

OPENNESS OF FID-LOCI

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ABSTRACT. Let R be a commutative Noetherian ring and M a finite R -module. In this paper, we consider Zariski-openness of the FID-locus of M , namely, the subset of $\text{Spec } R$ consisting of all prime ideals \mathfrak{p} such that $M_{\mathfrak{p}}$ has finite injective dimension as an $R_{\mathfrak{p}}$ -module. We prove that the FID-locus of M is an open subset of $\text{Spec } R$ whenever R is excellent.

1. INTRODUCTION

Throughout the present paper, we assume that all rings are commutative and Noetherian.

Let \mathbb{P} be a property of local rings. The \mathbb{P} -locus of a ring R is the set of prime ideals \mathfrak{p} of R such that the local ring $R_{\mathfrak{p}}$ satisfies the property \mathbb{P} . It is a natural question to ask whether the \mathbb{P} -locus of R is an open subset of $\text{Spec } R$ in the Zariski topology, and it has been considered for a long time. For example, it is known that the \mathbb{P} -locus of an excellent ring is open if \mathbb{P} is any of the regular property, the complete intersection property, the Gorenstein property, and the Cohen-Macaulay property. As to the details of openness of loci for properties of local rings, see [3], [4, §6–7], [6], [7, §24], [8], and [9].

On the other hand, let \mathbb{P} be a property of modules over a local ring. The \mathbb{P} -locus of a module M over a ring R is defined to be the subset of $\text{Spec } R$ consisting of all prime ideals \mathfrak{p} such that the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ satisfies \mathbb{P} . The locus of a finite module for the property of finite projective dimension is known to be an open subset [1, Corollary 9.4.7], and so is the locus of a finite module for the Gorenstein property if the base ring is acceptable, and therefore if it is excellent [5, Corollaries 4.6 and 4.7].

In this paper, we will consider openness of the locus of a finite module for the property of finite injective dimension, which we call the FID-locus. We shall prove that the FID-locus of a finite module satisfying certain conditions is an open subset. Using this result, we will show the following:

Theorem. *Let R be an excellent ring and M a finite R -module. Then the FID-locus*

$$\text{FID}_R(M) = \{ \mathfrak{p} \in \text{Spec } R \mid \text{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty \}$$

of M is an open subset of $\text{Spec } R$ in the Zariski topology.

Of course, this theorem implies the result of Greco and Marinari [3, Corollary 1.5] asserting that the Gorenstein locus of an excellent ring is open.

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2. THE RESULTS

Throughout this section, let R be a commutative Noetherian ring. Recall that a subset U of $\text{Spec } R$ is called *stable under generalization* provided that if $\mathfrak{p} \in U$ and $\mathfrak{q} \in \text{Spec } R$ with $\mathfrak{q} \subseteq \mathfrak{p}$ then $\mathfrak{q} \in U$. We begin by stating two lemmas. The former is called the ‘‘topological Nagata criterion’’; it is a criterion for Zariski-openness which is due to Nagata.

Lemma 2.1. [7, Theorem 24.2] *The following are equivalent for a subset U of $\text{Spec } R$:*

- (1) U is an open subset of $\text{Spec } R$;
- (2) U is stable under generalization, and contains a nonempty open subset of $V(\mathfrak{p})$ for any $\mathfrak{p} \in U$.

Lemma 2.2. [3, Lemma 1.1] *Let \mathfrak{p} be a minimal prime of a finite R -module M . Then there exist an element $f \in R \setminus \mathfrak{p}$ and a chain*

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = M_f$$

of R_f -submodules of M_f such that $N_i/N_{i-1} \cong R_f/\mathfrak{p}R_f$ for $1 \leq i \leq n$.

Next, we study an easy lemma.

Lemma 2.3. *Let \mathfrak{p} be a prime ideal of R and M a finite R -module. If $M_{\mathfrak{p}} = 0$, then $M_f = 0$ for some $f \in R \setminus \mathfrak{p}$.*

Proof. If $M_{\mathfrak{p}} = 0$, then \mathfrak{p} is not in the support of the R -module M , hence \mathfrak{p} does not contain the annihilator ideal $\text{Ann}_R M$. Therefore there is an element $f \in \text{Ann}_R M \setminus \mathfrak{p}$. We easily obtain $M_f = 0$. \square

We define the *FID-locus* of an R -module M to be the set of prime ideals \mathfrak{p} of R such that the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ has finite injective dimension, and denote it by $\text{FID}_R(M)$. Now, we can prove the following proposition, which will play a key role in the proof of our main result.

Proposition 2.4. *Let M be a finite R -module, and let $\mathfrak{p} \in \text{FID}_R(M)$. Suppose that the FID-locus $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$ contains a nonempty open subset of $\text{Spec } R/\mathfrak{p}$ for each integer j with $0 \leq j \leq \text{ht } \mathfrak{p}$. Then there exists an element $f \in R \setminus \mathfrak{p}$ such that the FID-locus $\text{FID}_R(M)$ contains $V(\mathfrak{p}) \cap D(f)$.*

Proof. First of all, we note that to prove the proposition we can freely replace our ring R with its localization R_g for an element $g \in R \setminus \mathfrak{p}$. In fact, we have $\mathfrak{p}R_g \in \text{FID}_{R_g}(M_g)$ and $\text{ht } \mathfrak{p}R_g = \text{ht } \mathfrak{p}$. Let U_j be a nonempty open subset of $\text{Spec } R/\mathfrak{p}$ which is contained in $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$ for $0 \leq j \leq \text{ht } \mathfrak{p}$. Write $U_j = D(I_j/\mathfrak{p})$ for some ideal I_j of R containing \mathfrak{p} , and we see that $D(I_jR_g/\mathfrak{p}R_g)$ is a nonempty open subset of $\text{Spec } R_g/\mathfrak{p}R_g$ which is contained in $\text{FID}_{R_g/\mathfrak{p}R_g}(\text{Ext}_{R_g}^j(R_g/\mathfrak{p}R_g, M_g))$. If there exists an element $\frac{h}{g^n} \in R_g \setminus \mathfrak{p}R_g$ with $h \in R$ and $n \geq 0$ such that $V(\mathfrak{p}R_g) \cap D(\frac{h}{g^n})$ is contained in $\text{FID}_{R_g}(M_g)$, then h is an element of $R \setminus \mathfrak{p}$ and $V(\mathfrak{p}) \cap D(gh)$ is contained in $\text{FID}_R(M)$.

Suppose that $M_{\mathfrak{p}} = 0$. Then we have $M_f = 0$ for some $f \in R \setminus \mathfrak{p}$ by Lemma 2.3. Hence the set $D(f)$ is itself contained in the locus $\text{FID}_R(M)$, and there is nothing more to prove. Therefore in what follows we consider the case where $M_{\mathfrak{p}} \neq 0$. Since $M_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$ -module of finite injective dimension, $R_{\mathfrak{p}}$ is a Cohen-Macaulay local

ring by virtue of [2, Corollary 9.6.2, Remark 9.6.4(a)]. Put $n = \dim R_{\mathfrak{p}}$, and take a sequence $\mathbf{x} = x_1, x_2, \dots, x_n$ of elements in \mathfrak{p} which forms an $R_{\mathfrak{p}}$ -regular sequence. Then, putting $H_i = (0 :_{R/(x_1, x_2, \dots, x_{i-1})} x_i)$, we have $(H_i)_{\mathfrak{p}} = 0$ for $1 \leq i \leq n$. Hence Lemma 2.3 implies that $(H_i)_{f_i} = 0$ for some $f_i \in R \setminus \mathfrak{p}$. Setting $f = f_1 f_2 \cdots f_n$, we see that f is in $R \setminus \mathfrak{p}$ and that \mathbf{x} is an R_f -regular sequence. Replacing R with R_f , we may assume that \mathbf{x} is an R -regular sequence.

Set $\bar{R} = R/(\mathbf{x})$ and $\bar{\mathfrak{p}} = \mathfrak{p}/(\mathbf{x})$. Then $\bar{\mathfrak{p}}$ is a minimal prime of \bar{R} , hence is an associated prime of \bar{R} . Let $\mathfrak{P}_1 = \bar{\mathfrak{p}}, \mathfrak{P}_2, \dots, \mathfrak{P}_s$ be the associated primes of \bar{R} . Taking an element of the set $\bigcap_{i=2}^s \mathfrak{P}_i \setminus \mathfrak{P}_1$, we easily see that there is an element $f \in R \setminus \mathfrak{p}$ such that $\text{Ass } \bar{R}_{\bar{f}} = \{\bar{\mathfrak{p}}\bar{R}_{\bar{f}}\}$, where \bar{f} denotes the residue class of f in \bar{R} . Replacing R with R_f , we may assume that $\text{Ass } \bar{R} = \{\bar{\mathfrak{p}}\}$.

On the other hand, since $\text{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$, we have $\text{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = n$ by [2, Theorem 3.1.17] and hence $\text{Ext}_{R_{\mathfrak{p}}}^{n+1}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = 0$, where $\kappa(\mathfrak{p})$ denotes the residue field of $R_{\mathfrak{p}}$. Therefore it follows from Lemma 2.3 that $\text{Ext}_{R_f}^{n+1}(R_f/\mathfrak{p}R_f, M_f) = 0$ for some $f \in R \setminus \mathfrak{p}$. Replacing R with R_f , we may assume that

$$(2.4.1) \quad \text{Ext}_R^{n+1}(R/\mathfrak{p}, M) = 0.$$

Here, we establish a claim.

Claim. *One may assume that $\text{Ext}_R^j(R/\mathfrak{p}, M) = 0$ for all integers $j > n$.*

Proof of Claim. If $\bar{\mathfrak{p}} = 0$, then $\mathfrak{p} = (\mathbf{x})$ and the R -module R/\mathfrak{p} has projective dimension n since \mathbf{x} is an R -regular sequence of length n . Hence $\text{Ext}_R^j(R/\mathfrak{p}, M) = 0$ for $j > n$, as desired. Assume $\bar{\mathfrak{p}} \neq 0$. Then we have $\emptyset \neq \text{Min}_{\bar{R}}(\bar{\mathfrak{p}}) \subseteq \text{Ass}_{\bar{R}}(\bar{\mathfrak{p}}) \subseteq \text{Ass } \bar{R} = \{\bar{\mathfrak{p}}\}$, and therefore $\text{Min}_{\bar{R}}(\bar{\mathfrak{p}}) = \{\bar{\mathfrak{p}}\}$. According to Lemma 2.2, for some element $f \in R \setminus \mathfrak{p}$ there is a chain

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_l = \bar{\mathfrak{p}}\bar{R}_{\bar{f}}$$

of $\bar{R}_{\bar{f}}$ -modules such that $N_i/N_{i-1} \cong \bar{R}_{\bar{f}}/\bar{\mathfrak{p}}\bar{R}_{\bar{f}} \cong R_f/\mathfrak{p}R_f$ for any $1 \leq i \leq l$. Replacing R with R_f , we may assume that there is a chain $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_l = \bar{\mathfrak{p}}$ of \bar{R} -modules such that each N_i/N_{i-1} is isomorphic to R/\mathfrak{p} .

We have obtained a series of exact sequences of R -modules

$$(2.4.2) \quad 0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow R/\mathfrak{p} \rightarrow 0 \quad (1 \leq i \leq l).$$

Using these sequences and (2.4.1), we can get $\text{Ext}_R^{n+1}(\bar{\mathfrak{p}}, M) = 0$. The natural exact sequence $0 \rightarrow \bar{\mathfrak{p}} \rightarrow \bar{R} \rightarrow R/\mathfrak{p} \rightarrow 0$ induces an exact sequence of Ext modules: $0 = \text{Ext}_R^{n+1}(\bar{\mathfrak{p}}, M) \rightarrow \text{Ext}_R^{n+2}(R/\mathfrak{p}, M) \rightarrow \text{Ext}_R^{n+2}(\bar{R}, M)$. Noting that \bar{R} has projective dimension n as an R -module, we have $\text{Ext}_R^i(\bar{R}, M) = 0$ for every $i > n$, and $\text{Ext}_R^{n+2}(R/\mathfrak{p}, M) = 0$. Using the sequences (2.4.2) again, we get $\text{Ext}_R^{n+2}(\bar{\mathfrak{p}}, M) = 0$. Iterating this procedure shows the claim. \square

The assumption of the proposition yields a nonempty open subset U_j of $\text{Spec } R/\mathfrak{p}$ contained in the locus $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$ for $0 \leq j \leq n$. We can write $U_j = D(I_j/\mathfrak{p})$ for some ideal I_j of R which strictly contains \mathfrak{p} . Hence there exists an element $f_j \in I_j \setminus \mathfrak{p}$, and setting $f = f_0 f_1 \cdots f_n$, we see that the set $D(f)$ is contained in $D(I_j)$ for any $0 \leq j \leq n$.

Fix a prime ideal $\mathfrak{q} \in V(\mathfrak{p}) \cap D(f)$. Then $\mathfrak{q}/\mathfrak{p}$ belongs to $D(I_j/\mathfrak{p}) = U_j$, which is contained in $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$ for $0 \leq j \leq n$. Hence $\text{Ext}_{R_{\mathfrak{q}}}^j(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}, M_{\mathfrak{q}})$ has finite injective dimension as an $R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}$ -module for any integer j with $0 \leq j \leq n$.

Put $m = \max\{\text{id}_{R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}}(\text{Ext}_{R_{\mathfrak{q}}}^j(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}), M_{\mathfrak{q}}) \mid 0 \leq j \leq n\}$. Consider the following spectral sequence:

$$E_2^{i,j} = \text{Ext}_{R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}}^i(\kappa(\mathfrak{q}), \text{Ext}_{R_{\mathfrak{q}}}^j(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}), M_{\mathfrak{q}}) \implies \text{Ext}_{R_{\mathfrak{q}}}^{i+j}(\kappa(\mathfrak{q}), M_{\mathfrak{q}}).$$

We have $E_2^{i,j} = 0$ if $i > m$, and the above claim shows that $E_2^{i,j} = 0$ if $j > n$. From this spectral sequence, we see that $\text{Ext}_{R_{\mathfrak{q}}}^i(\kappa(\mathfrak{q}), M_{\mathfrak{q}}) = 0$ for $i > m+n$. This implies that the $R_{\mathfrak{q}}$ -module $M_{\mathfrak{q}}$ has finite injective dimension (cf. [2, Proposition 3.1.14]), that is $\mathfrak{q} \in \text{FID}_R(M)$. It follows that $V(\mathfrak{p}) \cap D(f)$ is contained in $\text{FID}_R(M)$, which completes the proof of the proposition. \square

Now we state and prove our main result of this paper.

Theorem 2.5. *Let M be a finite R -module. Suppose that $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$ contains a nonempty open subset of $\text{Spec } R/\mathfrak{p}$ for any prime ideal $\mathfrak{p} \in \text{FID}_R(M)$ and any integer $0 \leq j \leq \text{ht } \mathfrak{p}$. Then $\text{FID}_R(M)$ is an open subset of $\text{Spec } R$.*

Proof. Proposition 2.4 shows that for any $\mathfrak{p} \in \text{FID}_R(M)$ there exists $f \in R \setminus \mathfrak{p}$ such that $\text{FID}_R(M)$ contains $V(\mathfrak{p}) \cap D(f)$. Note that $V(\mathfrak{p}) \cap D(f)$ is not an empty set since \mathfrak{p} belongs to it. On the other hand, it is easy to see from [2, Proposition 3.1.9] that $\text{FID}_R(M)$ is stable under generalization. Thus the theorem follows from Lemma 2.1. \square

We denote by $\text{Reg}(R)$ the *regular locus* of R , namely, the set of prime ideals \mathfrak{p} of R such that the local ring $R_{\mathfrak{p}}$ is regular. The following result can be obtained from the above theorem.

Corollary 2.6. *Let R be an excellent ring. Then $\text{FID}_R(M)$ is an open subset of $\text{Spec } R$ for any finite R -module M .*

Proof. Fix a prime ideal $\mathfrak{p} \in \text{FID}_R(M)$ and an integer j with $0 \leq j \leq \text{ht } \mathfrak{p}$. By the definition of an excellent ring, the regular locus $\text{Reg}(R/\mathfrak{p})$ is an open subset of $\text{Spec } R/\mathfrak{p}$. The zero ideal of R/\mathfrak{p} belongs to $\text{Reg}(R/\mathfrak{p})$, hence it is nonempty. Noting that any module over a regular local ring has finite injective dimension, we see that $\text{Reg}(R/\mathfrak{p})$ is contained in the locus $\text{FID}_{R/\mathfrak{p}}(\text{Ext}_R^j(R/\mathfrak{p}, M))$. Thus all the assumptions of Theorem 2.5 are satisfied, and it follows that $\text{FID}_R(M)$ is open in $\text{Spec } R$. \square

We denote by $\text{Gor}(R)$ the *Gorenstein locus* of R , that is, the subset of $\text{Spec } R$ consisting of all prime ideals \mathfrak{p} of R such that $R_{\mathfrak{p}}$ is a Gorenstein local ring. Since $\text{Gor}(R)$ coincides with $\text{FID}_R(R)$, the above corollary yields a result of Greco and Marinari [3, Corollary 1.5]:

Corollary 2.7 (Greco–Marinari). *Let R be an excellent ring. Then the Gorenstein locus $\text{Gor}(R)$ is open in $\text{Spec } R$.*

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