( R \)-module is a closed subset of \( \text{Spec}\, R \) in the Zariski topology.

(2) The nonfree locus of a subcategory of \( \text{mod}\, R \) is specialization-closed.

**Proof.** (1) It is seen from Proposition 2.10 that \( \text{NF}(X) = \text{Supp} \text{Ext}^1_R(X, \Omega X) \) for an \( R \)-module \( X \). As \( \text{Ext}^1_R(X, \Omega X) \) is a finitely generated \( R \)-module, the subset \( \text{NF}(X) \) of \( \text{Spec}\, R \) is closed.

(2) It is easy to see that in general any union of closed subsets of \( \text{Spec}\, R \) is specialization-closed. Hence this statement follows from (1).

Let \( Z \) be a closed subset of \( \text{Spec}\, R \). Then \( Z = V(I) \) for some ideal \( I \) of \( R \). We call such an ideal \( I \) the defining ideal of \( Z \). This is uniquely determined up to radical.
3. INDUCTIVE CONSTRUCTION OF RESOLVING CLOSURES

In this section, we build a filtration of subcategories in the resolving closure of a subcategory of mod R, and inductively construct the resolving closure. This is an imitation of the notion of thickenings in the thick closure of a subcategory of a triangulated category, which were introduced by Avramov, Buchweitz, Iyengar and Miller [6]. Using this construction of a resolving closure, we can obtain several properties of a resolving closure and its nonfree locus.

The additive closure \( \text{add} \mathcal{X} \) (or \( \text{add}_R \mathcal{X} \)) of a subcategory \( \mathcal{X} \) of mod \( R \) is defined to be the subcategory of mod \( R \) consisting of all direct summands of finite direct sums of modules in \( \mathcal{X} \). Note that \( \text{add} \mathcal{X} \) is closed under direct summands and finite direct sums, namely, \( R \)-modules \( M \) and \( N \) both belong to \( \text{add} \mathcal{X} \) if and only if so does \( M \oplus N \).

**Definition 3.1.** Let \( \mathcal{X} \) be a subcategory of mod \( R \). For a nonnegative integer \( n \), we inductively define a subcategory \( \text{res}^n \mathcal{X} \) (or \( \text{res}^n_R \mathcal{X} \)) of mod \( R \) as follows:

1. Set \( \text{res}^0 \mathcal{X} = \text{add} (\mathcal{X} \cup \{ R \}) \).
2. For \( n \geq 1 \), let \( \text{res}^n \mathcal{X} \) be the additive closure of the subcategory of mod \( R \) consisting of all \( R \)-modules \( Y \) having an exact sequence of either of the following two forms:
   
   \[
   0 \rightarrow A \rightarrow Y \rightarrow B \rightarrow 0,
   \]
   
   \[
   0 \rightarrow Y \rightarrow A \rightarrow B \rightarrow 0
   \]

   where \( A, B \in \text{res}^{n-1} \mathcal{X} \).

If \( \mathcal{X} \) consists of a single module \( X \), then we simply write \( \text{res}^n X \) instead of \( \text{res}^n \mathcal{X} \).

**Remark 3.2.** Let \( \mathcal{X}, \mathcal{Y} \) be subcategories of mod \( R \), and let \( n \) be a nonnegative integer. Then the following hold.

1. If \( \mathcal{X} \subseteq \mathcal{Y} \), then \( \text{res}^n \mathcal{X} \subseteq \text{res}^n \mathcal{Y} \).
2. One has equalities \( \text{res}^n (\text{add} \mathcal{X}) = \text{res}^n \mathcal{X} = \text{add}(\text{res}^n \mathcal{X}) \).
3. There is an ascending chain \( \{ 0 \} \subseteq \text{res}^0 \mathcal{X} \subseteq \text{res}^1 \mathcal{X} \subseteq \cdots \subseteq \text{res}^n \mathcal{X} \subseteq \cdots \subseteq \text{res} \mathcal{X} \) of subcategories of mod \( R \).
4. The equality \( \text{res} \mathcal{X} = \bigcup_{n \geq 0} \text{res}^n \mathcal{X} \) holds.

   The first and second statements follow by definition and induction on \( n \). As to the third statement, since \( \text{res}^n \mathcal{X} \) is closed under direct summands, it contains the zero module 0. For an \( R \)-module \( M \) in \( \text{res}^n \mathcal{X} \) there exists a short exact sequence \( 0 \rightarrow M \rightarrow 0 \), which shows that \( M \) is in \( \text{res}^{n+1} \mathcal{X} \) by definition. As for the fourth statement, it is easy to see by definition that \( \mathcal{X} \subseteq \bigcup_{n \geq 0} \text{res}^n \mathcal{X} \subseteq \mathcal{X} \). It remains to show that \( \bigcup_{n \geq 0} \text{res}^n \mathcal{X} \) is a resolving subcategory of mod \( R \). But this is also easy to check.

From its definition, we might think that there are not so many nonisomorphic indecomposable modules in \( \text{res}^n \mathcal{X} \). But, the following two examples say that this guess is not right.

**Example 3.3.** Let us consider the 1-dimensional complete local hypersurface \( R = \mathbb{C}[[x, y]]/(x^2) \) over the complex number field. Then the subcategory \( \text{res}^1 (xR) \) coincides with CM(\( R \)), and there exist infinitely many nonisomorphic indecomposable \( R \)-modules in \( \text{res}^1 (xR) \).
Indeed, set

\[
I_n = \begin{cases} 
R & (n = 0), \\
(x, y^n)R & (0 < n < \infty), \\
xR & (n = \infty).
\end{cases}
\]

It follows from [14, Example 6.5] that the set \(\{I_n\}_{0 \leq n \leq \infty}\) consists of all the nonisomorphic indecomposable maximal Cohen-Macaulay \(R\)-modules. For each integer \(n\) with \(0 < n < \infty\), we have isomorphisms

\[
((x, y^n)R)/xR \xrightarrow{f} R/xR \xrightarrow{g} xR,
\]

where \(f\) sends the residue class of \(a \in R\) in \(R/xR\) to the residue class of \(y^n a\) in \(((x, y^n)R)/xR\), and \(g\) sends the residue class of \(a \in R\) in \(R/xR\) to \(xa \in xR\). Hence we see that there is an exact sequence

\[
0 \to xR \to I_n \to xR \to 0
\]

for \(0 \leq n < \infty\), which implies that \(I_n\) is in \(\text{res}^1(xR)\) for \(0 \leq n \leq \infty\). On the other hand, \(xR\) is a maximal Cohen-Macaulay \(R\)-module, and \(\text{CM}(R)\) is a resolving subcategory of \(\text{mod} R\) by Example 2.4(4). Therefore \(\text{CM}(R)\) coincides with \(\text{res}^1(xR)\).

The localization \(R_p\) at the prime ideal \(p = xR\) is singular because it has a (nonzero) nilpotent \(x\). Hence the local ring \(R\) is not an isolated singularity, and thus there exist infinitely many isomorphism classes of indecomposable maximal Cohen-Macaulay \(R\)-modules by [2, §10] or [9, Corollary 2].

For a subcategory \(\mathcal{X}\) of \(\text{mod} R\), we denote by \(\text{ind} \mathcal{X}\) (or \(\text{ind}_R \mathcal{X}\)) the set of nonisomorphic indecomposable \(R\)-modules in \(\mathcal{X}\).

**Example 3.4.** Let \(k\) be a field. We consider the 2-dimensional hypersurface \(R = k[[x, y, z]]/(x^2)\). Put \(p(f) = (x, y - zf)R\) for an element \(f \in k[[z]] \subseteq R\). Then \(p(f)\) is an indecomposable \(R\)-module in \(\text{res}^1(xR)\), and there exist uncountably many nonisomorphic indecomposable \(R\)-modules in \(\text{res}^1(xR)\).

Indeed, note that \(xR\) is a maximal Cohen-Macaulay \(R\)-module. Hence the \(R\)-regular element \(y - zf\) is also \(xR\)-regular. Note also that \(p(f)\) is isomorphic to \(\Omega(xR)/(y - zf)xR)\). We can make the following pullback diagram:

\[
\begin{array}{cccccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
p(f) & \overline{=} & p(f) \\
\downarrow & & \downarrow \\
0 & \to & xR & \to & E & \to & R & \to & 0 \\
\| & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & xR & \xrightarrow{y-zf} & xR & \to & xR/(y - zf)xR & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & & & & & & 0
\end{array}
\]
Since the middle row splits, \( E \) is isomorphic to \( xR \oplus R \). We get an exact sequence
\[
0 \to p(f) \to xR \oplus R \to xR \to 0.
\]
The \( R \)-modules \( xR \) and \( xR \oplus R \) belong to \( \text{res}^0(xR) \), hence \( p(f) \) belongs to \( \text{res}^1(xR) \).

A similar argument to the proof of Claim 1 in [12, Example 4.3] shows that \( p(f) \) is an indecomposable \( R \)-module. Thus, we obtain a map from \( k[[z]] \) to \( \text{ind}(\text{res}^1(xR)) \) which is given by \( f \mapsto p(f) \). Along the same lines as in the proofs of Claims 2 and 3 in [12, Example 4.3], we can prove that this map is injective. Since the set \( k[[z]] \) is uncountably infinite, the assertion follows.

For a subcategory \( \mathcal{X} \) of \( \text{mod} \, R \) and a multiplicatively closed subset \( S \) of \( R \), we denote by \( \mathcal{X}_S \) the subcategory of \( \text{mod} \, R_S \) consisting of all \( R_S \)-modules \( X_S \) with \( X \in \mathcal{X} \). Our inductive construction of a resolving closure yields a relationship between a resolving closure and localization.

**Proposition 3.5.** Let \( \mathcal{X} \) be a subcategory of \( \text{mod} \, R \), and let \( S \) be a multiplicatively closed subset of \( R \). Then \( (\text{res}_R \mathcal{X})_S \) is contained in \( \text{res}_R \mathcal{X}_S \).

**Proof.** It is enough to show that \( (\text{res}_R^n \mathcal{X})_S \) is contained in \( \text{res}_R^n \mathcal{X}_S \) for each integer \( n \geq 0 \). We use induction on \( n \). The statement obviously holds when \( n = 0 \). Let \( n \geq 1 \), and take an \( R \)-module \( M \) in \( \text{res}_R^n \mathcal{X} \). Then there are a finite number of \( R \)-modules \( M_1, \ldots, M_t \) such that \( M \) is a direct summand of \( M_1 \oplus \cdots \oplus M_t \) and that for each \( 1 \leq i \leq t \) there exists an exact sequence of either of the following two forms:
\[
0 \to A_i \to M_i \to B_i \to 0,
\]
\[
0 \to M_i \to A_i \to B_i \to 0
\]
where \( A_i \) and \( B_i \) are in \( \text{res}_R^{n-1} \mathcal{X} \). Hence for each \( 1 \leq i \leq t \) there is an exact sequence of either of the following two forms:
\[
0 \to (A_i)_S \to (M_i)_S \to (B_i)_S \to 0,
\]
\[
0 \to (M_i)_S \to (A_i)_S \to (B_i)_S \to 0
\]
where \( (A_i)_S \) and \( (B_i)_S \) are in \( (\text{res}_R^{n-1} \mathcal{X})_S \). Induction hypothesis implies that \( (\text{res}_R^{n-1} \mathcal{X})_S \) is contained in \( \text{res}_R^{n-1} \mathcal{X}_S \). Since \( M_S \) is a direct summand of \( (M_1)_S \oplus \cdots \oplus (M_t)_S \), the \( R_S \)-module \( M_S \) belongs to \( \text{res}_R^n \mathcal{X}_S \).

Making use of the above result, we see that the nonfree locus of a subcategory is stable under taking its resolving closure.

**Corollary 3.6.** The equalities
\[
\text{NF}(\text{res} \mathcal{X}) = \text{NF}(\text{add} \mathcal{X}) = \text{NF}(\mathcal{X})
\]
hold for each subcategory \( \mathcal{X} \) of \( \text{mod} \, R \).

**Proof.** Note that there are inclusions \( \text{res}_R \mathcal{X} \supseteq \text{add}_R \mathcal{X} \supseteq \mathcal{X} \) of subcategories of \( \text{mod} \, R \). From this we see that there are inclusions \( \text{NF}(\text{res}_R \mathcal{X}) \supseteq \text{NF}(\text{add}_R \mathcal{X}) \supseteq \text{NF}(\mathcal{X}) \) of subsets of \( \text{Spec} \, R \). We have only to show that \( \text{NF}(\text{res}_R \mathcal{X}) \) is contained in \( \text{NF}(\mathcal{X}) \).

Let \( p \) be a prime ideal in \( \text{NF}(\text{res}_R \mathcal{X}) \). Then there is an \( R \)-module \( Y \in \text{res}_R \mathcal{X} \) such that \( p \) is in \( \text{NF}(Y) \). The localization \( Y_p \) belongs to \( (\text{res}_R \mathcal{X})_p \), and to \( \text{res}_{Rp} \mathcal{X}_p \) by Proposition
3.5. Assume that $p$ is not in $\text{NF}(\mathcal{X})$. Then for every $X \in \mathcal{X}$ the $R_p$-module $X_p$ is free. Hence the subcategory $\mathcal{X}_p$ of mod $R_p$ consists of all free $R_p$-modules, and in particular, $\mathcal{X}_p$ is resolving by Example 2.4(2). Therefore we have $\text{res}_{R_p} \mathcal{X}_p = \mathcal{X}_p$, and thus $Y_p$ is a free $R_p$-module. But this contradicts the choice of $p$. Consequently, the prime ideal $p$ must be in $\text{NF}(\mathcal{X})$, which completes the proof of the corollary. □

Using the above corollary, we can show that the nonfree locus of a subcategory is determined by the isomorphism classes of indecomposable modules in its resolving closure.

**Corollary 3.7.** Let $\mathcal{X}$ be a subcategory of mod $R$. Then one has $\text{NF}(\mathcal{X}) = \bigcup_{Y, Z \in \text{ind(Res} \mathcal{X})} \text{Supp} \text{Ext}^1_R(Y, Z)$.

**Proof.** By Corollary 3.6, replacing $\mathcal{X}$ with $\text{res} \mathcal{X}$, we may assume that the subcategory $\mathcal{X}$ is resolving. Under this assumption, we have only to show the equality $\text{NF}(\mathcal{X}) = \bigcup_{Y, Z \in \text{ind} \mathcal{X}} \text{Supp} \text{Ext}^1_R(Y, Z)$. If a prime ideal $p$ is such that $\text{Ext}^1_{R_p}(Y_p, Z_p) \neq 0$ for some modules $Y, Z \in \mathcal{X}$, then $Y_p$ is nonfree as an $R_p$-module, hence $p$ is in $\text{NF}(\mathcal{X})$. Conversely, let $p$ be a prime ideal in $\text{NF}(X)$ for some $X \in \mathcal{X}$. Then it follows from Proposition 2.10 that $\text{Ext}^1_{R_p}(X_p, \Omega X_p)$ is nonzero. Hence there are indecomposable summands $Y$ and $Z$ of $X$ and $\Omega X$ respectively such that $\text{Ext}^1_{R_p}(Y_p, Z_p)$ is nonzero. The modules $Y, Z$ are in $\text{ind} \mathcal{X}$. □

### 4. Closed subsets of nonfree loci

In this section, we study the structure of the nonfree locus of an $R$-module. The main result of this section is concerning closed subsets of a nonfree locus (in the relative topology induced by the Zariski topology of Spec $R$), which will often be referred in later sections. We begin with the following lemma, which is proved by taking advantage of an idea used in the proof of [9, Theorem 1].

**Lemma 4.1.** Let $R$ be a local ring with maximal ideal $m$. Let

$$\sigma : 0 \to L \xrightarrow{f} M \to N \to 0$$

be an exact sequence of $R$-modules. Let $x$ be an element in $m$. Then there is an exact sequence

$$0 \to L \xrightarrow{(\iota)} L \oplus M \to K \to 0.$$

If this splits, then so does $\sigma$.

**Proof.** There exists a homomorphism $(g, h) : L \oplus M \to L$ such that $1 = (g, h)(\tau) = xg + hf$. Applying $\text{Hom}_R(N, -)$ to $\sigma$, we have an exact sequence

$$\text{Hom}_R(N, N) \xrightarrow{\eta} \text{Ext}^1_R(N, L) \xrightarrow{\text{Ext}^1_R(N, f)} \text{Ext}^1_R(N, M),$$

and get $\text{Ext}^1_R(N, f)(\sigma) = (\text{Ext}^1_R(N, f) \cdot \eta)(1) = 0$. Set $\xi = \text{Ext}^1_R(N, g)$. There are equalities $1 = \text{Ext}^1_R(N, xg + hf) = x\xi + \text{Ext}^1_R(N, h) \cdot \text{Ext}^1_R(N, f)$, so we obtain $\sigma = x\xi(\sigma) + \text{Ext}^1_R(N, h)(\text{Ext}^1_R(N, f)(\sigma)) = x\xi(\sigma)$. Hence $\sigma = x^i \xi^i(\sigma)$ for any $i \geq 1$, and therefore $\sigma \in \bigcap_{i \geq 1} m^i \text{Ext}^1_R(N, L) = 0$ by virtue of Krull’s intersection theorem. Thus the exact sequence $\sigma$ splits. □
Using the above lemma, we prove the following proposition, which will play an essential role in the proofs of our main results.

**Proposition 4.2.** Let \( X \) be an \( R \)-module. Let \( \mathfrak{p} \) be a prime ideal in \( \text{NF}(X) \) and \( x \) an element in \( \mathfrak{p} \). Then there is a commutative diagram

\[
\begin{array}{ccc}
\sigma : 0 & \longrightarrow & \Omega X \\
\downarrow & \quad & \downarrow \\
x \sigma : 0 & \longrightarrow & \Omega X \end{array}
\]

of \( R \)-modules with exact rows, and the following statements hold:

1. \( X_1 \in \text{res}^2 X \),
2. \( V(\mathfrak{p}) \subseteq \text{NF}(X_1) \subseteq \text{NF}(X) \),
3. \( D(x) \cap \text{NF}(X_1) = \emptyset \).

**Proof.** Taking a free cover of \( X \), we get an exact sequence

\[
\sigma : 0 \rightarrow \Omega X \xrightarrow{f} R^n \rightarrow X \rightarrow 0
\]

of \( R \)-modules. Making a pushout diagram of \( f : \Omega X \rightarrow R^n \) and the multiplication map \( x : \Omega X \rightarrow \Omega X \), we obtain a commutative diagram (4.2.1).

(1) From the first row in (4.2.1) we see that \( \Omega X \) is in \( \text{res}^1 X \). It follows from the second row that \( X_1 \) is in \( \text{res}^2 X \).

(2) Assume that \( \mathfrak{p} \) is not in \( \text{NF}(X_1) \). Then \( (X_1)_p \) is free as an \( R_p \)-module, and the exact sequence

\[
0 \rightarrow \Omega X_p \xrightarrow{(f_p)} \Omega X_p \oplus R^n_p \rightarrow (X_1)_p \rightarrow 0
\]

splits. Lemma 4.1 implies that \( \sigma_p \) is a split exact sequence, and so the \( R_p \)-module \( X_p \) is free. This contradicts the assumption that \( \mathfrak{p} \) is in \( \text{NF}(X) \). Therefore \( \mathfrak{p} \) is in \( \text{NF}(X_1) \), and the set \( V(\mathfrak{p}) \) is contained in \( \text{NF}(X_1) \) by Corollary 2.11(2).

Take a prime ideal \( q \in \text{NF}(X_1) \). Suppose that \( q \) is not in \( \text{NF}(X) \). Then \( X_q \) is a free \( R_q \)-module, and the exact sequence

\[
0 \rightarrow \Omega X_q \xrightarrow{(f_q)} \Omega X_q \oplus R^n_q \rightarrow (X_1)_q \rightarrow 0
\]

both split. This implies that \( (X_1)_q \) is a free \( R_q \)-module, which contradicts the choice of \( q \). Thus \( \text{NF}(X_1) \) is contained in \( \text{NF}(X) \).

(3) Assume that the set \( D(x) \cap \text{NF}(X_1) \) is nonempty, and take a prime ideal \( q \) in \( D(x) \cap \text{NF}(X_1) \). Then the element \( x \) can be regarded as a unit of the local ring \( R_q \). Localizing the diagram (4.2.1) at \( q \), we see from the five lemma that \( (X_1)_q \) is a free \( R_q \)-module. Hence \( q \) is not in \( \text{NF}(X_1) \), which is a contradiction. \( \square \)

Now we can prove one of the main results of this paper.

**Theorem 4.3.** For any \( R \)-module \( X \) and any subset \( W \) of \( \text{NF}(X) \) which is closed in \( \text{Spec} \, R \), there exists an \( R \)-module \( Y \in \text{res} \, X \) such that \( W = \text{NF}(Y) \).
Proof. First of all, if $W$ is an empty set, then we can take $Y := R$. So, suppose that $W$ is nonempty. Take an irreducible decomposition $W = V(p_1) \cup \cdots \cup V(p_n)$ of $W$. Suppose that for each $1 \leq i \leq n$ we can find an $R$-module $Y_i \in \text{res} X$ such that $\text{NF}(Y_i)$ coincides with $V(p_i)$. Then, putting $Y = Y_1 \oplus \cdots \oplus Y_n$, which belongs to $\text{res} X$, we easily see that $W$ is equal to $\text{NF}(Y)$. So, we can assume without loss of generality that $W$ is an irreducible closed subset of $\text{Spec } R$; we write $W = V(p)$ for some $p \in \text{Spec } R$.

If $V(p)$ coincides with $\text{NF}(X)$, then we can take $Y := X$. So assume that $V(p)$ is strictly contained in $\text{NF}(X)$. Then there is a prime ideal $q \in \text{NF}(X)$ which is not in $V(p)$. Hence there exists an element $x \in p$ which is not in $q$. For this element $x$ of $R$, let $X_1$ be an $R$-module satisfying the three conditions in Proposition 4.2. Then it is obvious that $X_1$ is in $\text{res} X$. Since $q$ is in $D(x)$, it does not belong to $\text{NF}(X_1)$. Thus we have $V(p) \subseteq \text{NF}(X_1) \not\subseteq \text{NF}(X)$. If $V(p)$ coincides with $\text{NF}(X_1)$, then we can take $Y := X_1$. So we assume that $V(p)$ is strictly contained in $\text{NF}(X_1)$. Then, a similar argument to the above shows that there exists an $R$-module $X_2 \in \text{res} X$ which satisfies $V(p) \subseteq \text{NF}(X_2) \not\subseteq \text{NF}(X_1) \not\subseteq \text{NF}(X)$.

According to Corollary 2.11(1), all nonfree loci are closed subsets of $\text{Spec } R$. Since $\text{Spec } R$ is a noetherian space, every descending chain of closed subsets stablizes. This means that the above procedure to construct modules $X_i$ cannot be iterate infinitely many times. Hence there exists an $R$-module $Y \in \text{res} X$ such that $V(p)$ coincides with $\text{NF}(Y)$. \qed

A very special case of this theorem has already been obtained by the author; see [12, Lemma 3.4].

From the above theorem we see that the nonfree locus of a given nonfree module has an irreducible decomposition by the nonfree loci of a finite number of modules in its resolving closure.

Corollary 4.4. For every nonfree $R$-module $X$ there exists a decomposition

$$\text{NF}(X) = \text{NF}(Y_1) \cup \cdots \cup \text{NF}(Y_n)$$

with $Y_1, \ldots, Y_n \in \text{res} X$ such that $\text{NF}(Y_1), \ldots, \text{NF}(Y_n)$ are irreducible closed subsets of $\text{Spec } R$.

Proof. Since $\text{NF}(X)$ is a nonempty closed subset of $\text{Spec } R$, we have a decomposition

$$\text{NF}(X) = V(p_1) \cup \cdots \cup V(p_n)$$

for some prime ideals $p_1, \ldots, p_n$. We apply Theorem 4.3 to each $V(p_i)$ to see that there is an $R$-module $Y_i \in \text{res} X$ such that $V(p_i)$ coincides with $\text{NF}(Y_i)$. Then each $\text{NF}(Y_i)$ is irreducible and we have $\text{NF}(X) = \text{NF}(Y_1) \cup \cdots \cup \text{NF}(Y_n)$. \qed

We might think that the above corollary predicts that all closed subsets of $\text{Spec } R$ are the nonfree loci of some modules. But, as the proposition below says, this statement does not hold. Here, for a subset $W$ of $\text{Spec } R$, we denote by $\min W$ the set of minimal elements of $W$ with respect to inclusion relation.

Proposition 4.5. Let $W$ be a nonempty closed subset of $\text{Spec } R$. Then the following are equivalent:

1. $W$ is an $R$-module.
2. $W$ is a nonfree $R$-module.
3. $W$ is irreducible.
4. $W$ is of the form $\text{NF}(Y)$ for some $Y$.
5. $W$ is a minimal element of $\text{Spec } R$.
6. $W$ is a nonempty closed subset of $\text{Spec } R$.
7. $W$ is a nonfree $R$-module.
8. $W$ is irreducible.
9. $W$ is of the form $\text{NF}(Y)$ for some $Y$.
10. $W$ is a minimal element of $\text{Spec } R$.
(1) One has $W = \text{NF}(X)$ for some $R$-module $X$;
(2) One has $W = \text{NF}(X)$ for some $R$-module $X \in \text{res}_R(\bigoplus_{p \in \text{min} W} R/p)$;
(3) For every $p \in W$, the local ring $R_p$ is not a field.

Proof. (2) $\Rightarrow$ (1): This implication is trivial.
(1) $\Rightarrow$ (3): Let $p$ be a prime ideal in $W$. Then $X_p$ is not a free $R_p$-module. In particular, $R_p$ is not a field.
(3) $\Rightarrow$ (2): Take an irreducible decomposition $W = V(p_1) \cup \cdots \cup V(p_n)$ of $W$. Fix an integer $i$ with $1 \leq i \leq n$. By assumption, the local ring $R_{p_i}$ is not a field. It is easy to see that $p_i$ belongs to $\text{NF}_R(R/p_i)$, hence $V(p_i)$ is contained in $\text{NF}_R(R/p_i)$ by Corollary 2.11(2). Theorem 4.3 implies that there exists an $R$-module $Y_i \in \text{res}_R(R/p_i)$ such that $V(p_i) = \text{NF}(Y_i)$. Setting $Y = Y_1 \oplus \cdots \oplus Y_n$, we see that $Y$ is in $\text{res}_R(\bigoplus_{i=1}^n R/p_i)$ and that $W$ coincides with $\text{NF}(Y)$.

5. Walks in resolving subcategories

In this section, we investigate the structure of the resolving closure of an $R$-module by means of the inductive construction of the resolving closure which we obtained in Section 3. More precisely, let $X$ be an $R$-module. For an $R$-module $Y \in \text{res} X$, we consider how many resolving operations are needed to take to construct $Y$ from $X$. Here, resolving operations mean extensions and kernels of epimorphisms. For this purpose, we introduce the following invariant which measures the minimum number of required resolving operations. This is an imitation of a level in a triangulated category defined in [6].

Definition 5.1. For two $R$-modules $X$ and $Y$, we define
\[
\text{step}(X, Y) = \inf \{ n \geq 0 \mid Y \in \text{res}_R^n X \}.
\]

Remark 5.2. Let $X$ be an $R$-module.
(1) One has $\text{step}(X, Y) = 0$ for every $R$-module $Y \in \text{res}^0 X = \text{add}(X \oplus R)$. In particular, $\text{step}(X, X) = \text{step}(X, R) = \text{step}(X, 0) = 0$.
(2) One has $\text{step}(X, Y) < \infty$ for an $R$-module $Y$ if and only if $Y$ belongs to $\text{res} X$.

In general, the invariant $\text{step}(-, -)$ does not induce a distance function. However, it satisfies the triangle inequality.

Proposition 5.3. Let $X, Y, Z$ be $R$-modules.
(1) Let $m, n$ be nonnegative integers. If $Y \in \text{res}^m X$ and $Z \in \text{res}^n Y$, then $Z \in \text{res}^{m+n} X$.
(2) The inequality $\text{step}(X, Z) \leq \text{step}(X, Y) + \text{step}(Y, Z)$ holds.

Proof. (1) Let us prove this assertion by induction on $n$.

When $n = 0$, the module $Z$ is in $\text{add}(Y \oplus R)$. Note that both $Y$ and $R$ are in $\text{res}^m X$. Since $\text{res}^m X$ is an additive closure, it contains $\text{add}(Y \oplus R)$. Hence $Z$ belongs to $\text{res}^m X = \text{res}^{m+n} X$.

Let $n \geq 1$. By definition, there are a finite number of $R$-modules $M_1, \ldots, M_s$ such that $Z$ is a direct summand of $M_1 \oplus \cdots \oplus M_s$ and that for each $1 \leq i \leq s$ there exists an exact
sequence of either of the following two forms:

\[ 0 \to A_i \to M_i \to B_i \to 0, \]
\[ 0 \to M_i \to A_i \to B_i \to 0 \]

where \( A_i, B_i \in \text{res}^{n+1} Y \). Induction hypothesis implies that the modules \( A_i, B_i \) are in \( \text{res}^{m+n+1} X \) for \( 1 \leq i \leq s \). Hence each \( M_i \) is in \( \text{res}^{m+n} X \), and therefore so is \( Z \).

(2) Set \( p = \text{step}(X, Y) \) and \( q = \text{step}(Y, Z) \). Then \( Y \) is in \( \text{res}^{p} X \) and \( Z \) is in \( \text{res}^{q} Y \). The assertion (1) implies that \( Z \) is in \( \text{res}^{p+q} X \), which says that \( \text{step}(X, Z) \leq p + q = \text{step}(X, Y) + \text{step}(Y, Z) \). \( \square \)

Let \( Z \) be a subset of \( \text{Spec} R \). For a prime ideal \( p \) in \( Z \), we define the \textit{height} of \( p \) with respect to \( Z \) as the supremum of \( \text{ht}(p/q) \) where \( q \) runs through all prime ideals in \( Z \) that are contained in \( p \). We denote it by \( \text{ht}_Z(p) \).

\textbf{Remark 5.4.} The following statements are straightforward.

1. One has \( \text{ht}_{\text{Spec} R}(p) = \text{ht} p \) for any \( p \in \text{Spec} R \).
2. For a prime ideal \( p \) in a subset \( Z \) of \( \text{Spec} R \), it holds that \( 0 \leq \text{ht}_Z(p) \leq \text{ht} p \).
3. Let \( Z \) be a closed subset of \( \text{Spec} R \), and let \( I \) be the defining ideal of \( Z \). For each prime ideal \( p \in Z \), one has \( \text{ht}_Z(p) = \text{ht}(p/I) \).
4. Let \( p, q \) be prime ideals in a subset \( Z \) of \( \text{Spec} R \). If \( p \subseteq q \), then \( \text{ht}_Z(p) \leq \text{ht}_Z(q) \).
5. Let \( Z, W \) be subsets of \( \text{Spec} R \). If \( Z \subseteq W \), then \( \text{ht}_Z(p) \leq \text{ht}_W(p) \) for any \( p \in Z \).
6. Let \( R \) be a local ring with maximal ideal \( m \). Then the equality \( \text{ht}_Z(m) = \dim Z \) holds for every subset \( Z \) of \( \text{Spec} R \) containing \( m \). (Recall that the \textit{dimension} \( \dim Z \) of a subset \( Z \) of \( \text{Spec} R \) is defined as the supremum of \( \dim R/p \) where \( p \) runs over all prime ideals in \( Z \).)
7. Let \( Z \) be a subset of \( \text{Spec} R \) and let \( p \) be a prime ideal in \( Z \). Then \( \text{ht}_Z(p) = 0 \) if and only if \( p \) is minimal in \( Z \).

Now we state and prove one of the main results of this paper.

\textbf{Theorem 5.5.} Let \( X \) be an \( R \)-module and let \( p \) be a prime ideal in \( \text{NF}(X) \). Then there exists an \( R \)-module \( Y \in \text{res} X \) satisfying the following three conditions:

1. \( \text{step}(X, Y) \leq 2 \text{ht}_{\text{NF}(X)}(p) \),
2. \( p \in \text{NF}(Y) \),
3. \( \text{ht}_{\text{NF}(Y)}(p) = 0 \).

\textit{Proof.} We prove the theorem by induction on \( n := \text{ht}_{\text{NF}(X)}(p) \). (Note that \( \text{ht}_{\text{NF}(X)}(p) \) is finite because \( R \) is a noetherian ring.)

When \( n = 0 \), we set \( Y := X \). Then \( Y \) is in \( \text{res}^0 X \), so we have \( \text{step}(X, Y) = 0 = 2 \text{ht}_{\text{NF}(X)}(p) \). We also have \( p \in \text{NF}(Y) \) and \( \text{ht}_{\text{NF}(Y)}(p) = 0 \).

When \( n \geq 1 \), put

\[ S = \{ q \in \text{NF}(X) \mid \text{ht}_{\text{NF}(X)}(q) = 0 \}. \]

Corollary 2.11(1) implies that \( \text{NF}(X) \) is a closed subset of \( \text{Spec} R \). Letting \( I \) be the defining ideal of \( \text{NF}(X) \), we have \( \text{ht}_{\text{NF}(X)}(q) = \text{ht}(q/I) \) for every \( q \in \text{NF}(X) \). Hence \( S \) coincides with the set of minimal prime ideals of \( I \), and therefore \( S \) is a finite set. As \( n \) is positive now, the prime ideal \( p \) is not contained in all prime ideals in \( S \). By prime
avoidance, we can choose an element $x \in p$ which is not contained in all prime ideals in $S$.

For this element $x$, take an $R$-module $X_1$ which satisfies the conditions in Proposition 4.2. Namely, the module $X_1$ satisfies the following three conditions:

$$X_1 \in \text{res}^2 X,$$
$$V(p) \subseteq \text{NF}(X_1) \subseteq \text{NF}(X),$$
$$D(x) \cap \text{NF}(X_1) = \emptyset.$$ 

Hence $\text{step}(X, X_1) \leq 2$ and $X_1 \in \text{res} X$. Since $S$ is contained in $D(x)$, we have $S \cap \text{NF}(X_1) = \emptyset$.

Let $q$ be a prime ideal in $\text{NF}(X_1)$ which is contained in $p$. Then $q$ does not belong to $S$, so $\text{ht}_{\text{NF}(X)}(q) > 0$. Hence $\text{ht}(q/r) > 0$ for some prime ideal $r \in \text{NF}(X)$ which is contained in $q$. There are inequalities

$$\text{ht}(p/q) < \text{ht}(p/q) + \text{ht}(q/r) \leq \text{ht}(p/r) \leq \text{ht}_{\text{NF}(X)}(p) = n.$$ 

Therefore we have $\text{ht}_{\text{NF}(X_1)}(p) < n$. The induction hypothesis implies that there exists an $R$-module $Y \in \text{res} X_1$ such that $\text{step}(X_1, Y) \leq 2 \text{ht}_{\text{NF}(X_1)}(p)$, that $p \in \text{NF}(Y)$ and that $\text{ht}_{\text{NF}(Y)}(p) = 0$. According to Proposition 5.3(2), there are inequalities

$$\text{step}(X, Y) \leq \text{step}(X, X_1) + \text{step}(X_1, Y)$$
$$\leq 2 + 2 \text{ht}_{\text{NF}(X_1)}(p)$$
$$\leq 2 + 2(n - 1) = 2n = 2 \text{ht}_{\text{NF}(X)}(p).$$

Thus the proof of the theorem is completed.

Applying the above theorem to a local ring $R$, we get the following result. This result contains Theorem A from the introduction.

**Corollary 5.6.** Let $R$ be a local ring. Then for every nonfree $R$-module $X$, there exists a nonfree $R$-module $Y$ in $\text{res} X$ satisfying the following conditions:

1. $\text{step}(X, Y) \leq 2 \dim \text{NF}(X),$
2. $Y$ is free on the punctured spectrum of $R$.

**Proof.** Let $m$ be the unique maximal ideal of $R$. We observe that $m$ is in $\text{NF}(X)$. Letting $p = m$ in Theorem 5.5, we see that there is an $R$-module $Y \in \text{res} X$ such that $\text{step}(X, Y) \leq 2 \text{ht}_{\text{NF}(X)}(m)$, that $m \in \text{NF}(Y)$ and that $\text{ht}_{\text{NF}(Y)}(m) = 0$. These three conditions imply that the inequality $\text{step}(X, Y) \leq 2 \dim \text{NF}(X)$ holds, that $Y$ is a nonfree $R$-module and that $Y_p$ is a free $R_p$-module for every prime ideal $p \neq m$, respectively.

Restricting the above corollary to the Cohen-Macaulay case, we obtain the following result on maximal Cohen-Macaulay modules.

**Corollary 5.7.** Let $R$ be a Cohen-Macaulay local ring. Then for any nonfree maximal Cohen-Macaulay $R$-module $X$, there exists a nonfree maximal Cohen-Macaulay $R$-module $Y$ satisfying the following two conditions:

1. $\text{step}(X, Y) \leq 2 \dim \text{Sing} R,$
2. $Y$ is free on the punctured spectrum of $R$. 


Proof. By virtue of Corollary 5.6, we find an $R$-module $Y \in \text{res} \ X$ which is free on the punctured spectrum of $R$ and satisfies the inequality $\text{step}(X, Y) \leq 2 \dim \text{NF}(X)$. Since $X$ is in $\text{CM}(R)$ and $\text{CM}(R)$ is resolving by Example 2.4(4), the module $Y$ is also in $\text{CM}(R)$, that is, $Y$ is maximal Cohen-Macaulay. Since $\text{NF}(\text{CM}(R)) = \text{Sing} \ R$ by Example 2.9, the assertion follows. □

Forgetting the first condition on the module $Y$ in Corollary 5.6, we obtain the following result.

**Corollary 5.8.** Let $R$ be a local ring and $\mathcal{X}$ a resolving subcategory of $\text{mod} \ R$. If there exists a nonfree $R$-module in $\mathcal{X}$, then there exists a nonfree $R$-module in $\mathcal{X}$ which is free on the punctured spectrum of $R$.

**Remark 5.9.** In the case where $R$ is a Cohen-Macaulay local ring, a nonfree $R$-module in $\mathcal{X} := \text{CM}(R)$ which is free on the punctured spectrum can be constructed explicitly as follows. Let $R$ be a $d$-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. Then it is well-known and easy to see that there exists a nonfree $R$-module in $\text{CM}(R)$ if and only if $R$ is singular. When this is the case, the $R$-module $\Omega^d(R/\mathfrak{m})$ is a nonfree $R$-module in $\text{CM}(R)$ which is free on the punctured spectrum of $R$.

Applying Corollary 5.8 to the resolving subcategory $\mathcal{X} = \mathcal{G}(R)$ (cf. Example 2.4(6)), we have the following.

**Corollary 5.10.** Let $R$ be a local ring. If there exists a nonfree totally reflexive $R$-module, then there exists a nonfree totally reflexive $R$-module which is free on the punctured spectrum.

**Remark 5.11.** A local ring over which all totally reflexive modules are free is called $G$-regular. G-regular local rings have been studied by several authors. One of the main problems for G-regular local rings is to establish necessary and/or sufficient conditions for a given local ring to be G-regular. For the details of G-regular local rings, see [13]. The above corollary should give some contribution to this problem.

### 6. Resolving subcategories of countable type

In this section, we investigate resolving subcategories in which there exist only countably many nonisomorphic indecomposable modules. The following proposition plays a key role for this goal, which is proved by using Theorem 4.3. It actually gives a refinement of one inclusion in the equality given in Corollary 3.7.

**Proposition 6.1.** For a subcategory $\mathcal{X}$ of $\text{mod} \ R$ one has an inclusion of sets:

$$\text{NF}(\mathcal{X}) \subseteq \left\{ \sqrt{\text{Ann} \ \text{Ext}_R^1(Y, Z)} \mid Y, Z \in \text{ind}(\text{res} \ \mathcal{X}) \right\}.$$

**Proof.** Let $\mathfrak{p}$ be a prime ideal in $\text{NF}(\mathcal{X})$. Then $\mathfrak{p}$ is in $\text{NF}(X)$ for some $R$-module $X \in \mathcal{X}$. As $\text{NF}(X)$ is specialization-closed by Corollary 2.11(2), the irreducible set $V(\mathfrak{p})$ is contained in $\text{NF}(X)$. According to Theorem 4.3, there exists an $R$-module $Y \in \text{res} \ X$ such that $V(\mathfrak{p})$ coincides with $\text{NF}(Y)$. Since $\text{NF}(Y) =$
Supp $\text{Ext}_R^1(Y, \Omega Y) = V(\text{Ann} \text{Ext}_R^1(Y, \Omega Y))$ by Proposition 2.10, the prime ideal $p$ is equal to $\sqrt{\text{Ann} \text{Ext}_R^1(Y, \Omega Y)}$. Take indecomposable decompositions $Y = \bigoplus_{i=1}^m Y_i$ and $\Omega Y = \bigoplus_{j=1}^n Z_j$. Then we have

$$p = \sqrt{\text{Ann} \text{Ext}_R^1(Y, \Omega Y)}$$

$$= \sqrt{\text{Ann} \text{Ext}_R^1(\bigoplus_{i=1}^m Y_i, \bigoplus_{j=1}^n Z_j)}$$

$$= \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} \sqrt{\text{Ann} \text{Ext}_R^1(Y_i, Z_j)}.$$  

Since $p$ is a prime ideal, $p$ is equal to $\sqrt{\text{Ann} \text{Ext}_R^1(Y_a, Z_b)}$ for some integers $a, b$. As $\text{res} \mathcal{X}$ is a resolving subcategory of $\text{mod} R$, the $R$-modules $Y_a, Z_b$ are in $\text{res} \mathcal{X}$, hence in $\text{res} \mathcal{X}$ and therefore in $\text{ind}(\text{res} \mathcal{X})$. Thus we obtain the desired inclusion.  

A very special case of the above proposition has already been obtained by the author; see [12, Proposition 3.5].

**Definition 6.2.** We say that a subcategory $\mathcal{X}$ of $\text{mod} R$ has **countable type** if the set $\text{ind} \mathcal{X}$ is countable.

We say that a Cohen-Macaulay local ring $R$ has **countable Cohen-Macaulay representation type** if $\text{CM}(R)$ has countable type.

The result below is a direct consequence of Proposition 6.1.

**Corollary 6.3.** Let $\mathcal{X}$ be a subcategory of $\text{mod} R$. If $\text{res} \mathcal{X}$ has countable type, then $\text{NF}(\mathcal{X})$ is at most a countable set.

The converse of this corollary does not necessarily hold. Indeed, we have the following example.

**Example 6.4.** We consider a 1-dimensional local hypersurface $R = \mathbb{C}[[x, y]]/(x^4 + y^5)$. Let $m = (x, y)$ be the maximal ideal of $R$, and set $\mathcal{X} = \text{CM}(R)$. Then, since this ring $R$ is an integral domain of dimension 1, we have $\text{NF}(\mathcal{X}) = \text{Sing} R = \{m\}$ (cf. Example 2.9). In particular, the set $\text{NF}(\mathcal{X})$ is finite, hence at most countable. Since $\mathcal{X}$ is resolving by Example 2.4(4), we have $\text{res} \mathcal{X} = \mathcal{X}$. This subcategory $\mathcal{X}$ does not have countable type by virtue of the classification theorem [8, Theorem B] of hypersurfaces of finite and countable Cohen-Macaulay representation type.

The lemma below is proved by using so-called countable prime avoidance; see [12, Lemma 2.2] for the proof.

**Lemma 6.5.** Let $R$ be a local ring with residue field $k$, and assume either that $R$ is complete or that $k$ is uncountable. Let $Z$ be a specialization-closed subset of $\text{Spec} R$. If $Z$ is at most countable, then $\dim Z \leq 1$.

Corollaries 6.3, 2.11(2) and Lemma 6.5 yield the following theorem, which is one of the main results of this paper.
Theorem 6.6. Let $R$ be a local ring with residue field $k$, and assume either that $R$ is complete or that $k$ is uncountable. Let $\mathcal{X}$ be a subcategory of $\text{mod} R$ such that $\text{res} \mathcal{X}$ has countable type. Then $\dim \text{NF}(\mathcal{X}) \leq 1$.

Combining this theorem with Corollary 5.6 gives the following result.

Corollary 6.7. Let $R$ be a local ring with residue field $k$, and assume either that $R$ is complete or that $k$ is uncountable. Let $X$ be a nonfree $R$-module such that $\text{res} X$ has countable type. Then there exists a nonfree $R$-module $Y \in \text{res} X$ which is free on the punctured spectrum of $R$ and satisfies $\text{step}(X, Y) \leq 2$.

Remark 6.8. Along the lines in the proof of Theorem 5.5, we can actually construct such a module $Y$ as in the above corollary. Let $m$ be the unique maximal ideal of $R$. Since $X$ is a nonfree $R$-module, $m$ belongs to $\text{NF}(X)$. Theorem 6.6 guarantees that $\text{ht} \text{NF}(X)(m) = \dim \text{NF}(X) \leq 1$. If $\text{ht} \text{NF}(X)(m) = 0$, then $X$ is free on the punctured spectrum, so we can take $Y := X$. In this case we have $\text{step}(X, Y) = 0$. If $\text{ht} \text{NF}(X)(m) = 1$, then the proof of Theorem 5.5 implies that there exists an element $x \in m$ which is not in each prime ideal $q \in \text{NF}(X)$ with $\text{ht} \text{NF}(X)(q) = 0$. Applying Proposition 4.2 to this element $x$, we obtain an $R$-module $X_1$ satisfying the three conditions in the proposition. The proof of Theorem 5.5 shows that $\text{ht} \text{NF}(X_1)(m) < 1$, which implies that $X_1$ is free on the punctured spectrum of $R$. Thus we can take $Y := X_1$. In this case we have $\text{step}(X, Y) \leq 2$.

We immediately get the following corollary from Theorem 6.6. This result is nothing but Theorem B from the introduction.

Corollary 6.9. Let $R$ be a local ring with residue field $k$, and assume either that $R$ is complete or that $k$ is uncountable. Let $\mathcal{X}$ be a resolving subcategory of $\text{mod} R$ of countable type. Then $\dim \text{NF}(\mathcal{X}) \leq 1$.

Applying this corollary to the subcategory of maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring (cf. Example 2.4(4)), we can recover a theorem of Huneke and Leuschke.

Corollary 6.10. [9, Theorem 1.3][12, Theorem 2.4] Let $R$ be a Cohen-Macaulay local ring of countable Cohen-Macaulay representation type. Assume either that $R$ is complete or that the residue field is uncountable. Then $\dim \text{Sing} R \leq 1$.

Applying Corollary 6.9 to the subcategory of totally $C$-reflexive modules where $C$ is a semidualizing module (cf. Example 2.4(5)), we obtain a refinement of the main theorem of [12].

Corollary 6.11. (cf. [12, Theorem 3.6]) Let $R$ be a local ring which either is complete or has uncountable residue field. Let $C$ be a semidualizing $R$-module. Suppose that $\mathcal{G}_{C}(R)$ has countable type. Then $\dim \text{NF}(\mathcal{G}_{C}(R)) \leq 1$.

References


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