SOME CHARACTERIZATIONS OF GORENSTEIN LOCAL RINGS IN TERMS OF G-DIMENSION

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Abstract. In this paper, we shall characterize Gorenstein local rings by the existence of special modules of finite G-dimension.

1. Introduction and Preliminary

Throughout the present paper, R always denotes a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. All modules considered in this paper are finitely generated.

Gorenstein dimension (abbr. G-dimension), which was defined by Auslander [1], has played an important role in the classification of modules and rings together with projective dimension. Let us recall the definition of G-dimension.

Definition. Let M be an R-module.

(1) If the following conditions hold, then we say that the G-dimension of M is zero, and write $\text{G-dim}_R M = 0$.
   i) The natural homomorphism $M \to \text{Hom}_R(\text{Hom}_R(M, R), R)$ is isomorphic.
   ii) $\text{Ext}^i_R(M, R) = 0$ for every $i > 0$.
   iii) $\text{Ext}^i_R(\text{Hom}_R(M, R), R) = 0$ for every $i > 0$.

(2) Let n be a non-negative integer. If the G-dimension of the nth syzygy module $\Omega^n_R M$ of M is zero, then we say that the G-dimension of M is not bigger than n, and write $\text{G-dim}_R M \leq n$.

G-dimension has a lot of properties that are similar to those of projective dimension. We state here several properties of G-dimension. For the proofs, we refer to [2], [4], [7], and [11].

Proposition 1.1. *Let R be a local ring with residue field k.*

(1) *Let M be a non-zero R-module. If* $\text{G-dim}_R M < \infty$*, then* $\text{G-dim}_R M = \text{depth}\, R - \text{depth}_R M$.

(2) *The following conditions are equivalent.*
   i) *R is Gorenstein.*
   ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*
   iii) $\text{G-dim}_R k < \infty$.

(3) *For any R-module M,* $\text{G-dim}_R M \leq \text{pd}_R M$. *Hence any module of finite projective dimension is also of finite G-dimension.*

(4) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. If two of L, M, N are of finite G-dimension, then so is the third.*

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(5) *Let M be an R-module and let $\boldsymbol{x} = x_1, \cdots, x_n$ be a sequence of elements of R. Then the following hold.*
  i) *If $\boldsymbol{x}$ is an M-sequence, then* $\text{G-dim}_R M/\boldsymbol{x}M = \text{G-dim}_R M + n$.
  ii) *If $\boldsymbol{x}$ is an R-sequence in* $\text{Ann}_R M$, *then* $\text{G-dim}_{R/(\boldsymbol{x})} M = \text{G-dim}_R M - n$.

The Peskine-Szpiro intersection theorem is one of the main results in commutative algebra in the 1980's.

The Peskine-Szpiro Intersection Theorem. *Let R be a commutative noetherian local ring, and let M, N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Then*

$$\dim_R N \leq \text{pd}_R M.$$

This was proved by Peskine and Szpiro [8] in the positive characteristic case, by Hochster [6] in the equicharacteristic case, and by Roberts [9] in the general case. The following theorem is directly given as a corollary of the Peskine-Szpiro intersection theorem.

Theorem 1.2. [10, Proposition 6.2.4] *Let R be a commutative noetherian local ring. Suppose that there exists a Cohen-Macaulay R-module of finite projective dimension. Then the local ring R is Cohen-Macaulay.*

As we have observed in Proposition 1.1, G-dimension shares many properties with projective dimension. So we are naturally leaded to the following conjecture:

Conjecture. Let R be a commutative noetherian local ring. Suppose that there exists a Cohen-Macaulay R-module M of finite G-dimension. Then the local ring R is Cohen-Macaulay.

If this conjecture is proved, as the assertion is itself very interesting, its proof might give another easier proof of the Peskine-Szpiro intersection theorem. However, nothing has known about this conjecture until now.

In the next section, we shall give several conditions in terms of G-dimension that are equivalent to the condition that R is Gorenstein. The main result of this paper is Theorem 2.3. This theorem says that the above conjecture is true if the type of R or M is one.

2. Results

In this section, we consider when the local ring R is Gorenstein, by using G-dimension. Theorem 2.3 is the main result. For an R-module M, denote by $\mu_R^i(M)$ the ith Bass number of M and by $r_R(M)$ the type of M, i.e., $\mu_R^i(M) = \dim_k \text{Ext}_R^i(k, M)$ and $r_R(M) = \mu_R^t(M)$ where $t = \text{depth}_R M$. Type has the following properties:

Proposition 2.1. *Let M be an R-module and let $\boldsymbol{x} = x_1, \cdots, x_n$ be an M-sequence. Then $r_R(M/\boldsymbol{x}M) = r_R(M)$. If, in addition, $\boldsymbol{x}$ is also an R-sequence, then $r_{R/(\boldsymbol{x})}(M/\boldsymbol{x}M) = r_R(M)$.*

We omit the proof of the above proposition because it is standard. Refer to the proof of [3, Lemma 1.2.4].

In order to prove our main theorem, we prepare the following lemma.

Lemma 2.2. [5, Theorem 1.1] *Let R be a commutative noetherian local ring with residue class field k, and M be a finitely generated R-module. Then $\mathrm{Ext}_R^i(k, M) \neq 0$ for all i, $\mathrm{depth}_R M \leq i \leq \mathrm{id}_R M$.*

Here, we remark that the assertion of the lemma holds without finiteness of the injective dimension of M. In other words, if the R-module M is of infinite injective dimension, then one has $\mathrm{Ext}_R^i(k, M) \neq 0$ for all $i \geq \mathrm{depth}_R M$.

We denote by $l_R(M)$ the length of an R-module M, and by $\nu_R(M)$ the minimal number of generators of an R-module M, that is to say, $\nu_R(M) = \dim_k(M \otimes_R k)$. We shall state the main result of this paper.

Theorem 2.3. *The following conditions are equivalent.*

i) *R is Gorenstein.*
ii) *R admits an ideal I of finite G-dimension such that the factor ring R/I is Gorenstein.*
iii) *R admits a Cohen-Macaulay module of type 1 and of finite G-dimension.*
iv) *R is a local ring of type 1 admitting a Cohen-Macaulay module of finite G-dimension.*

Proof. i) $\Rightarrow$ ii): Both the zero ideal and the maximal ideal of R satisfy the condition ii) by Proposition 1.1.2.

ii) $\Rightarrow$ iii): Let $\boldsymbol{x}$ be a sequence of elements of R which forms a maximal R/I-sequence. Then the factor ring $R/I + (\boldsymbol{x})$ is an artinian Gorenstein ring, so one has $\mathrm{Hom}_R(k, R/I + (\boldsymbol{x})) \cong \mathrm{Hom}_{R/I+(\boldsymbol{x})}(k, R/I + (\boldsymbol{x})) \cong k$. Hence one sees from Proposition 2.1 that $r_R(R/I) = r_R(R/I+(\boldsymbol{x})) = 1$. On the other hand, Proposition 1.1.4 yields that $\text{G-dim}_R R/I < \infty$. Thus the R-module R/I is a Cohen-Macaulay module which has type one and finite G-dimension.

iii) $\Rightarrow$ iv): Let M be a Cohen-Macaulay module of type 1 and of finite G-dimension. Taking a maximal M-sequence $\boldsymbol{x}$, we see from Proposition 2.1 and Proposition 1.1.5 that $M/\boldsymbol{x}M$ is an R-module of finite length, of type 1, and of finite G-dimension. Hence, replacing M by $M/\boldsymbol{x}M$, one may assume that the length of M is finite. Then since the ideal $\mathrm{Ann}_R M$ is $\mathfrak{m}$-primary, one can choose an R-sequence $\boldsymbol{y} = y_1, \cdots, y_t$ in $\mathrm{Ann}_R M$, where $t = \mathrm{depth}\, R$. Noting that $\mathrm{Hom}_{R/(\boldsymbol{y})}(k, M) \cong \mathrm{Hom}_R(k, M)$, one sees that $r_{R/(\boldsymbol{y})}(M) = r_R(M) = 1$. On the other hand, Proposition 1.1.5 implies that $\text{G-dim}_{R/(\boldsymbol{y})} M < \infty$. Therefore, replacing R by $R/(\boldsymbol{y})$, we may assume that $\mathrm{depth}\, R = 0$. Thus we see from Proposition 1.1.1 that the G-dimension of the R-module M is zero, which especially yields that $M^{**} \cong M$. Hence we have the following isomorphisms.

$$\begin{aligned} \mathrm{Hom}_R(k, M) &\cong \mathrm{Hom}_R(k, M^{**}) \\ &\cong \mathrm{Hom}_R(k \otimes_R M^*, R) \\ &\cong \mathrm{Hom}_R((k \otimes_R M^*) \otimes_k k, R) \\ &\cong \mathrm{Hom}_k(k \otimes_R M^*, \mathrm{Hom}_R(k, R)). \end{aligned}$$

It follows from these isomorphisms that $1 = r_R(M) = \nu_R(M^*)\, r_R(R)$. Hence $r_R(R) = 1$.

iv) $\Rightarrow$ i): Let M be a Cohen-Macaulay R-module of finite G-dimension, and let $\boldsymbol{x}$ be a maximal M-sequence. Then we see that the residue R-module $M/\boldsymbol{x}M$ is of finite length and of finite G-dimension by Proposition 1.1.5. Hence, replacing M by $M/\boldsymbol{x}M$, we may assume that $l_R(M) < \infty$. Set $t = \mathrm{depth}\, R$. Since $\mathrm{Ann}_R M$ is an $\mathfrak{m}$-primary ideal, one can take an R-sequence $\boldsymbol{x} = x_1, \cdots, x_t$ in $\mathrm{Ann}_R M$.

It then follows from Proposition 1.1.5 that $\text{G-dim}_{R/(\boldsymbol{x})}M = \text{G-dim}_R M - t < \infty$. Since $r_{R/(\boldsymbol{x})}(R/(\boldsymbol{x})) = r_R(R) = 1$ by Proposition 2.1, replacing R by $R/(\boldsymbol{x})$, we may assume that $\operatorname{depth} R = 0$. Suppose that R is not Gorenstein. Then the R-module R has infinite injective dimension. Hence Lemma 2.2 especially yields that $\operatorname{Ext}^1_R(k,R) \neq 0$. Put $s = \dim_k \operatorname{Ext}^1_R(k,R)(> 0)$ and let

$$0 = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_n = M$$

be a composition series of M. Decompose this series to short exact sequences

$$0 \to M_{i-1} \to M_i \to k \to 0$$

for $1 \leq i \leq n$. Since $r(R) = 1$, one has $k^* \cong k$. Hence, applying the R-dual functor $(-)^* = \operatorname{Hom}_R(-,R)$ to these sequences, one obtains exact sequences

$$\begin{cases} 0 \to k \to M_i^* \to M_{i-1}^* & \text{for } \ 1 \leq i \leq n-1, \quad \text{and} \\ 0 \to k \to M^* \to M_{n-1}^* \to k^s \to \operatorname{Ext}^1_R(M,R). \end{cases}$$

Here, as $\text{G-dim}_R M = 0$ by Proposition 1.1.1, it follows from definition that $\operatorname{Ext}^1_R(M,R) = 0$. Therefore we get

$$\begin{cases} l_R(M_i^*) \leq l_R(M_{i-1}^*) + 1 & \text{for } \ 1 \leq i \leq n-1, \quad \text{and} \\ l_R(M^*) = l_R(M_{n-1}^*) + 1 - s. \end{cases}$$

Since $s > 0$, we have the following inequalities.

$$\begin{array}{rcl} l_R(M^*) & < & l_R(M_{n-1}^*) + 1 \\ & \leq & l_R(M_{n-2}^*) + 2 \\ & \leq & \cdots \\ & \leq & l_R(M_0^*) + n \\ & = & n. \end{array}$$

That is to say, $l_R(M^*) < l_R(M)$. Because M^* is also of G-dimension zero by definition, the same argument for M^* shows that $l_R(M^{**}) < l_R(M^*)$. However, since $\text{G-dim}_R M = 0$, one has $M^{**} \cong M$. Thus, we obtain $l_R(M) < l_R(M^*) < l_R(M)$, which is contradiction. Hence R is Gorenstein. $\square$

Remark 2.4. Each of the criteria for the Gorenstein property in the above theorem requires that the Bass number of a certain module is one. With relation to this, we should remark that the local ring R is Gorenstein if and only if $\mu_R^d(R) = 1$ where $d = \dim R$. This result is due to Foxby and Roberts; see [3, Corollary 9.6.3, Remark 9.6.4].

Now, let us study the following proposition, which will be used as a lemma to give the other characterizations of Gorenstein local rings. This proposition says that, the existence of a module M of finite G-dimension with exact sequence $0 \to W \to M \to V \to 0$ where V, W are annihilated by $\mathfrak{m}$, determines the higher Bass numbers of the base ring. Yoshino [12], observing this, gives a characterization of artinian local rings of low Loewy length admitting modules of finite G-dimension.

Proposition 2.5. *Let $0 \to W \to M \to V \to 0$ be a short exact sequence of R-modules. Suppose that R is not Gorenstein,* $\operatorname{depth} R = 0$*, $M \neq 0$,* $\text{G-dim}_R M < \infty$ *and V, W are annihilated by the maximal ideal $\mathfrak{m}$ of R. Set $m = \dim_k V$, $n = \dim_k W$, and $r = r_R(R)$. Then the following hold.*

i) *$n = rm$.*

ii) $\mu_R^i(R) = \begin{cases} r & \textit{for} \ \ i=0, \\ r^{i+1}-r^{i-1} & \textit{for} \ \ i>0. \end{cases}$

iii) $l_R(M^*) = l_R(M) = m+n$.

Proof. First of all, note from Proposition 1.1.1 that G-dim$_R M = 0$. Since $V \cong k^m$ and $W \cong k^n$, we have a short exact sequence

$$0 \to k^n \to M \to k^m \to 0. \tag{1}$$

Put $s_i = \mu_R^i(R)$ and $s = s_1$. As R is non-Gorenstein and M is non-zero, Proposition 1.1.2 and Lemma 2.2 imply that $m, n \neq 0$ and $s_i \neq 0$ for every $i > 0$. Note from definition that $\mathrm{Ext}_R^i(M,R) = 0$ for every $i > 0$. Applying the R-dual functor $(-)^* = \mathrm{Hom}_R(-,R)$ to this sequence, we obtain an exact sequence $0 \to k^{mr} \to M^* \to k^{nr} \to k^{ms} \to 0$ and isomorphisms $k^{ns_i} \cong k^{ms_{i+1}}$ for all $i > 0$. Hence one gets a short exact sequence

$$0 \to k^{mr} \to M^* \to k^{nr-ms} \to 0, \tag{2}$$

and equalities

$$ns_i = ms_{i+1} \tag{3}$$

for all $i > 0$. Note by definition that $M^{**} \cong M$ and that $\mathrm{Ext}_R^i(M^*,R) = 0$ for every $i > 0$. Applying the R-dual functor to the sequence (2), one gets an exact sequence

$$0 \to k^{(nr-ms)r} \to M \to k^{mr^2} \to k^{(nr-ms)s} \to 0, \tag{4}$$

and isomorphisms $k^{mrs_i} \cong k^{(nr-ms)s_{i+1}}$ for all $i > 0$. Therefore one obtains equalities

$$mrs_i = (nr-ms)s_{i+1} \tag{5}$$

for all $i > 0$. It follows from (3) and (5) that

$$s = \frac{n^2-m^2}{nm} r. \tag{6}$$

On the other hand, the sequences (4) and (1) yield the following equalities.

$$\begin{aligned} (nr-ms)r + mr^2 &= l_R(k^{(nr-ms)r}) + l_R(k^{mr^2}) \\ &= l_R(M) + l_R(k^{(nr-ms)s}) \\ &= l_R(k^n) + l_R(k^m) + l_R(k^{(nr-ms)s}) \\ &= n + m + (nr-ms)s. \end{aligned} \tag{7}$$

From the equalities (6) and (7), we easily see that $n = rm$ and $s = r^2 - 1$. Hence we obtain $s_i = r^{i-1}s = r^{i+1} - r^{i-1}$ for any $i > 0$ by (3), and $l_R(M^*) = n + m = l_R(M)$ by (2) and (1), as desired. □

Using the above proposition, let us induce another characterization of Gorensteinness, which is a corollary of Theorem 2.3.

Corollary 2.6. *The following conditions are equivalent.*

i) *R is Gorenstein.*

ii) *R admits a non-zero module of length ≤ 2 and of finite G-dimension.*

iii) *R admits a non-zero module M satisfying $\mathfrak{m}^2 M = 0$, $l_R(M) \leq 2\,\nu_R(M)$, and* G-dim$_R M < \infty$.

Proof. i) $\Rightarrow$ ii): Proposition 1.1.2 implies that the simple R-module k satisfies the condition ii).

ii) $\Rightarrow$ iii): Let $M \neq 0$ be an R-module with $l_R(M) \leq 2$ and $\text{G-dim}_R M < \infty$. Then $\mathfrak{m}M$ is a non-trivial submodule of M by Nakayama's lemma. Since $l_R(M) \leq 2$, we see that the submodule $\mathfrak{m}^2 M$ of M must be the zero module. On the other hand, we have $l_R(M) \leq 2 \leq 2\,\nu_R(M)$.

iii) $\Rightarrow$ i): Since the length of the R-module M is finite, the ideal $\mathrm{Ann}_R M$ is $\mathfrak{m}$-primary. Hence there exists an R-sequence $\boldsymbol{x} = x_1, \cdots, x_t$ in $\mathrm{Ann}_R M$, where $t = \operatorname{depth} R$. By Proposition 1.1.5, the $R/(\boldsymbol{x})$-module M is of finite G-dimension. So, replacing R by $R/(\boldsymbol{x})$, we may assume that the depth of R is zero. Consider the natural short exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0.$$

By the assumption, the module $\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, as well as $M/\mathfrak{m}M$. Suppose that R is non-Gorenstein. Then we see from Proposition 2.5 that

$$\begin{array}{rcl} 2\,\nu_R(M) & \geq & l_R(M) \\ & = & l_R(\mathfrak{m}M) + \nu_R(M) \\ & = & r_R(R)\,\nu_R(M) + \nu_R(M) \\ & = & (r_R(R) + 1)\,\nu_R(M). \end{array}$$

Therefore we get $2 \geq r_R(R) + 1$, that is, $r_R(R) \leq 1$, hence $r_R(R) = 1$. Thus, Theorem 2.3 implies that R is Gorenstein, which contradicts the present assumption. This contradiction proves that R is Gorenstein, and we are done. $\square$

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