POWERS OF THE MAXIMAL IDEAL AND VANISHING OF (CO)HOMOLOGY

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ABSTRACT. We prove that each positive power of the maximal ideal of a commutative Noetherian local ring is Tor-rigid, and strongly-rigid. This gives new characterizations of regularity and, in particular, shows that such ideals satisfy the torsion condition of a long-standing conjecture of Huneke and Wiegand.

1. INTRODUCTION

Throughout R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and all R-modules are assumed to be finitely generated.

In this paper we are motivated by the following result of Levin and Vasconcelos:

Theorem 1.1 (Levin and Vasconcelos [9]). *Let M be an R-module. Assume $\mathfrak{m}^t M \neq 0$ for some $t \geq 1$. Then R is regular if and only if $\mathsf{pd}_R(\mathfrak{m}^t M) < \infty$ if and only if $\mathsf{id}_R(\mathfrak{m}^t M) < \infty$.*

Theorem 1.1 was examined previously in the literature. For example, Asadollahi and Puthenpurakal obtained beautiful characterizations of local rings in terms of various homological dimensions: if M is an R-module of positive depth with $\text{H-dim}_R(\mathfrak{m}^n M) < \infty$ for some $n \gg 0$, then R satisfies the property H, where H-dim denotes a homological dimension such as projective dimension; see [1, Theorem 1] for details. The main purpose of this short note is to prove an analogous result for nonzero modules of the form $\mathfrak{m}^t M$. However, our main result, stated as Theorem 1.2, concerns the vanishing of Ext and Tor rather than homological dimensions.

Theorem 1.2. *Let M be an R-module with $\mathsf{depth}_R(M) \geq 1$, and let $t \geq 0$ be an integer. If $\mathsf{Tor}_n^R(\mathfrak{m}^t M, N) = 0$ (respectively, $\mathsf{Ext}_R^n(N, \mathfrak{m}^t M) = 0$) for some R-module N and some $n \geq 1$, then $\mathsf{Tor}_n^R(M, N) = 0$ (respectively, $\mathsf{Ext}_R^n(N, M) = 0$).*

Theorem 1.2 does not hold for modules of zero depth in general; see Examples 2.3 and 2.4. In section 2 we give a proof of Theorem 1.2 and discuss its consequences. We should mention that one such consequence of Theorem 1.2 is Theorem 1.1 in the positive depth case; see the paragraph after Example 2.4. Moreover, Theorem 1.2 implies that each positive power of the maximal ideal is Tor-rigid and strongly-rigid; see Definition 2.1. More precisely, we have:

Corollary 1.3. *Assume $\mathsf{depth}(R) \geq 1$, and let $t \geq 1$. If $\mathsf{Tor}_n^R(\mathfrak{m}^t, N) = 0$ for some $n \geq 0$ and some R-module N, then $\mathsf{pd}_R(N) \leq n$, and hence $\mathsf{Tor}_i^R(\mathfrak{m}^t, N) = 0$ for all $i \geq n$.*

Although the behavior of powers of the maximal ideal obtained in Corollary 1.3 may seem expectable, to the best of our knowledge, the conclusion of the corollary is new. Note that Corollary 1.3 follows immediately by letting $M = \mathfrak{m}$ in Theorem 1.2.

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Corollary 1.3 yields a new characterization of regularity for local rings of positive depth: we record the result as Corollary 1.4 and prove it in the paragraph preceding Corollary 2.5.

Corollary 1.4. *Assume* $\mathsf{depth}(R) \geq 1$. *If M is an R-module such that $\mathfrak{m}^s M \neq 0$ for some $s \geq 1$, then R is regular if and only if $\mathsf{Ext}^n_R(\mathfrak{m}^s M, \mathfrak{m}^t) = 0$ for some $n, t \geq 1$. In particular, R is regular if and only if $\mathsf{Ext}^n_R(\mathfrak{m}^s, \mathfrak{m}^t) = 0$ for some $n, s, t \geq 1$.*

As another consequence of Corollary 1.3, we conclude by [3, 2.15] that each positive power of the maximal ideal satisfies the torsion condition proposed in a long-standing conjecture of Huneke and Wiegand; see [8, pages 473-474] for details.

Corollary 1.5. *Assume R is one-dimensional, non-regular, and reduced. Then $\mathfrak{m}^t \otimes_R (\mathfrak{m}^t)^*$ has torsion for each $t \geq 1$, where $(\mathfrak{m}^t)^* = \mathsf{Hom}_R(\mathfrak{m}^t, R)$.*

Levin and Vascencelos [9, Lemma, page 316] proved, if M and N are R-modules such that $\mathfrak{m}M \neq 0$ and $\mathsf{Tor}^R_n(\mathfrak{m}M, N) = \mathsf{Tor}^R_{n+1}(\mathfrak{m}M, N) = 0$ for some $n \geq 0$, then $\mathsf{pd}_R(N) \leq n$ and $\mathsf{Tor}^R_i(\mathfrak{m}M, N) = 0$ for all $i \geq n$; see also [4, 2.9]. While proving Theorem 1.2, we have discovered that we can extend the result of Levin and Vascencelos by considering nonzero modules of the form $\mathfrak{m}M \otimes_R N$: at the end of Section 2, we will observe that each such module is isomorphic to $\mathfrak{m}C$ for some R-module C, and we will prove the following:

Proposition 1.6. *Assume R is not Artinian and let $n \geq 1$ be an integer. Then the following conditions are equivalent:*

(i) *R is Gorenstein.*
(ii) $\mathsf{Ext}^i_R(\mathfrak{m}^{\otimes n}, R) = 0$ *for all* $i \gg 0$.
(iii) $\mathsf{Ext}^i_R(\mathfrak{m}^n, R) = 0$ *for all* $i \gg 0$.

2. PROOF OF THE MAIN RESULT AND CONSEQUENCES

We start by recalling some definitions.

Definition 2.1. Let M be an R-module. Recall that:

(i) ([2]) M is *Tor-rigid* provided that the following holds: whenever N is an R-module with $\mathsf{Tor}^R_j(M, N) = 0$ for some $j \geq 1$, one has that $\mathsf{Tor}^R_v(M, N) = 0$ for all $v \geq j$.
(ii) ([7]) M is *strongly-rigid* provided that the following holds: whenever N is an R-module with $\mathsf{Tor}^R_n(M, N) = 0$ for some $n \geq 1$, one has that $\mathsf{pd}_R(N) < \infty$.

Let us point out that, it is not known whether strongly-rigid modules are Tor-rigid; see [11, Question 2.5]. In general it is quite subtle to determine whether a given module is strongly-rigid or Tor-rigid, but various characterizations of local rings have already been obtained in terms of such classes of modules. For example, existence of a nonzero Tor-rigid module of finite injective dimension forces the ring to be Gorenstein; see [11, 4.13(i)].

The following, straightforward albeit quite useful, observation is implicit in [9].

2.2. Let $C = (C_i, \partial_i)_{i \in \mathbb{Z}}$ be a minimal complex with C_i are R-modules, i.e., $\mathsf{im}(\partial_{i+1}) \subseteq \mathfrak{m} \cdot C_i$ for each i. Assume $\mathsf{H}_n(\mathfrak{m}C) = 0$ for some $n \in \mathbb{Z}$. As $\mathsf{H}_n(\mathfrak{m}C) = \mathsf{ker}(\partial_n) \cap \mathfrak{m}C_n / \mathfrak{m} \cdot \mathsf{im}(\partial_{n+1})$, we have $\mathsf{im}(\partial_{n+1}) \subseteq \mathsf{ker}(\partial_n) \cap \mathfrak{m} \cdot C_n = \mathfrak{m} \cdot \mathsf{im}(\partial_{n+1})$. By Nakayama's lemma, we conclude that $\mathsf{im}(\partial_{n+1}) = 0$, i.e., $\partial_{n+1} = 0$.

We can now use 2.2 and prove our main result:

Proof of Theorem 1.2. We will only prove the statement about the vanishing of Tor; the one about Ext follows similarly.

Note that $\mathsf{depth}_R(\mathfrak{m}^j M) \geq 1$ for any $j \geq 0$. Hence it suffices to consider the case where $t = 1$ and $n \geq 1$. Assume $\mathsf{Tor}_n^R(\mathfrak{m}M, N) = 0$, and consider the exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0.$$

This yields the exact sequence $0 = \mathsf{Tor}_n^R(\mathfrak{m}M, N) \to \mathsf{Tor}_n^R(M, N) \to \mathsf{Tor}_n^R(M/\mathfrak{m}M, N)$, which shows that $\mathsf{Tor}_n^R(M, N)$ has finite length.

Let $C = M \otimes_R F$, where $F = (F_i, \partial_i^F)_{i \geq 0}$ is a minimal free resolution of N. It follows that $\mathsf{im}(\partial_{i+1}^C) \subseteq \mathfrak{m} \cdot C_i$ for each i. Since $0 = \mathsf{Tor}_n^R(\mathfrak{m}M, N) = \mathsf{H}_n(\mathfrak{m}M \otimes_R F)$, we see from 2.2 that $\partial_{n+1}^C = 1_M \otimes_R \partial_{n+1}^F = 0$. Therefore, we have

$$\mathsf{Tor}_n^R(M, N) = \mathsf{ker}\left(1_M \otimes_R \partial_n^F\right) / \left(\mathsf{im}(1_M \otimes_R \partial_{n+1}^F)\right) = \mathsf{ker}\left(1_M \otimes_R \partial_n^F\right).$$

Now suppose $\mathsf{Tor}_n^R(M, N) \neq 0$. Then, since it embeds into a finite direct sum of copies of M, we conclude that $1 \leq \mathsf{depth}_R\left(\mathsf{Tor}_n^R(M, N)\right) < \infty$. However, $\mathsf{Tor}_n^R(M, N)$ has finite length so that $\mathsf{depth}_R\left(\mathsf{Tor}_n^R(M, N)\right) = 0$. This shows that $\mathsf{Tor}_n^R(M, N)$ must vanish, as claimed. ■

It is also worth noting that Theorem 1.2 may fail if the module in question has zero depth: we give two such examples over rings of depth one and two, respectively.

Example 2.3. Let k be a field, $R = k[\![x, y]\!]/(xy)$, $M = k \oplus R$, and let $N = R/(x+y)$. Then $\mathsf{depth}_R(M) = 0$, $\mathfrak{m}M = \mathfrak{m}$ and $\mathsf{pd}_R(N) = 1$. However, $\mathsf{Tor}_1^R(\mathfrak{m}M, N) = 0 \neq \mathsf{Tor}_1^R(M, N)$.

Example 2.4. Let k be a field, $R = k[\![x, y]\!]$, $\mathfrak{m} = (x, y)$, $M = \mathfrak{m}/(x^2, xy)$ and $N = R/(y)$. Then $\mathfrak{m}M = \mathfrak{m}^2/(x^2, xy) \cong R/\left((x^2, xy) :_R y^2\right) = R/(x)$. Hence $\mathsf{Tor}_1^R(\mathfrak{m}M, N) = 0$. But $\mathsf{Tor}_1^R(M, N) \neq 0$ since multiplication by y on M is not injective. Note that $\mathsf{depth}_R(M) = 0$.

Next we discuss several corollaries of Theorem 1.2. First we deduce from Theorem 1.2 the positive depth case of Theorem 1.1, and prove Corollary 1.4.

Proof of the positive depth case of Theorem 1.1 by using Theorem 1.2. Assume M has positive depth and $\mathsf{pd}_R(\mathfrak{m}^t M)$ is finite, say s. Then we have $\mathsf{Tor}_{s+1}^R(\mathfrak{m}^t M, k) = 0$ so that $\mathsf{Tor}_{s+1}^R(\mathfrak{m}^{t-1} M, k) = 0$ by Theorem 1.2. Hence, $\mathsf{pd}_R(\mathfrak{m}^{t-1} M)$ is also finite. There is an exact sequence

$$0 \to \mathfrak{m}^t M \to \mathfrak{m}^{t-1} M \to k^{\oplus u} \to 0,$$

with $u \geq 1$ as $\mathfrak{m}^{t-1} M \neq 0$. It follows that $\mathsf{pd}_R(k) < \infty$, and R is regular. The assertion on injective dimension is shown similarly. ■

Proof of Corollary 1.4. Assume $\mathfrak{m}^s M \neq 0$ and $\mathsf{Ext}_R^n(\mathfrak{m}^s M, \mathfrak{m}^t) = 0$ for some $n, s, t \geq 1$. Note, since $\mathsf{depth}(R) \geq 1$, Corollary 1.3 implies that $\mathfrak{m}^t$ is strongly-rigid and Tor-rigid. As $\mathsf{depth}_R(\mathfrak{m}^t) = 1$, we conclude from [11, 1.1] that $\mathsf{pd}_R(\mathfrak{m}^s M) < \infty$. Now Theorem 1.1 shows that R is regular. ■

Corollary 2.5. *Let M be an R-module such that* $\mathsf{depth}_R(M) \geq 1$.

(i) *If M strongly-rigid, then $\mathfrak{m}^t M$ is strongly-rigid for each $t \geq 1$.*
(ii) *If M is Tor-rigid, then $\mathfrak{m}M$ is strongly-rigid and Tor-rigid.*

Proof. Part (i) is an immediate corollary of Theorem 1.2. So we will prove part (ii).

Let N be an R-module with $\mathsf{Tor}_n^R(\mathfrak{m}M, N) = 0$ for some $n \geq 1$. Then it follows from Theorem 1.2 that $\mathsf{Tor}_n^R(M, N) = 0$. Since M is Tor-rigid, we have that $\mathsf{Tor}_{n+1}^R(M, N) = 0$. Hence, tensoring the exact sequence $0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0$ with N, we obtain the exact sequence $0 = \mathsf{Tor}_{n+1}^R(M, N) \to \mathsf{Tor}_{n+1}^R(M/\mathfrak{m}M, N) \to \mathsf{Tor}_n^R(\mathfrak{m}M, N) = 0$. This shows $\mathsf{Tor}_{n+1}^R(k, N) = 0$ so that $\mathsf{pd}_R(N) \leq n$, and $\mathsf{Tor}_i^R(\mathfrak{m}M, N) = 0$ for all $i \geq n$. ■

Theorem 1.2 allows us to find out new classes of Tor-rigid modules over hypersurfaces:

Corollary 2.6. *Let $R = S/(f)$ be a hypersurface ring, where $(S, \mathfrak{n})$ is an unramified regular local ring and $0 \neq f \in \mathfrak{n}$. If M is a finite length R-module, then $\mathfrak{m}\Omega^i(M)$ is Tor-rigid for each $i \geq 1$.*

Proof. Note that each finite length module is Tor-rigid [8, 2.4]. Hence we may assume R has positive depth. Then, given $i \geq 1$, since $\Omega^i(M)$ is a Tor-rigid module that has positive depth, we conclude by Corollary 2.5(ii) that $\mathfrak{m}\Omega^i(M)$ is Tor-rigid, as claimed. ■

If R has positive depth and N is an R-module, it is known, and easy to see, that N is free if and only if $\mathfrak{m} \otimes_R N$ is torsion-free; see, for example, [6, page 842]. Thanks to Theorem 1.2, we can extend this result under mild conditions:

Corollary 2.7. *Assume $\mathsf{depth}(R) \geq 1$, and let N be an R-module. Assume $N_{\mathfrak{p}}$ is torsionless for each associated prime ideal $\mathfrak{p}$ of R (e.g., R is reduced). Then N is free if and only if $\mathfrak{m}^t \otimes_R N$ is torsion-free for some $t \geq 1$.*

Proof. Let X be the torsion-free part of N. Then $X_{\mathfrak{p}} \cong N_{\mathfrak{p}}$ for each associated prime $\mathfrak{p}$ of R. So $\mathsf{Ext}^1_R(\mathsf{Tr}X, R) = 0$, and hence there is an exact sequence $0 \to X \to F \to C \to 0$, where F is a free module; see, for example, [10, Prop. 5].

Tensoring X with the short exact sequence $0 \to \mathfrak{m}^t \to R \to R/\mathfrak{m}^t \to 0$, we conclude that there is an injection $\mathsf{Tor}^R_1(X, R/\mathfrak{m}^t) \hookrightarrow \mathfrak{m}^t \otimes_R X \cong \mathfrak{m}^t \otimes_R N$; see [8, 1.1]. This implies that $\mathsf{Tor}^R_1(X, R/\mathfrak{m}^t) = 0$. Therefore we have $0 = \mathsf{Tor}^R_2(C, R/\mathfrak{m}^t) \cong \mathsf{Tor}^R_1(C, \mathfrak{m}^t)$, and hence $\mathsf{pd}_R(C) \leq 1$; see Corollary 1.3. Thus, X is free and this implies N is free; see [8, 1.1]. ■

We finish this section by showing that modules of the form $\mathfrak{m}M \otimes_R N$ is strongly-rigid and Tor-rigid. This will allow us to establish Proposition 1.6 advertised in the introduction.

2.8. Let M and N be R-modules such that $\mathfrak{m}M \neq 0 \neq N$. Then consider the minimal free presentations of M and N, respectively: $R^{\oplus a} \twoheadrightarrow M$ and $R^{\oplus b} \twoheadrightarrow N$.

It follows that we have the surjection: $\mathfrak{m}^{\oplus a} = \mathfrak{m}R^{\oplus a} \twoheadrightarrow \mathfrak{m}M$. Tensoring this surjection with N, we obtain another surjection: $\mathfrak{m}^{\oplus a} \otimes_R N \twoheadrightarrow \mathfrak{m}M \otimes_R N$. Consequently, we have the following isomorphisms and surjective maps:

$$\mathfrak{m}R^{\oplus ab} = \mathfrak{m}^{\oplus ab} \cong \mathfrak{m}^{\oplus a} \otimes_R R^{\oplus b} \twoheadrightarrow \mathfrak{m}^{\oplus a} \otimes_R N \twoheadrightarrow \mathfrak{m}M \otimes_R N.$$

Therefore, there is an R-submodule C of $\mathfrak{m}R^{\oplus ab}$ such that

$$\mathfrak{m}M \otimes_R N \cong \frac{\mathfrak{m}R^{\oplus ab}}{C} = \mathfrak{m}\left(\frac{R^{\oplus ab}}{C}\right).$$

The rigidity property (mentioned preceding Proposition 1.6) of nonzero modules of the form $\mathfrak{m}M$, in view of 2.8, yields:

2.9. If $\mathsf{Tor}^R_n(\mathfrak{m}M \otimes_R N, X) = \mathsf{Tor}^R_{n+1}(\mathfrak{m}M \otimes_R N, X) = 0$ for some R-modules M, N, X and $n \geq 0$ such that $\mathfrak{m}M \neq 0 \neq N$, then $\mathsf{pd}_R(X) \leq n$, and $\mathsf{Tor}^R_i(\mathfrak{m}M \otimes_R N, X) = 0$ for all $i \geq n$.

The observations in 2.8 and 2.9, in particular, show that tensor powers of the maximal ideal have rigidity:

2.10. Assume R is not Artinian, $t \geq 1$ and $\mathfrak{m}^{\otimes 0} = R$. Then, letting $M = R$ and $N = \mathfrak{m}^{\otimes(t-1)}$ in 2.9, we conclude that, if $\mathsf{Tor}^R_n(\mathfrak{m}^{\otimes t}, X) = \mathsf{Tor}^R_{n+1}(\mathfrak{m}^{\otimes t}, X) = 0$ for some R-module X and some $n \geq 0$, then $\mathsf{pd}_R(X) \leq n$, and $\mathsf{Tor}^R_i(\mathfrak{m}^{\otimes t}, X) = 0$ for all $i \geq n$.

We can now note that Proposition 1.6 is a consequence of 2.10 and [5, 4.4]. We finish this section by recording a special case of Proposition 1.6:

Proposition 2.11. *R is Gorenstein if and only if* $\text{G-dim}_R(\mathfrak{m} \otimes_R \mathfrak{m}) < \infty$.

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