ON THE EXISTENCE OF EMBEDDINGS INTO MODULES OF FINITE HOMOLOGICAL DIMENSIONS

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Abstract. Let R be a commutative Noetherian local ring. We show that R is Gorenstein if and only if every finitely generated R-module can be embedded in a finitely generated R-module of finite projective dimension. This extends a result of Auslander and Bridger to rings of higher Krull dimension, and it also improves a result due to Foxby where the ring is assumed to be Cohen-Macaulay.

1. Introduction

Throughout this paper, let R be a commutative Noetherian local ring. All R-modules in this paper are assumed to be finitely generated.

In [1, Proposition 2.6 (a) and (d)] Auslander and Bridger proved the following.

Theorem 1.1 (Auslander-Bridger). *The following are equivalent:*

(1) *R is quasi-Frobenius (i.e. Gorenstein with Krull dimension zero).*
(2) *Every R-module can be embedded in a free R-module.*

On the other hand, in [4, Theorem 2] Foxby showed the following.

Theorem 1.2 (Foxby). *The following are equivalent:*

(1) *R is Gorenstein.*
(2) *R is Cohen-Macaulay, and every R-module can be embedded in an R-module of finite projective dimension.*

For an R-module C we denote by $\operatorname{add}_R C$ the class of R-modules which are direct summands of finite direct sums of copies of C. The *C-dimension* of an R-module X, $C\text{-}\dim_R X$, is defined as the infimum of nonnegative integers n such that there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to X \to 0$$

of R-modules with $C_i \in \operatorname{add}_R C$ for all $0 \leq i \leq n$.

In this paper, we prove the following theorem. This result removes from Theorem 1.2 the assumption that R is Cohen-Macaulay, and it extends Theorem 1.1 to rings of higher Krull dimension. It should be noted that our proof of this result is different from Foxby's proof for the special case $C = R$.

Theorem 1.3. *Let R be a commutative Noetherian local ring with residue field k. Let C be a semidualizing R-module of depth t. Then the following are equivalent:*

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(1) *C is dualizing.*
(2) *Every R-module can be embedded in an R-module of finite C-dimension.*
(3) *The R-module $\operatorname{Tr}\Omega^t k \otimes_R C$ can be embedded in an R-module of finite C-dimension. (Here $\operatorname{Tr}\Omega^t k$ denotes the transpose of the t-th syzygy of the R-module k.)*

Moreover, if one of these three conditions holds, then R is Cohen-Macaulay.

2. Proof of Theorem 1.3 and its applications

First of all, we recall the definition of a semidualizing module.

Definition 2.1. An R-module C is called *semidualizing* if the natural homomorphism $R \to \operatorname{Hom}_R(C, C)$ is an isormophism and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$.

Note that a dualizing module is nothing but a semidualizing module of finite injective dimension. Another typical example of a semidualizing module is a free module of rank one. Recently a considerable number of authors have studied semidualizing modules and have obtained many results concerning these modules.

We denote by $\mathfrak{m}$ the maximal ideal of R and by k the residue field of R. To prove our main theorem, we establish two lemmas.

Lemma 2.2. *Let C be a semidualizing R-module. Let $g : M \to X$ be an injective homomorphism of R-modules with $C\text{-}\dim_R X < \infty$. If $\operatorname{Ext}^i_R(M, C) = 0$ for any $1 \le i \le C\text{-}\dim_R X$, then the natural map $\lambda_M : M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ is injective.*

Proof. First of all we prove that M can be embedded in a module C_0 in $\operatorname{add}_R C$. For this we set $n = C\text{-}\dim_R X$. If $n = 0$, then this is obvious from the assumption, since $X \in \operatorname{add}_R C$. If $n > 0$, then there exists an exact sequence

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} X \to 0$$

with $C_i \in \operatorname{add}_R C$ for $0 \le i \le n$. Putting $X_i = \operatorname{Im} d_i$, we have exact sequences

$$0 \to X_{i+1} \to C_i \to X_i \to 0 \quad (0 \le i \le n-1).$$

Then we have $\operatorname{Ext}^1_R(M, X_1) = 0$, since there are isomorphisms $\operatorname{Ext}^1_R(M, X_1) \cong \operatorname{Ext}^2_R(M, X_2) \cong \cdots \cong \operatorname{Ext}^n_R(M, X_n) \cong \operatorname{Ext}^n_R(M, C_n) = 0$. Hence $\operatorname{Hom}_R(M, d_0) : \operatorname{Hom}_R(M, C_0) \to \operatorname{Hom}_R(M, X)$ is surjective. This implies that the homomorphism $g \in \operatorname{Hom}_R(M, X)$ is lifted to $f \in \operatorname{Hom}_R(M, C_0)$, i.e. $d_0 \cdot f = g$. Since g is injective, f is injective as well. Therefore M has an embedding f into C_0.

To prove that λ_M is injective, we note that λ_{C_0} is an isomorphism, because of $C_0 \in \operatorname{add}_R C$. Since there is an injective homomorphism $f : M \to C_0$, the following commutative diagram forces λ_M to be injective:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\lambda_M"'] & C_0 \arrow[d, "\lambda_{C_0}"', "\cong"] \\
\operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C) \arrow[r, "\operatorname{Hom}_R(\operatorname{Hom}_R(f,C),C)"] & \operatorname{Hom}_R(\operatorname{Hom}_R(C_0, C), C).
\end{tikzcd}$$

$\square$

Lemma 2.3. *Let C be a semidualizing R-module and let M be an R-module. Assume that M is free on the punctured spectrum of R. Then there is an isomorphism*

$$\operatorname{Ext}_R^i(M,R) \cong \operatorname{Ext}_R^i(M \otimes_R C, C)$$

for each integer $i \le \operatorname{depth}_R C$.

Proof. Set $t = \operatorname{depth}_R C$. Since C is semidualizing, we have a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p(\operatorname{Tor}_q^R(M,C),C) \Rightarrow \operatorname{Ext}_R^{p+q}(M,R).$$

Note by assumption that the R-module $\operatorname{Tor}_q^R(M,C)$ has finite length for $q > 0$. By [2, Proposition 1.2.10(e)] we have $E_2^{p,q} = 0$ if $p < t$ and $q > 0$. Hence

$$\operatorname{Ext}_R^i(M \otimes_R C, C) = E_2^{i,0} \cong \operatorname{Ext}_R^i(M,R)$$

for $i \le t$. □

Let M be an R-module. Take a free resolution

$$F_\bullet = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \to 0)$$

of M. Then for a nonnegative integer n we define the *n-th syzygy* of M by the image of d_n and denote it by $\Omega_R^n M$ or simply $\Omega^n M$. We also define the *(Auslander) transpose* of M by the cokernel of the map $\operatorname{Hom}_R(d_1,R) : \operatorname{Hom}_R(F_0,R) \to \operatorname{Hom}_R(F_1,R)$ and denote it by $\operatorname{Tr}_R M$ or simply $\operatorname{Tr} M$. Note that the n-th syzygy and the transpose of M are uniquely determined up to free summand. Note also that they commute with localization; namely, for every prime ideal $\mathfrak{p}$ of R there are isomorphisms $(\Omega_R^n M)_\mathfrak{p} \cong \Omega_{R_\mathfrak{p}}^n M_\mathfrak{p}$ and $(\operatorname{Tr}_R M)_\mathfrak{p} \cong \operatorname{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}$ up to free summand.

Recall that for a positive integer n an R-module is called *n-torsionfree* if

$$\operatorname{Ext}_R^i(\operatorname{Tr} M, R) = 0$$

for all $1 \le i \le n$. Now we can prove our main theorem.

Proof of Theorem 1.3. (1) ⇒ (2): By virtue of [6, Theorem (3.11)], the local ring R is Cohen-Macaulay. Now assertion (2) follows from [4, Theorem 1].

(2) ⇒ (3): This implication is obvious.

(3) ⇒ (1): We denote by $(-)^\dagger$ the C-dual functor $\operatorname{Hom}_R(-,C)$. Put $t = \operatorname{depth}_R C$ and set $M = \operatorname{Tr}\Omega^t k$. Then we have $\operatorname{depth} R = t$ by [5]. Since

$$\operatorname{grade}_R \operatorname{Ext}_R^i(k,R) \ge i-1$$

for $1 \le i \le t$, the module $\Omega^t k$ is t-torsionfree by [1, Proposition (2.26)]. Hence $\operatorname{Ext}_R^i(M,R) = 0$ for $1 \le i \le t$. As M is free on the punctured spectrum of R, Lemma 2.3 implies $\operatorname{Ext}_R^i(M \otimes_R C, C) = 0$ for $1 \le i \le t$. By assumption (3), the module $M \otimes_R C$ has an embedding into a module X with $C\text{-}\dim_R X < \infty$. According to [7, Lemma 4.3], we have $C\text{-}\dim_R X \le t$. Lemma 2.2 shows that the natural map $\lambda_{M\otimes_R C} : M \otimes_R C \to (M \otimes_R C)^{\dagger\dagger}$ is injective. On the other hand, since there are natural isomorphisms

$$\begin{aligned}(M \otimes_R C)^{\dagger\dagger} &= \operatorname{Hom}_R(\operatorname{Hom}_R(M \otimes_R C, C), C) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(M, \operatorname{Hom}_R(C,C)), C)\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(M,R),C),\end{aligned}$$

we see from [1, Proposition (2.6)(a)] that

$$\begin{aligned}\operatorname{Ker}\lambda_{M\otimes_R C} &\cong \operatorname{Ext}_R^1(\operatorname{Tr} M, C) \cong \operatorname{Ext}_R^1(\Omega^t k, C)\\ &\cong \operatorname{Ext}_R^{t+1}(k,C).\end{aligned}$$

Thus we obtain $\mathrm{Ext}_R^{t+1}(k, C) = 0$. By [3, Theorem (1.1)], the R-module C must have finite injective dimension.

As we observed in the proof of the implication (1) $\Rightarrow$ (2), assertion (1) implies that R is Cohen-Macaulay. Thus the last assertion follows. $\square$

Now we give applications of our main theorem. Letting $C = R$ in Theorem 1.3, we obtain the following result. This improves Theorem 1.2 and extends Theorem 1.1.

Corollary 2.4. *The following are equivalent:*
(1) *R is Gorenstein.*
(2) *Every R-module can be embedded in an R-module of finite projective dimension.*

Combining Corollary 2.4 with [4, Theorem 1], we have the following.

Corollary 2.5. *If every finitely generated R-module can be embedded in a finitely generated R-module of finite projective dimension, then every finitely generated R-module can be embedded in a finitely generated R-module of finite injective dimension.*

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