

FLAT RING EPIMORPHISMS AND UNIVERSAL LOCALISATIONS OF COMMUTATIVE RINGS

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ABSTRACT. We study different types of localisations of a commutative noetherian ring. More precisely, we provide criteria to decide: (a) if a given flat ring epimorphism is a universal localisation in the sense of Cohn and Schofield; and (b) when such universal localisations are classical rings of fractions. In order to find such criteria, we use the theory of support and we analyse the specialisation closed subset associated to a flat ring epimorphism. In case the underlying ring is locally factorial or of Krull dimension one, we show that all flat ring epimorphisms are universal localisations. Moreover, it turns out that an answer to the question of when universal localisations are classical depends on the structure of the Picard group. We furthermore discuss the case of normal rings, for which the divisor class group plays an essential role to decide if a given flat ring epimorphism is a universal localisation. Finally, we explore several (counter)examples which highlight the necessity of our assumptions.

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1. INTRODUCTION

Flat epimorphisms of commutative noetherian rings is a very classical topic which was studied in the 1960's in the context of algebraic geometry (see [8, 19]). Since then, powerful ring theoretic and homological techniques such as universal localisation [30] and triangulated localisation [24] were developed. With their help, we are now able to obtain a rather complete picture of how flat epimorphisms look like and how they are controlled by Picard groups, divisor class groups and local cohomology.

In commutative algebra, different types of localisations can be classified geometrically by studying certain subsets of the prime spectrum. More precisely, for a commutative noetherian ring A , it was shown in [24] that the assignment of support yields a bijection between localising subcategories of the derived category $\mathcal{D}(A)$ and arbitrary subsets of $\text{Spec}(A)$. This bijection restricts to a correspondence between smashing subcategories and specialisation closed subsets, which again are in bijection with hereditary torsion classes of the module category $\text{Mod-}A$. Classically, these torsion classes are linked to the so-called Gabriel localisations and hence to flat ring epimorphisms starting in A (see [8, 32]).

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One of the key observations here is that a different type of ring theoretical localisation, so-called universal localisation, is very closely connected to flat ring epimorphisms in the context of commutative noetherian rings. In fact, on many occasions flat epimorphisms and universal localisations coincide.

Universal localisations were first introduced in [30] in order to extend classical localisation theory in a meaningful way to non-commutative rings. Universal localisations are constructed by adjoining universally the inverses to a set Σ of maps between finitely generated projective A -modules. As a result, we obtain a ring epimorphism $A \rightarrow A_\Sigma$ satisfying $\mathrm{Tor}_1^A(A_\Sigma, A_\Sigma) = 0$. Universal localisations have proved to be useful in many different areas of mathematics (see [1, 25, 27]), but the concept still remains rather mysterious and classification results are only available in a few cases.

In the context of commutative noetherian rings, however, universal localisation acquires a very concrete geometric meaning: instead of making universally invertible an element of the ring, that is, a section of the structure sheaf of the corresponding affine scheme, we rather make invertible a section in a general invertible sheaf.

Following this point of view, we prove the following result. The first assertion does not require A to be noetherian, but the second does; otherwise, there is a counterexample. For the sake of completeness, we remind the reader that if $A \rightarrow B$ is a ring epimorphism and A is commutative, so is B . If, moreover, B is flat over A and A is noetherian, so is B (see [31, 19]).

Theorem 1.1 (Corollary 4.4 and Proposition 4.5). *Let A be a commutative ring.*

- (1) *Any universal localisation of A is a flat ring epimorphism.*
- (2) *A ring epimorphism $A \rightarrow B$ with A noetherian is flat if and only if $\mathrm{Tor}_1^A(B, B) = 0$.*

Flat ring epimorphisms of commutative noetherian rings are related to the notion of coherent subsets of $\mathrm{Spec}(A)$, which has been introduced by Krause (see [16]). More precisely, it turns out that flat ring epimorphisms $A \rightarrow B$ bijectively correspond to the smashing subcategories in $\mathcal{D}(A)$ whose orthogonal class is closed under cohomologies, which in the terminology of [16] precisely means that the corresponding specialisation closed subset of $\mathrm{Spec}(A)$ has coherent complement. We characterise such specialisation closed subsets in terms of vanishing of local cohomology and we obtain the following classification of flat ring epimorphisms.

Theorem 1.2 (Theorem 4.9 and Corollary 4.10). *Let A be a commutative noetherian ring. Let U be a generalisation closed subset of $\mathrm{Spec}(A)$. Then U is coherent if and only if $H_V^i(A) = 0$ for all $i > 1$, where $V = \mathrm{Spec}(A) \setminus U$, and the latter condition implies that the minimal primes in V are of height ≤ 1 . In particular, there is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{flat ring epimorphisms from } A \end{array} \right\} \xrightarrow{1-1} \left\{ \begin{array}{l} \text{generalisation closed} \\ \text{coherent subsets of } \mathrm{Spec}(A) \end{array} \right\}.$$

Note that the concept of coherent subset of the spectrum, originally introduced in module-theoretic terms, now, acquires a geometric interpretation via local cohomology.

Recall from Theorem 1.1 that universal localisations always yield flat ring epimorphisms. Clearly, classical localisations at a multiplicative set S are always universal localisations with respect to the set of maps $A \rightarrow A$ given by multiplication with an element from S . The reverse implications, however, generally do not hold – we obtain the following hierarchy of inclusions

$$\left\{ \begin{array}{l} \text{equivalence classes} \\ \text{of classical} \\ \text{localisations} \\ f: A \rightarrow B \end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{l} \text{equivalence classes} \\ \text{of universal} \\ \text{localisations} \\ f: A \rightarrow B \end{array} \right\} \xrightarrow{\Psi} \left\{ \begin{array}{l} \text{equivalence classes} \\ \text{of flat ring} \\ \text{epimorphisms} \\ f: A \rightarrow B \end{array} \right\} \xrightarrow{\Theta} \left\{ \begin{array}{l} \text{specialisation closed} \\ \text{subsets } V \subseteq \mathrm{Spec}(A) \\ \text{with minimal primes} \\ \text{of height } \leq 1 \end{array} \right\}$$

where Θ is injective and sends $f: A \rightarrow B$ to the complement of the image of $f^b: \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$.

In order to distinguish universal localisations from classical rings of fractions (inclusion Φ), we will analyse the structure of the Picard group $\mathrm{Pic}(A)$ formed by isoclasses of invertible (hence projective) modules. In order to decide if a given flat ring epimorphism is a universal localisation (inclusion Ψ), we will study the corresponding specialisation closed subset (injection Θ). The condition on the height of the minimal primes suggests to consider Weil divisors on A , i.e. formal integer combinations of prime ideals of height one, together with the associated divisor class group $\mathrm{Cl}(A)$, which is the quotient group obtained by factoring out the principal Weil divisors. This group will be of particular relevance in the case of normal rings, where it contains the Picard group as a subgroup. We obtain the following main results.

Theorem 1.3. *Let A be a commutative noetherian ring.*

- (1) (Theorems 5.4 and 5.13) *If $\text{Pic}(A)$ is torsion, the inclusion Φ is an equality, i.e. every universal localisation is a classical ring of fractions. Moreover, the converse holds true if A is locally factorial.*
- (2) (Theorems 5.7, 5.13 and 5.17) *Suppose that A satisfies one of the following conditions:*
 - (a) *A has Krull dimension at most one, or*
 - (b) *A is locally factorial, or (more generally)*
 - (c) *A is normal and $\text{Cl}(A)/\text{Pic}(A)$ is torsion.**Then the inclusions Ψ and Θ are bijections. In particular, flat ring epimorphisms are universal localisations.*
- (3) (Proposition 6.4 and Theorem 5.17) *If A is normal, has Krull dimension two and is a G -ring (e.g. a flat epimorphic image of a finitely generated algebra over a field or of a complete local ring), then Θ is a bijection. In this case, Ψ is an equality if and only if $\text{Cl}(A)/\text{Pic}(A)$ is torsion, and both Φ and Ψ are equalities if and only if $\text{Cl}(A)$ is torsion.*

We will show, through various examples, that most assumptions in the theorem cannot be omitted.

In case A is normal, it follows that every flat ring epimorphism is a classical localisation whenever $\text{Cl}(A)$ is torsion (and so are $\text{Pic}(A)$ and $\text{Cl}(A)/\text{Pic}(A)$). This was essentially already observed in [19, IV, Proposition 4.5]. In case A is even locally factorial, we obtain that all flat ring epimorphisms are classical localisations if and only if $\text{Pic}(A) \cong \text{Cl}(A)$ is torsion. For Dedekind domains this goes back to [31, Remark on p. 47] and [19, IV, Proposition 4.6].

Finally, observe that statement (2), case (c) above is not presented in its most general form. In fact, for a specialisation closed subset V it is enough to know that all minimal prime ideals are of height one and torsion in $\text{Cl}(A)/\text{Pic}(A)$ to guarantee that V is associated with a universal localisation f . Conversely, whenever we know that V has a unique minimal prime ideal \mathfrak{p} , then the existence of a corresponding universal localisation f already implies that \mathfrak{p} is torsion in $\text{Cl}(A)/\text{Pic}(A)$.

Structure of the paper: We begin in Section 2 with some preliminaries on ring theoretical and categorical localisations. In Section 3, we focus on commutative rings and recall some facts on supports and local cohomology, followed by a discussion of several groups associated to a commutative ring: the Picard group, the Cartier divisors, the Weil divisors and the divisor class group. Section 4 explores some general facts on the relation between flat ring epimorphisms and universal localisations of commutative noetherian rings, while Section 5 is devoted to developing the theory behind Theorem 1.3 in this introduction. Finally, in Section 6 we explore some examples showing that our assertions of Section 5 cannot be naively generalised.

Notation. Let A be a ring (in most occasions, commutative noetherian), and let $\text{Mod-}A$ be the category of all right A -modules. We denote by $\mathcal{D}(A)$ the unbounded derived category of $\text{Mod-}A$. All subcategories considered are strict, i.e. closed under isomorphic images. For a subcategory \mathcal{B} of a category \mathcal{A} , we denote by \mathcal{B}^\perp the full subcategory formed by the objects X of \mathcal{A} such that $\text{Hom}_{\mathcal{A}}(B, X) = 0$ for all B in \mathcal{B} . When A is a commutative ring with prime spectrum $\text{Spec}(A)$, and \mathfrak{p} is in $\text{Spec}(A)$, we denote by $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residue field of A at \mathfrak{p} .

2. PRELIMINARIES: LOCALISATIONS

2.1. Localisations of rings. A **ring epimorphism** is a ring homomorphism which is an epimorphism in the category of rings. It is well-known that $f: A \rightarrow B$ is a ring epimorphism if and only if $B \otimes_A \text{Coker } f = 0$ or equivalently, if and only if the associated restriction functor $f_*: \text{Mod-}B \rightarrow \text{Mod-}A$ is fully faithful. We say that two ring epimorphisms $f_1: A \rightarrow B_1$ and $f_2: A \rightarrow B_2$ are **equivalent** if $f_1 = \psi f_2$ for a ring isomorphism ψ . The equivalence classes of this relation are called **epiclasses**.

A full subcategory \mathcal{Y} of $\text{Mod-}A$ is **bireflective** if the inclusion functor $i: \mathcal{Y} \rightarrow \text{Mod-}A$ has both a left adjoint and a right adjoint functor, or equivalently, \mathcal{Y} is closed under products, coproducts, kernels, and cokernels. There is a well-known bijection (see [9, Theorem 1.2][30, Theorem 4.8]) between epiclasses of ring epimorphisms $A \rightarrow B$ (with $\text{Tor}_1^A(B, B) = 0$) and bireflective (extension-closed) subcategories of $\text{Mod-}A$. This assignment sends an epimorphism $f: A \rightarrow B$ to the essential image of the restriction functor $f_*: \text{Mod-}B \rightarrow \text{Mod-}A$.

A ring epimorphism $f: A \rightarrow B$ is said to be a **(left) flat epimorphism** if B is a flat (left) A -module. More generally, f is a **homological ring epimorphism** if $\text{Tor}_i^A(B, B) = 0$ for all $i > 0$, or equivalently, if the restriction functor $f_*: \mathcal{D}(B) \rightarrow \mathcal{D}(A)$ induced by f is fully faithful ([10, Theorem 4.4]). An important tool to construct ring epimorphisms is the concept of universal localisations. As shown in [30,

Theorem 4.1], if Σ is a set of morphisms between finitely generated projective right A -modules, then there are a ring A_Σ and a morphism of rings $f_\Sigma: A \rightarrow A_\Sigma$, called the **universal localisation of A at Σ** , such that

- (1) f_Σ is **Σ -invertig**, i.e. if $\alpha: P \rightarrow Q$ belongs to Σ , then $\alpha \otimes_A 1_{A_\Sigma}: P \otimes_A A_\Sigma \rightarrow Q \otimes_A A_\Sigma$ is an isomorphism of right A_Σ -modules, and
- (2) f_Σ is **universal Σ -invertig**, i.e. for any Σ -invertig ring homomorphism $\psi: A \rightarrow B$, there is a unique ring homomorphism $\bar{\psi}: A_\Sigma \rightarrow B$ such that $\bar{\psi}f_\Sigma = \psi$.

The homomorphism f_Σ is a ring epimorphism and, moreover, $\text{Tor}_1^A(A_\Sigma, A_\Sigma) = 0$.

Easy examples of universal localisations can be obtained from the classical construction of a ring of fractions of a commutative ring A with denominators in a multiplicative set S (which we will often refer to as **classical localisations**). Indeed, these are universal localisations at the set of A -module endomorphisms of A given by multiplication by the elements of S . Note, however, that universal localisations of an arbitrary ring A in general are not flat, and not even homological ring epimorphisms. We will, however, prove in Section 4 that universal localisations of commutative rings yield flat ring epimorphisms.

2.2. Categorical localisations. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\text{Mod-}A$ is said to be a **torsion pair** if $\text{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$ and, for every A -module M , there is a short exact sequence

$$0 \rightarrow T \rightarrow M \rightarrow F \rightarrow 0$$

with T in \mathcal{T} and F in \mathcal{F} . It turns out that the modules T and F in such a sequence depend functorially on M , and the endofunctor of $\text{Mod-}A$ sending M to T is said to be the **torsion radical** associated to \mathcal{T} . The subcategory \mathcal{T} is said to be a **torsion class** and \mathcal{F} is said to be a **torsionfree class**. Such classes in $\text{Mod-}A$ can be characterised by closure conditions (we refer to [32, Chapter VI] for details).

A torsion class \mathcal{T} in $\text{Mod-}A$ is said to be **hereditary** if it is closed under submodules, or equivalently, if the corresponding torsionfree class $\mathcal{F} := \mathcal{T}^\perp$ is closed under injective envelopes. A hereditary torsion class yields an (exact) quotient functor of abelian categories $i^*: \text{Mod-}A \rightarrow \text{Mod-}A/\mathcal{T}$. The inclusion functor $j_!: \mathcal{T} \rightarrow \text{Mod-}A$ and the quotient functor $i^*: \text{Mod-}A \rightarrow \text{Mod-}A/\mathcal{T}$ induce a **localising sequence of abelian categories**

$$(\mathcal{L}_{\mathcal{T}}) \quad \begin{array}{ccc} & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{array} & \\ \text{Mod-}A/\mathcal{T} & \xrightarrow{i_*} & \text{Mod-}A \\ & \begin{array}{c} \xrightarrow{j_!} \\ \xleftarrow{j^*} \end{array} & \\ & & \mathcal{T} \end{array}$$

i.e. (i^*, i_*) and $(j_!, j^*)$ are adjoint pairs, $j_!$ and i_* are fully faithful and $\text{Ker}(i^*) = \text{Im}(j_!)$. It follows that $\text{Im}(i_*)$ can be identified with $\mathcal{T}^\perp \cap \text{Ker Ext}_A^1(\mathcal{T}, -)$ and that the composition $j_!j^*$ is the torsion radical associated to \mathcal{T} . A well-known theorem states that every flat ring epimorphism $f: A \rightarrow B$ arises in this way, that is, the subcategory $\mathcal{T}_f := \text{Ker}(- \otimes_A B)$ is a hereditary torsion class inducing a localising sequence such that $\text{Mod-}A/\mathcal{T}_f \cong \text{Mod-}B$. Moreover, it is easy to see that the localising sequences $(\mathcal{L}_{\mathcal{T}})$ that arise from left flat ring epimorphisms are precisely those for which i_* has a right adjoint (see [8] and [32, XI, Theorem 2.1] for details).

A triangulated subcategory of $\mathcal{D}(A)$ is called **localising** if it is closed under coproducts, and a localising subcategory \mathcal{S} of $\mathcal{D}(A)$ is called **smashing** whenever the inclusion functor $j_!: \mathcal{S} \rightarrow \mathcal{D}(A)$ admits a right adjoint that preserves coproducts. In this case, the inclusion $j_!$ and the quotient functor $i^*: \mathcal{D}(A) \rightarrow \mathcal{D}(A)/\mathcal{S}$ induce a so-called **recollement of triangulated categories**

$$\begin{array}{ccccc} & \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{array} & & \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \end{array} & \\ \mathcal{D}(A)/\mathcal{S} & \xrightarrow{i_*} & \mathcal{D}(A) & \xrightarrow{j^*} & \mathcal{S} \\ & \begin{array}{c} \xrightarrow{i^!} \\ \xleftarrow{j_*} \end{array} & & \begin{array}{c} \xrightarrow{j_!} \\ \xleftarrow{j_*} \end{array} & \end{array}$$

i.e. $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples, i_* , $j_!$, and j_* are fully faithful and $\text{Ker}(i^*) = \text{Im}(j_!)$. It follows that $\text{Im}(i_*)$ can be identified with \mathcal{S}^\perp .

Example 2.1. If $f: A \rightarrow B$ is a left flat ring epimorphism, then $\mathcal{S}_f := \text{Ker}(- \otimes_A^{\mathbb{L}} B)$ is a smashing subcategory and there is an associated recollement as above, where $\mathcal{S} = \mathcal{S}_f$, $j_!$ is the inclusion functor, $\mathcal{D}(A)/\mathcal{S}_f$ is identified with $\mathcal{D}(B)$ and i^* is the derived tensor functor $- \otimes_A^{\mathbb{L}} B$. Moreover, \mathcal{S}_f can be identified with the smallest localising subcategory of $\mathcal{D}(A)$ containing the cone of f when viewed as a morphism in $\mathcal{D}(A)$, and we have $i^! = \text{RHom}_A(B, -)$ (see, for example, [26]).

In Section 4, we will show that for a commutative noetherian ring A a smashing subcategory \mathcal{S} of $\mathcal{D}(A)$ arises from a flat ring epimorphism in the way above if and only if its orthogonal class \mathcal{S}^\perp is closed

under cohomologies. This observation will provide a derived analogue of the well-known characterisation of flat ring epimorphisms as localisations mentioned above.

3. PRELIMINARIES: COMMUTATIVE ALGEBRA

In this section, we assume A to be a commutative noetherian ring. Recall that the Zariski-closed sets are of the form $V(I) := \{\mathfrak{p} \in \text{Spec}(A) \mid I \subseteq \mathfrak{p}\}$, for an ideal I of A . If I is principal ideal generated by an element x of A , we simply write $V(x)$.

3.1. Supports and local cohomology. Recall that a prime ideal \mathfrak{p} is **associated** to an A -module M if A/\mathfrak{p} is isomorphic to a submodule of M . We denote by $\text{Ass } M$ the set of prime ideals associated to M . Following [16], we define the **support** of a complex of A -modules X as

$$\text{supp } X = \{\mathfrak{p} \in \text{Spec}(A) \mid X \otimes_A^{\mathbb{L}} k(\mathfrak{p}) \neq 0\}.$$

It is well-known (see [7, Proposition 2.8 and Remark 2.9] or [16, Lemma 3.3]) that for an A -module M , $\text{supp } M$ coincides with the subset of $\text{Spec}(A)$ formed by those prime ideals that occur as associated primes of the injective modules appearing in a minimal injective coresolution of M .

A subset V of $\text{Spec}(A)$ is said to be **specialisation closed** if for any primes $\mathfrak{p} \subseteq \mathfrak{q}$, if \mathfrak{p} lies in V , then so does \mathfrak{q} . Equivalently, V is specialisation closed if it is a union of Zariski-closed subsets of $\text{Spec}(A)$. In particular, every specialisation closed subset is the union of the Zariski-closures of the primes \mathfrak{p} which are minimal in V . Observe that, by Nakayama's lemma, the support of a finitely generated module M coincides with the prime ideals \mathfrak{p} in $\text{Spec}(A)$ such that $M \otimes_A A_{\mathfrak{p}} \neq 0$ (sometimes referred to as the **classical support of M**). In general, the classical support of a module M is the specialisation-closure of $\text{supp } M$, see [3, Lemma 2.2]. This shows that a specialisation closed subset V of $\text{Spec}(A)$ contains $\text{supp } M$ if and only if it contains the classical support of M . Since the latter has the same minimal elements as $\text{Ass } M$, it also follows that $\text{supp } M \subseteq V$ if and only if $\text{Ass } M \subseteq V$.

In some cases, the support of a complex X is determined by the support of its cohomology modules, that is $\text{supp } X = \bigcup_{i \in \mathbb{Z}} \text{supp } H^i(X)$. For instance, if X is a bounded complex of finitely generated A -modules, we can compute $\text{supp } X$ by taking the classical support of its cohomology modules. For a two-term complex of finitely generated projective A -modules $X: P_{-1} \xrightarrow{\sigma} P_0$, this means that $\text{supp } X$ consists of the prime ideals \mathfrak{p} in $\text{Spec}(A)$ for which $\sigma_{\mathfrak{p}} = \sigma \otimes_A A_{\mathfrak{p}}$ is not an isomorphism. We will often denote the two-term complex X simply by the map σ , writing, for example, $\text{supp } \sigma$ instead of $\text{supp } X$.

The notion of support turns out to be fundamental in the understanding of hereditary torsion classes in $\text{Mod-}A$ and smashing subcategories of $\mathcal{D}(A)$.

Theorem 3.1. [32, Ch.VI,§5,6][24] *Let A be a commutative noetherian ring. There are bijections between*

- (1) *hereditary torsion classes in $\text{Mod-}A$;*
- (2) *smashing subcategories of $\mathcal{D}(A)$;*
- (3) *specialisation closed subsets of $\text{Spec}(A)$.*

More precisely, the bijections are given by assigning to a hereditary torsion class or to a smashing subcategory its support, which is then specialisation closed in $\text{Spec}(A)$.

Remark 3.2. The bijective correspondence between (1) and (2) in the theorem above can also be made explicit. To a hereditary torsion class \mathcal{T} in $\text{Mod-}A$ we assign the smallest localising subcategory of $\mathcal{D}(A)$ containing \mathcal{T} . In the other direction, to a smashing subcategory \mathcal{S} of $\mathcal{D}(A)$ we associate the subcategory $\mathcal{S} \cap \text{Mod-}A$ of $\text{Mod-}A$.

Given a specialisation closed subset V , we denote the associated hereditary torsion class by \mathcal{T}_V and the associated smashing subcategory by \mathcal{S}_V .

Remark 3.3. It also follows from [24] that \mathcal{S}_V^{\perp} coincides with the subcategory of $\mathcal{D}(A)$ formed by the objects with support contained in $\text{Spec}(A) \setminus V$. Indeed, in [24] it is shown that the assignment of support yields a bijection between localising subcategories of $\mathcal{D}(A)$ and subsets of $\text{Spec}(A)$. Moreover, the support of a localising subcategory \mathcal{L} can be detected as the subset of $\text{Spec}(A)$ formed by the prime ideals \mathfrak{p} whose residue field $k(\mathfrak{p})$ lies in \mathcal{L} . As shown in [24, Lemma 3.5], either $k(\mathfrak{p})$ lies in \mathcal{L} or $k(\mathfrak{p})$ lies in \mathcal{L}^{\perp} . Now, if \mathcal{L} is a smashing subcategory, then \mathcal{L}^{\perp} is localising and, by the arguments above, the associated subsets of the spectrum are complementary.

If we consider the recollement

$$(3.1) \quad \begin{array}{ccccc} & & i^* & & j_! \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{S}_V^\perp & \xrightarrow{i_*} & \mathcal{D}(A) & \xrightarrow{j^*} & \mathcal{S}_V \\ & \curvearrowleft & & \curvearrowright & \\ & & i^! & & j_* \end{array}$$

with $j_!$ and i_* being just the inclusion functors, it turns out that the functor $j_!j^*$ coincides with the right derived functor of the torsion radical Γ_V of \mathcal{T}_V . Indeed, if X is a homotopically injective complex, we have a (by [4, Corollary 2.1.5] even componentwise split) short exact sequence

$$0 \longrightarrow \Gamma_V(X) \longrightarrow X \longrightarrow X/\Gamma_V(X) \longrightarrow 0,$$

which induces a triangle in $\mathcal{D}(A)$. Now it follows from [24, Lemma 2.10] that $\Gamma_V(X) \in \mathcal{S}_V$ and $X/\Gamma_V(X) \in \mathcal{S}_V^\perp$, so $\Gamma_V(X) \longrightarrow X$ is an \mathcal{S}_V -reflection, as claimed.

Example 3.4. Let $f: A \longrightarrow B$ be a flat ring epimorphism, and let V be the specialisation closed subset corresponding to the smashing subcategory $\mathcal{S}_f := \text{Ker}(- \otimes_A^{\mathbb{L}} B)$ from Example 2.1. It follows from Remark 3.3 that a prime ideal \mathfrak{p} belongs to $\text{Spec}(A) \setminus V$ if and only if $B \otimes_A k(\mathfrak{p}) \cong k(\mathfrak{p})$, and it belongs to V if and only if $B \otimes_A k(\mathfrak{p}) = 0$, or equivalently, $B = \mathfrak{p}B$.

Moreover, it is shown in [19, IV, Proposition 2.1] that all ideals $J \subseteq B$ are of the form $J = IB$ where $I = f^{-1}(J)$. Hence the map of spectra $f^b: \text{Spec}(B) \longrightarrow \text{Spec}(A)$, $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$ is a homeomorphic embedding of $\text{Spec}(B)$ onto $\text{Spec}(A) \setminus V = \text{supp } B$, see also [19, IV, Proposition 1.4 and Corollaire 2.2].

Definition 3.5. The **local cohomology** of an A -module M with respect to a specialisation closed subset V is $H_V^i(M) := H^i(\mathbb{R}\Gamma_V M)$, where Γ_V is the torsion radical of \mathcal{T}_V . We will denote $\Gamma_{V(I)}(M)$ and $H_{V(I)}^i(M)$ simply by $\Gamma_I(M)$ and $H_I^i(M)$, respectively.

Note that $\Gamma_V(M) = \{x \in M \mid \text{supp } xA \subseteq V\}$ and, thus, we may recover from above the classical sheaf-theoretic definition of local cohomology from [12, Chapter IV]. Alternatively, we also have that $H_V^i(M) = H^i(j_!j^*M)$, where $j_!$ and j^* are functors appearing in the recollement of $\mathcal{D}(A)$ induced by \mathcal{S}_V .

Two well-known properties of local cohomology are the independence of the base (see, for example, [4, §4.2]) and the flat base change (see, for example, [4, §4.3]). These properties were classically studied for closed subsets and, in the following, we generalise the statement to specialisation closed subsets.

First, for each A -module M and each specialisation closed subset V of $\text{Spec}(A)$ we have the equality

$$\Gamma_V(M) = \bigcup_{I \subseteq A, V(I) \subseteq V} \Gamma_I(M).$$

Indeed, if x is an element of M with $\text{supp } xA \subseteq V$, then we can write $\text{supp } xA = V(I) \subseteq V$ where I is the annihilator of x . Next, we note that this union is directed since the union of finitely many Zariski closed subsets of $\text{Spec}(A)$ is again Zariski closed. Now recall that the local cohomology functors of M are computed by applying the torsion radical to the injective resolution of M . Since cohomology commutes with direct limits, we conclude (cf. [12, Page 219, Motif D]) that for all $i \geq 0$

$$H_V^i(M) = \varinjlim_{I \subseteq A, V(I) \subseteq V} H_I^i(M).$$

Taking direct limits of the isomorphisms given by the Independence of the Base Theorem and the Flat Base Change Theorem we obtain the following statement.

Proposition 3.6. *Let $f: A \longrightarrow B$ be a homomorphism of commutative noetherian rings. Consider the induced map of spectra $f^b: \text{Spec}(B) \longrightarrow \text{Spec}(A)$ and the associated restriction functor $f_*: \text{Mod-B} \longrightarrow \text{Mod-A}$. Suppose that $V \subseteq \text{Spec}(A)$ is specialisation closed and $W = (f^b)^{-1}(V)$. Then, for all $i \geq 0$,*

- (1) *there is a natural equivalence of functors $f_*(H_W^i(-)) \cong H_V^i(f_*(-)): \text{Mod-B} \longrightarrow \text{Mod-A}$;*
- (2) *if B is flat over A , then there is a natural equivalence of functors*

$$H_W^i(- \otimes_A B) \cong H_V^i(-) \otimes_A B: \text{Mod-A} \longrightarrow \text{Mod-B}.$$

A classic result of Grothendieck states that, for a commutative noetherian ring A , $H_V^i(A) = 0$ for all degrees i larger than the Krull dimension of A (see, for example, [4, Theorem 6.1.2]). The *Hartshorne-Lichtenbaum Vanishing Theorem*, on the other hand, is a useful tool to detect the non-vanishing of local cohomology when the degree equals the Krull dimension. Recall that for a commutative noetherian local ring A with maximal ideal \mathfrak{m} , the completion \widehat{A} of A at \mathfrak{m} (defined as the ring $\varprojlim A/\mathfrak{m}^n$) is again a commutative noetherian local ring of the same Krull dimension.

Theorem 3.7. [4, Theorem 8.2.1] *Let A be a local commutative noetherian ring of Krull dimension d and with maximal ideal \mathfrak{m} . Given a proper ideal I of A , the following statements are equivalent.*

- (1) $H_I^d(A) = 0$;
- (2) *for every prime ideal \mathfrak{q} of \widehat{A} satisfying $\dim \widehat{A}/\mathfrak{q} = d$, we have $\dim \widehat{A}/(I\widehat{A} + \mathfrak{q}) > 0$.*

Remark 3.8. If A is a local commutative noetherian ring of Krull dimension d and with maximal ideal \mathfrak{m} , since $\dim \widehat{A}/(\mathfrak{m}\widehat{A} + \mathfrak{q}) = 0$ for any prime \mathfrak{q} of \widehat{A} , it follows from the theorem above that $H_{\mathfrak{m}}^d(A) \neq 0$. This statement is known as *Grothendieck's non-vanishing Theorem* (see [4, Theorem 6.1.4]).

Note also that the primes \mathfrak{q} of \widehat{A} satisfying $\dim \widehat{A}/\mathfrak{q} = d$ must necessarily be minimal primes of \widehat{A} .

3.2. Invertible ideals and divisors. An element of A is **regular** if it is not a zero-divisor. Let K denote the (classical) localisation of A at all regular elements, that is, the total ring of fractions of A . A finitely generated A -module M is said to be **invertible** if it is locally free of rank one (i.e. $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(A)$), and it is said to be a **fractional ideal of A** if it is an A -submodule of K . Note that, since flatness can be checked locally, an invertible A -module is, by definition, flat, and since it is also finitely generated, it must be projective. Invertible modules are so called due to the well-known fact that the evaluation map yields an isomorphism $M \otimes_A M^* \cong A$, where $M^* = \text{Hom}_A(M, A)$ ([6, Theorem 11.6a]). Isoclasses of invertible A -modules form an abelian group under the operation \otimes_A . This group is called the **Picard group of A** and it is denoted by $\text{Pic}(A)$. Fractional ideals are so named due to the fact that, once we choose a common denominator $g \in A$ for the generators of such a module M , we see that gM can be identified with an actual ideal of A , which is isomorphic to M as an A -module.

We quickly recall some properties of these modules over a commutative noetherian ring A with total ring of fractions K . We refer to [6, Theorem 11.6 and Corollary 11.7] for details.

- Every invertible A -module is isomorphic to an invertible fractional ideal of A , and every invertible fractional ideal of A contains a regular element.
- Invertible fractional ideals form an abelian group for the operation of multiplication of A -submodules of K . The inverse of an invertible fractional ideal M is $M^{-1} := \{s \in K : sM \subseteq A\}$. This is the group of **Cartier divisors of A** and is denoted by $C(A)$. It can be shown that the group $C(A)$ is generated by the invertible ideals of A .
- The assignment sending an invertible fractional ideal M to its isoclass yields a surjective homomorphism of abelian groups $C(A) \longrightarrow \text{Pic}(A)$, whose kernel is isomorphic to K^\times/A^\times .

The free abelian group with basis formed by the prime ideals of A of height one is denoted by $\text{Div}(A)$ and its elements (i.e. formal integer combinations of such prime ideals) are called **(Weil) divisors**. A Weil divisor is said to be **effective** if it is a non-negative integer combination of prime ideals of height one. Given a Weil divisor $x = \sum n_{\mathfrak{p}}\mathfrak{p}$, we say that x is **supported on** the (finite) subset of $\text{Spec}(A)$ formed by the primes \mathfrak{p} with $n_{\mathfrak{p}} \neq 0$.

Let I be an invertible ideal of A and let $\mathfrak{p} \in V(I)$ be a prime ideal of height one (we write $\text{ht}(\mathfrak{p}) = 1$). Since I contains a regular element, \mathfrak{p} corresponds to one of the finitely many minimal primes in A/I , and $\dim A_{\mathfrak{p}}/I_{\mathfrak{p}} = 0$. Hence the $A_{\mathfrak{p}}$ -module $A_{\mathfrak{p}}/I_{\mathfrak{p}}$ has finite length $\ell(A_{\mathfrak{p}}/I_{\mathfrak{p}}) > 0$ for a finite number of primes \mathfrak{p} of height one, and vanishes for the others. Therefore, we may define a map $\text{div}: C(A) \longrightarrow \text{Div}(A)$ sending an invertible ideal I of A to

$$\text{div}(I) := \sum_{\text{ht}(\mathfrak{p})=1} \ell(A_{\mathfrak{p}}/I_{\mathfrak{p}}) \cdot \mathfrak{p}.$$

It turns out that this map is a homomorphism of abelian groups (see [6, Theorem 11.10]). A general formula for div can be derived from the fact that the invertible ideals of A generate $C(A)$. If I is an invertible fractional ideal, then

$$\text{div}(I) := \text{div}(gI) - \text{div}(gA)$$

where g is a regular element of A such that $gI \subseteq A$.

Remark 3.9. If I is an invertible ideal of A (hence a projective A -module) and σ denotes its inclusion into A , then $\ell(A_{\mathfrak{p}}/I_{\mathfrak{p}})$ in the expression above is non-zero precisely on the primes \mathfrak{p} of height one which lie in $\text{supp } \sigma$. In other words, we have that $\text{div}(I)$ is supported on $\text{supp } \sigma \cap \{\mathfrak{p} \in \text{Spec}(A) : \text{ht}(\mathfrak{p}) = 1\}$.

A divisor is said to be **principal** if it is the image under div of a cyclic invertible fractional ideal, that is, it has the form $\text{div}(sA)$ for some $s \in K^\times$. We write $\text{div}(s)$ for short. The **divisor class group** $\text{Cl}(A)$ of A is defined to be the quotient of $\text{Div}(A)$ by the subgroup $\text{PDiv}(A)$ of principal divisors.

Remark 3.10. Our reference [6, §11.5] uses a different terminology and notation. The group $\text{Cl}(A)$ is called the codimension-one Chow group and is denoted by $\text{Chow}(A)$ there.

Note that in particular, for an invertible fractional ideal I and g regular in A such that $gI \subseteq A$, the divisors $\text{div}(I)$ and $\text{div}(gI)$ get identified in $\text{Cl}(A)$. In fact, it is easy to see that the image of $\text{div}(I)$ in $\text{Cl}(A)$ depends only on the isoclass of I . Therefore, the composition $\text{C}(A) \rightarrow \text{Div}(A) \rightarrow \text{Cl}(A)$ factors through a group homomorphism $\bar{\text{div}}: \text{Pic}(A) \rightarrow \text{Cl}(A)$, and we get a commutative diagram:

$$(3.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \{sA : s \in K^\times/A^\times\} & \longrightarrow & \text{C}(A) & \longrightarrow & \text{Pic}(A) \longrightarrow 0 \\ & & \downarrow \text{div} & & \downarrow \text{div} & & \downarrow \bar{\text{div}} \\ 0 & \longrightarrow & \text{PDiv}(A) & \longrightarrow & \text{Div}(A) & \longrightarrow & \text{Cl}(A) \longrightarrow 0 \end{array}$$

In Section 6.2 we will also discuss variants of groups of divisors for graded rings and their induced projective schemes.

4. FLAT RING EPIMORPHISMS AND UNIVERSAL LOCALISATIONS: GENERAL FACTS

In this section, we discuss some general facts about ring epimorphisms and universal localisations of commutative rings. Note that if A is a commutative ring and $A \rightarrow B$ is a ring epimorphism, then also B is commutative ([31, Corollary 1.2]). Moreover, if A is noetherian and $A \rightarrow B$ is flat, then also B is noetherian ([19, IV, Corollary 2.3]). We begin with a general construction of new ring epimorphisms from given ones.

Lemma 4.1. *Let A be an arbitrary ring, $f: A \rightarrow B$ and $g: A \rightarrow C$ be ring epimorphisms with associated bireflective subcategories \mathcal{X}_B and \mathcal{X}_C . Denote by $B \sqcup_A C$ (together with the morphisms $h_B: B \rightarrow B \sqcup_A C$ and $h_C: C \rightarrow B \sqcup_A C$) the pushout of f and g in the category of rings. Then h_B and h_C are ring epimorphisms and the bireflective subcategory associated to the composition $h_B f = h_C g$ is given by $\mathcal{X}_B \cap \mathcal{X}_C$.*

Proof. It follows from the very definition of the pushout that h_B and h_C are ring epimorphisms (and thus so is the composition $h_B f = h_C g$). Moreover, an A -module X belongs to $\mathcal{X}_B \cap \mathcal{X}_C$ if and only if the A -action $\alpha_X: A \rightarrow \text{End}_{\mathbb{Z}}(X)$ factors through both $f: A \rightarrow B$ and $g: A \rightarrow C$. Using the universal property of the pushout, this is further equivalent to α_X factorising through $A \rightarrow B \sqcup_A C$ or, in other words, to X lying in the bireflective subcategory $\mathcal{X}_{B \sqcup_A C}$ associated to the composition $h_B f = h_C g$. \square

Note that in the lemma above, if A is a commutative ring, then the pushout $B \sqcup_A C$ is given by $B \otimes_A C$ (together with the two natural maps $h_B = \text{id}_B \otimes_A 1_C$ and $h_C = 1_B \otimes_A \text{id}_C$). In particular, given a ring epimorphism $f: A \rightarrow B$ and a prime \mathfrak{p} in $\text{Spec}(A)$, there is a ring epimorphism $f_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ and, consequently, also a ring epimorphism $A \rightarrow B_{\mathfrak{p}}$ which we shall denote by $\hat{f}_{\mathfrak{p}}$.

4.1. Universal localisations. Let us now turn our attention to universal localisations of a commutative ring A . Our first target is to show that these localisations always yield flat ring epimorphisms (independently of A being noetherian or not). In order to do so, we begin with studying universal localisations from a local point of view. We need the following auxiliary results.

Lemma 4.2. *Let A be a commutative ring and let $\sigma: A^n \rightarrow A^m$ be a linear map with associated matrix $M \in M_{m \times n}(A)$.*

- (1) *If $n = m$, then σ is an isomorphism if and only if $\det M$ is invertible in A .*
- (2) *If $n \neq m$ and σ is an isomorphism, then $A = 0$.*

Proof. (1) is well known. (2) follows from the fact that any non-zero commutative ring has the invariant basis number property (i.e. isomorphic finitely generated free modules must have the same rank). \square

Lemma 4.3. *Any universal localisation of a local commutative ring is a classical localisation.*

Proof. This follows from the previous lemma, since finitely generated projective modules over local rings are well-known to be free. \square

Given a prime ideal \mathfrak{p} in $\text{Spec}(A)$ and a universal localisation $f_{\Sigma}: A \rightarrow A_{\Sigma}$, we can construct two further ring epimorphisms, namely $(f_{\Sigma})_{\mathfrak{p}}: A_{\mathfrak{p}} \rightarrow (A_{\Sigma})_{\mathfrak{p}}$ and $(\hat{f}_{\Sigma})_{\mathfrak{p}}: A \rightarrow (A_{\Sigma})_{\mathfrak{p}}$. Note that it follows from Lemma 4.1 and the very definition of universal localisation that $(f_{\Sigma})_{\mathfrak{p}}$ is the universal (and hence classical) localisation of $A_{\mathfrak{p}}$ at the set $\Sigma_{\mathfrak{p}} := \{A_{\mathfrak{p}} \otimes_A \sigma \mid \sigma \in \Sigma\}$ and that $(\hat{f}_{\Sigma})_{\mathfrak{p}}$ is the universal localisation of A at the set $\Sigma \cup \{A \xrightarrow{s} A \mid s \in A \setminus \mathfrak{p}\}$. Now we have the desired result.

Corollary 4.4. *Any universal localisation of a commutative ring is a flat ring epimorphism.*

Proof. We can check flatness of A_Σ locally. Now it is enough to observe that by Lemma 4.3 the universal localisation $(f_\Sigma)_\mathfrak{p} : A_\mathfrak{p} \rightarrow (A_\Sigma)_\mathfrak{p}$ must be classical and, hence, $(A_\Sigma)_\mathfrak{p}$ is a flat $A_\mathfrak{p}$ -module. \square

4.2. Flat ring epimorphisms. Throughout this subsection, we consider a commutative noetherian ring A and we study flat ring epimorphisms $A \rightarrow B$. We begin with an easy homological description.

Proposition 4.5. *Let A be a commutative noetherian ring. Then a ring epimorphism $f : A \rightarrow B$ is flat if and only if $\mathrm{Tor}_1^A(B, B) = 0$.*

Proof. We only prove the if-part as the other implication is trivial. In view of [22, Theorem 7.1], it is sufficient to prove that for each $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} = f^\flat(\mathfrak{q})$, the induced ring epimorphism $A_\mathfrak{p} \rightarrow B_\mathfrak{q}$ is flat. Thus, we may without loss of generality assume that A and B are local and f is a local homomorphism (i.e. $f(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m} \subseteq A$ and $\mathfrak{n} \subseteq B$ are the maximal ideals). We will prove that under these assumptions, f is even an isomorphism.

To this end, let \mathcal{X}_B denote the essential image of the restriction functor in $\mathrm{Mod}\text{-}A$. By assumption, \mathcal{X}_B is an exact abelian subcategory closed under products and extensions. Since $\bar{f} : A/\mathfrak{m} \rightarrow B/\mathfrak{n}$ is a non-zero ring epimorphism and A/\mathfrak{m} is a field, \bar{f} is an isomorphism by [31, Corollary 1.2] (see also [19, IV, Corollaire 1.3]). In particular, A/\mathfrak{m} lies in \mathcal{X}_B . It then follows inductively that A/\mathfrak{m}^n is in \mathcal{X}_B for each $n \geq 1$ and also that $\hat{A} = \varprojlim A/\mathfrak{m}^n$ belongs to \mathcal{X}_B . Thus, $f \otimes_A \hat{A}$ is an isomorphism. Since A is noetherian, \hat{A} is a faithfully flat A -module by [22, Theorems 7.2 and 8.8] and the conclusion follows since $\hat{A} \otimes_A -$ reflects isomorphisms. \square

Remark 4.6. Note that the fact that A is noetherian is needed only in the very last step of the proof to infer that $\hat{A} \otimes_A -$ reflects isomorphisms. This may fail for non-noetherian commutative rings as may the conclusion of Proposition 4.5 (see [2, Theorem 7.2]).

Remark 4.7. Alternatively, one can deduce Proposition 4.5 as a consequence of [16, Lemma 3.5], which implies that every injective B -module is injective as an A -module. It then suffices to apply this to the character module $B^+ := \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$.

Corollary 4.8. *Let A be a commutative noetherian ring. Then every bireflective extension-closed subcategory of $\mathrm{Mod}\text{-}A$ is equivalent to $\mathrm{Mod}\text{-}A/\mathcal{T}$ for some hereditary torsion class \mathcal{T} .*

Proof. A bireflective extension-closed subcategory corresponds to a ring epimorphism $f : A \rightarrow B$ with $\mathrm{Tor}_1^A(B, B) = 0$ (see Subsection 2.1). The proposition above then asserts that f is a flat ring epimorphism and thus arises from a categorical localisation (see Subsection 2.2). \square

Next, we are aiming for a restricted version of Theorem 3.1 identifying the smashing subcategories of $\mathcal{D}(A)$ and the specialisation closed subsets of $\mathrm{Spec}(A)$ that correspond to flat ring epimorphisms.

Theorem 4.9. *Let A be a commutative noetherian ring and let V be a specialisation closed subset of $\mathrm{Spec}(A)$ with associated smashing subcategory \mathcal{S}_V . The following statements are equivalent.*

- (1) \mathcal{S}_V^\perp is closed under taking cohomologies.
- (2) $H_V^k(A) = 0$ for all $k > 1$.
- (3) There is a flat ring epimorphism $f : A \rightarrow B$ such that $\mathcal{S}_V = \mathrm{Ker}(- \otimes_A^{\mathbb{L}} B)$ and $\mathcal{S}_V^\perp \cong \mathcal{D}(B)$.
- (4) The modules supported in $\mathrm{Spec}(A) \setminus V$ form an exact abelian extension-closed subcategory of $\mathrm{Mod}\text{-}A$.

In particular, if V satisfies the conditions above, then the minimal primes of V have height zero or one.

Proof. Consider the recollement induced by the smashing subcategory \mathcal{S}_V and fix the notation for its functors as in (3.1). The units and counits of the adjunctions in the recollement induce a triangle

$$j_!j^*A \rightarrow A \rightarrow i_*i^*A \rightarrow j_!j^*A[1].$$

The long exact sequence of cohomologies associated to this triangle tells us that, for all $k \geq 1$,

$$H^k(i_*i^*A) \cong H^{k+1}(j_!j^*A) = H_V^{k+1}(A).$$

(1) \Rightarrow (2): Suppose that \mathcal{S}_V^\perp is closed under taking cohomologies. Since $\mathcal{T}_V \subseteq \mathcal{S}_V$ by Remark 3.2, we infer that $H_V^{k+1}(A)$ lies in both \mathcal{S}_V and its orthogonal \mathcal{S}_V^\perp for all $k \geq 1$, that is, $H_V^k(A) = 0$ for all $k > 1$.

(2) \Rightarrow (3): Recall from [26] that every recollement of $\mathcal{D}(A)$ is equivalent to one induced by a homological epimorphism of differential graded algebras $f : A \rightarrow B$, where B is identified, as an object in $\mathcal{D}(A)$,

with i_*i^*A . Let us compute the cohomologies of i_*i^*A . By definition of local cohomology $H^k(j_!j^*A) = H_V^k(A) = 0$ for all $k < 0$, and $H^0(j_!j^*A) = \Gamma_V(A)$ embeds in A . It follows from the triangle above that $H^k(i_*i^*A) = 0$ for all $k < 0$. Now the assumption (2) guarantees that i_*i^*A is an A -module and f is a homological epimorphism of rings up to quasi-isomorphism. Then f is a flat ring epimorphism by Proposition 4.5, and we can invoke Example 2.1.

(3) \Rightarrow (4): By Remark 3.3, the complexes supported in $\text{Spec}(A) \setminus V$ are precisely those in \mathcal{S}_V^\perp . Hence, the A -modules supported in $\text{Spec}(A) \setminus V$ are those in $\mathcal{S}_V^\perp \cap \text{Mod-}A$. By assumption (3), these are precisely the A -modules in the essential image of the restriction functor f_* , which naturally form an exact abelian and extension-closed subcategory of $\text{Mod-}A$.

(4) \Rightarrow (1): This follows from [16, Theorem 1.1].

Finally, let \mathfrak{p} be a minimal prime in V . In particular, the preimage of V under $\text{Spec}(A_{\mathfrak{p}}) \rightarrow \text{Spec}(A)$ is just $\{\mathfrak{p}A_{\mathfrak{p}}\}$. Then, by Proposition 3.6(2), we have $(H_V^k(A))_{\mathfrak{p}} \cong H_{\mathfrak{p}A_{\mathfrak{p}}}^k(A_{\mathfrak{p}})$. Now, if condition (2) is satisfied, Remark 3.8 guarantees that \mathfrak{p} has height at most one. \square

In [16], the subsets $W \subseteq \text{Spec}(A)$ satisfying that the modules supported in W form an exact abelian and extension-closed subcategory of $\text{Mod-}A$ are called **coherent**. Note that the class of modules supported in any subset W is always extension-closed and that every specialisation closed subset V is coherent (the modules supported there form the hereditary torsion class \mathcal{T}_V). Theorem 4.9 describes when the complement of a specialisation closed subset is coherent in terms of local cohomology. This may be regarded as a geometric interpretation of coherence.

The following corollary provides a restriction of the bijections in Theorem 3.1.

Corollary 4.10. *Let A be a commutative noetherian ring. There are bijections between*

- (1) *epiclasses of flat ring epimorphisms $A \rightarrow B$;*
- (2) *smashing subcategories \mathcal{S} of $\mathcal{D}(A)$ for which \mathcal{S}^\perp is closed under taking cohomologies;*
- (3) *specialisation closed subsets of $\text{Spec}(A)$ with coherent complement.*

Proof. This is a direct consequence of Theorems 3.1 and 4.9. \square

Example 4.11. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a family of elements of A . The set $V = \bigcup_{\lambda \in \Lambda} V(x_\lambda)$ is a specialisation closed subset of $\text{Spec}(A)$ with coherent complement that corresponds to the classical localisation of A at the multiplicative subset S generated by the x_λ (see also [16, §4]). Indeed, a prime ideal \mathfrak{p} lies in $\text{supp } A_S$ if and only if $A_S \otimes_A k(\mathfrak{p}) \neq 0$ if and only if $\mathfrak{p} \cap \{x_\lambda\}_{\lambda \in \Lambda} = \emptyset$.

Example 4.12. Let k be a field, $A = k[[X, Y]]$ and consider the specialisation closed subset $V = \{\mathfrak{m}\}$ formed by the unique maximal ideal $\mathfrak{m} = (X, Y)$ in A . Since \mathfrak{m} has height 2, by Theorem 4.9, $\text{Spec}(A) \setminus \{\mathfrak{m}\}$ is not coherent (see also [16]) and the recollement induced by the smashing subcategory \mathcal{S}_V does not arise from a flat ring epimorphism.

Example 4.13. Let k be a field, and let $A = k[[X, Y, U]]/(XU)$ be a quotient of a formal power series ring. Then A is a 2-dimensional complete local hypersurface with maximal ideal $\mathfrak{m} = (X, Y, U)$. The ideal $\mathfrak{p} = (X, Y)$ is a prime ideal of A of height 1, but we claim that $\text{Spec}(A) \setminus V(\mathfrak{p})$ is not coherent. To see that, consider a hypothetical flat ring epimorphism $f: A \rightarrow B$ with $\text{supp } B = \text{Spec}(A) \setminus V(\mathfrak{p})$. If we put $\bar{A} = A/(U) \cong k[[X, Y]]$ and $\bar{B} = B \otimes_A \bar{A} \cong B/(f(U))$, then $\bar{f}: \bar{A} \rightarrow \bar{B}$ is a flat ring epimorphism and, by Remark 3.4, $\text{supp } \bar{B} = \text{Spec}(\bar{A}) \setminus \{\bar{\mathfrak{p}}\}$, where $\bar{\mathfrak{p}} = \mathfrak{p} + (U)/(U)$ is the maximal ideal of \bar{A} . However, such a flat ring epimorphism cannot exist by Example 4.12.

5. FLAT RING EPIMORPHISMS AND UNIVERSAL LOCALISATIONS: COMPARISONS

We have seen that universal localisations yield flat ring epimorphisms over any commutative ring A . In this section, we assume furthermore that A is noetherian and we approach the question of when the converse is true.

After some preliminaries, we start in §5.1 by showing that all universal localisations of A are classical rings of fractions whenever the Picard group $\text{Pic}(A)$ is torsion. We exhibit examples showing that this is not always the case, not even for Dedekind domains. In §5.2 we focus on rings of Krull dimension one. For such rings, flat ring epimorphisms coincide with universal localisations. §5.3 is devoted to normal rings. We show that the divisor class group $\text{Cl}(A)$ and its quotient $\text{Cl}(A)/\text{Pic}(A)$ by the Picard group determine whether a flat ring epimorphism is a classical ring of fractions or a universal localisation. In the special case of a locally factorial ring this means that flat ring epimorphisms coincide with universal localisations, and they further coincide with classical rings of fractions if and only if $\text{Pic}(A)$ is torsion.

We begin with a lemma which will be crucial in this context.

Lemma 5.1. *Let A be a commutative noetherian ring. Let further $f: A \rightarrow B$ be a flat ring epimorphism with associated specialisation closed subset V and let Σ be a set of maps between finitely generated projective A -modules. Then f is the universal localisation of A at Σ if and only if $V = \text{supp } \Sigma$.*

Proof. Let $\mathcal{S}_V = \text{Ker}(- \otimes_A^{\mathbb{L}} B)$ be the smashing subcategory associated to V . It follows from [25] that f is the universal localisation of A at Σ if and only if \mathcal{S}_V identifies with $\text{Loc } \Sigma$, the smallest localising subcategory of $\mathcal{D}(A)$ containing Σ . But the latter is further equivalent to $V = \text{supp } \Sigma$. \square

Remark 5.2. Similarly, we know from Example 4.11 that the universal localisation of A at a set Σ coincides with the classical localisation at a multiplicative subset $S \subset A$ if and only if $\text{supp } \Sigma = \bigcup_{s \in S} V(s)$.

5.1. Universal versus classical localisation. In order to provide a sufficient condition for a universal localisation to be a classical ring of fractions, we establish a relation with invertible ideals.

Proposition 5.3. *Let A be a commutative noetherian ring and Σ be a set of maps between finitely generated projective A -modules. Then the universal localisation of A at Σ coincides with the universal localisation of A*

- *at a set Γ of maps of the form $\gamma: A \rightarrow J_\gamma$, where J_γ is an invertible ideal of A ;*
- *at a set Γ' of maps of the form $\gamma: J_\gamma \rightarrow A$, where J_γ is an invertible ideal of A .*

Moreover, if A is a domain and $A_\Sigma \neq 0$, then the set Γ' above can be chosen to consist of inclusion maps.

Proof. Without loss of generality, we may assume that A is connected and, hence, all finitely generated projective modules have a well-defined rank (i.e. localisations of a projective module at prime ideals have constant rank). First, we claim that if Σ contains a map $\sigma: P \rightarrow Q$ between modules of different rank, then the localisation at Σ is zero and, in particular, it can be realised as claimed above. Observe that the difference between the ranks of $P_{\mathfrak{p}}$ and $Q_{\mathfrak{p}}$ and the fact that $(A_\Sigma)_{\mathfrak{p}} \otimes_A \sigma_{\mathfrak{p}}$ is an isomorphism imply by Lemma 4.2 that $(A_\Sigma)_{\mathfrak{p}} = 0$ for all primes \mathfrak{p} . Therefore, we have $A_\Sigma = 0$.

Hence, we assume that Σ contains only maps between projective modules of the same rank. Given a map $\sigma: P \rightarrow Q$ between modules of rank n , we claim that a homomorphism of commutative rings $f: A \rightarrow B$ inverts σ if and only if it inverts the n -fold exterior product $\wedge^n \sigma$. Indeed, this is the case because locally at a prime \mathfrak{p} , the map $\wedge^n \sigma_{\mathfrak{p}}$ is the multiplication by the determinant of $\sigma_{\mathfrak{p}}$, and the matrix $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \sigma_{\mathfrak{p}} \in M_{n \times n}(B_{\mathfrak{p}})$ is invertible if and only if so is its determinant (which is $f_{\mathfrak{p}}(\det(\sigma_{\mathfrak{p}}))$). We conclude that the universal localisation at Σ coincides with the universal localisation at $\wedge \Sigma := \{\wedge^{\text{rank } P} \sigma, (\sigma: P \rightarrow Q) \in \Sigma\}$.

Without loss of generality we may then assume that Σ is a set of maps between invertible A -modules. Note that for any A -module M , and for any $\sigma: P \rightarrow Q$ in Σ , the map $M \otimes_A \sigma$ is an isomorphism if and only if $M \otimes_A (\sigma \otimes_A P^*)$ is an isomorphism. Hence, we may once again replace the set Σ by the set $\Gamma := \{\sigma \otimes_A P^* : (\sigma: P \rightarrow Q) \in \Sigma\}$. Finally, observe that, up to isomorphism, each map $\sigma \otimes_A P^*$ is of the form $A \rightarrow J$ for some invertible ideal J of A . Analogously, the universal localisation of A at Σ identifies with the universal localisation of A at $\Gamma' := \{Q^* \otimes_A \sigma : (\sigma: P \rightarrow Q) \in \Sigma\}$.

Assume now that A is a domain. Clearly any non-zero map $\gamma: A \rightarrow J$ to an invertible ideal J is injective since J is a torsion-free module. Since the image of $\gamma \otimes_A J^*$ is an invertible ideal of A , the statement follows. \square

Theorem 5.4. *Let A be a commutative noetherian ring and Σ be a set of maps of the form $\sigma: A \rightarrow J$, with J an invertible ideal of A . If the isoclass of each J is a torsion element in $\text{Pic}(A)$, then the universal localisation at Σ is a classical ring of fractions. In particular, if $\text{Pic}(A)$ is torsion, then every universal localisation is a classical ring of fractions.*

Proof. For a map $\sigma: A \rightarrow J$ in Σ , let n_J denote the order of J in $\text{Pic}(A)$. Clearly, $J^{\otimes n_J} \cong A$ and, therefore, $\sigma^{\otimes n_J}: A^{\otimes n_J} \rightarrow J^{\otimes n_J}$ can be identified with the multiplication map by an element of A . We claim that the universal localisation of A at Σ coincides with the universal localisation at $\{\sigma^{\otimes n_J} : (\sigma: A \rightarrow J) \in \Sigma\}$ and that, therefore, it is an ordinary ring of fractions.

Analogously to the proof of the previous proposition, consider a ring homomorphism $f: A \rightarrow B$ and let us check that locally at any prime, f inverts a map $\sigma: A \rightarrow J$ as above if and only if it inverts $\sigma^{\otimes n_J}$. Indeed, at a prime \mathfrak{p} in $\text{Spec}(A)$, $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \sigma_{\mathfrak{p}}$ is given by multiplication by an element b of $B_{\mathfrak{p}}$ and, thus, $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \sigma_{\mathfrak{p}}^{\otimes n_J}$ is given by multiplication by b^{n_J} . Hence, $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \sigma_{\mathfrak{p}}^{\otimes n_J}$ is invertible if and only if $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \sigma_{\mathfrak{p}}$ is invertible, as wanted. \square

We will see below that for locally factorial rings the converse holds true as well: $\text{Pic}(A)$ is torsion if and only if every universal localisation is a classical ring of fractions.

Example 5.5. The ring $A = \mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain (hence, locally factorial) which is not a unique factorisation domain. Its Picard group is $\mathbb{Z}/2\mathbb{Z}$ and, thus, every universal localisation is a classical ring of fractions. The latter property is in fact common to all number fields A , i.e. the rings of integers of finite field extensions of \mathbb{Q} , as they are always Dedekind domains with finite (hence torsion) Picard group, [20, Ch. 5, Corollary 2].

Example 5.6. On the other hand, every abelian group occurs as the ideal class group of some Dedekind domain [5], so there are (non-local) Dedekind domains A admitting a universal localisation $A \rightarrow B$ which is not classical. Such a universal localisation is by [31, p. 47] always the inclusion of A to an intermediate ring $A \subseteq B \subseteq K$, where K is the quotient field of A (see Proposition 5.12 and Theorem 5.13 below for more information).

5.2. Rings of Krull dimension at most one. Next, we prove a classification result for rings satisfying a certain geometric condition, namely having Krull dimension at most one. The situation is similar to the results obtained in [17, Theorem 6.1] for possibly non-commutative hereditary rings and in [2, Theorem 6.8] for commutative semihereditary rings. Note that for rings of Krull dimension at most one, every subset of $\text{Spec}(A)$ is coherent (see [16, Corollary 4.3]) and, by Corollary 4.10, epiclassses of flat ring epimorphisms correspond bijectively to specialisation closed subsets of the spectrum. Given a prime ideal \mathfrak{p} in $\text{Spec}(A)$, we denote by $\Lambda(\mathfrak{p})$ the set of prime ideals of A which are contained in \mathfrak{p} . Note that $\Lambda(\mathfrak{p})$ with the induced topology is homeomorphic to $\text{Spec}(A_{\mathfrak{p}})$.

Theorem 5.7. *Let A be a commutative noetherian ring of Krull dimension at most one and let V be a specialisation closed subset of $\text{Spec}(A)$. Then there is a set of maps Σ between finitely generated projective A -modules such that $\text{supp}(\Sigma) = V$. As a consequence, every flat ring epimorphism is a universal localisation.*

Proof. For each minimal element \mathfrak{p} in the set V we find a map σ that is supported exactly on $V(\mathfrak{p})$. Since V is the union of such sets, the theorem will then follow. Let \mathfrak{p} be such a prime ideal. By prime avoidance, we can choose x to be an element of \mathfrak{p} that is not contained in any associated prime that does not lie in $V(\mathfrak{p})$, i.e. we can choose x in $\mathfrak{p} \setminus \bigcup\{\mathfrak{q} \mid \mathfrak{q} \in \text{Ass}(A) \setminus V(\mathfrak{p})\}$.

Suppose that $V(x) = V(\mathfrak{p})$. In this case, we show that the map $\sigma: A \rightarrow A$ given by multiplication by x is as wanted (see also [16, Remark 4.2(1)]). Indeed, given a prime ideal \mathfrak{q} , if x does not lie in \mathfrak{q} , then x is invertible in $A_{\mathfrak{q}}$ and, thus, $\sigma_{\mathfrak{q}}$ is an isomorphism. If x lies in \mathfrak{q} , then x lies in the unique maximal ideal of $A_{\mathfrak{q}}$ and, therefore, $\sigma_{\mathfrak{q}}$ is not surjective (and, thus, not an isomorphism). This shows that the support of σ is precisely $V(x)$, which by assumption is $V(\mathfrak{p})$.

Suppose now that $V(x) \supsetneq V(\mathfrak{p})$. The fact that A has Krull dimension at most one guarantees that all ideals in $V(x) \setminus V(\mathfrak{p})$ are both minimal and maximal prime ideals over A/xA . In particular, there are only finitely many such primes. Let

$$V(x) \setminus V(\mathfrak{p}) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}.$$

We observe that the localisation map $\pi: A/xA \rightarrow (A/xA)_{\mathfrak{m}_i}$ is surjective. Indeed, this can be checked locally using the standard fact that for a commutative noetherian ring R and $\mathfrak{a}, \mathfrak{b} \in \text{Spec}(R)$, we have $R_{\mathfrak{a}} \otimes_R R_{\mathfrak{b}} = 0$ if and only if $\Lambda(\mathfrak{a}) \cap \Lambda(\mathfrak{b}) = \emptyset$. In our setting, it then follows that $\pi_{\mathfrak{q}} = 0$ for any \mathfrak{q} in $\text{Spec}(A/xA) \setminus \{\mathfrak{m}_i\}$, and $\pi_{\mathfrak{m}_i}$ is an isomorphism. So π is locally surjective, and thus it is surjective.

As a consequence, also the composition $A \rightarrow A/xA \rightarrow (A/xA)_{\mathfrak{m}_i}$ is surjective. We consider its kernel I_i together with the short exact sequence

$$0 \rightarrow I_i \rightarrow A \rightarrow (A/xA)_{\mathfrak{m}_i} \rightarrow 0.$$

Since localisations are exact, and $(A/xA)_{\mathfrak{m}_i} \cong A_{\mathfrak{m}_i}/xA_{\mathfrak{m}_i}$, we have $(I_i)_{\mathfrak{q}} = A_{\mathfrak{q}}$ whenever $\mathfrak{q} \neq \mathfrak{m}_i$ and $(I_i)_{\mathfrak{m}_i} = xA_{\mathfrak{m}_i}$. An easy consequence is that the ideals I_1, \dots, I_n are pairwise coprime. Indeed, if $1 \leq i \neq j \leq n$, then $(I_i + I_j)_{\mathfrak{q}} = A_{\mathfrak{q}}$ for any \mathfrak{q} in $\text{Spec}(A)$, showing that $I_i + I_j = A$. Thus, we conclude that the product of ideals $I := I_1 I_2 \dots I_n$ equals the intersection $I_1 \cap I_2 \cap \dots \cap I_n$ and contains the element x .

We will show that I is a projective A -module and that the map $\sigma: A \rightarrow I$ given by multiplication by x is as wanted. To check the projectivity of I we see that I is locally projective. Indeed, since $I_{\mathfrak{q}} = (I_1)_{\mathfrak{q}}(I_2)_{\mathfrak{q}} \dots (I_n)_{\mathfrak{q}}$, it follows that $I_{\mathfrak{q}} = A_{\mathfrak{q}}$ if \mathfrak{q} does not lie in $\{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$, and $I_{\mathfrak{m}_i} = xA_{\mathfrak{m}_i}$. Note that since $\text{Ass}(A_{\mathfrak{m}_i}) = \text{Spec}(A_{\mathfrak{m}_i}) \cap \text{Ass}(A)$, by choice of x we have that x does not lie in the union of the ideals in $\text{Ass}(A_{\mathfrak{m}_i})$ and, thus, it is a regular element in $A_{\mathfrak{m}_i}$. This shows that also $I_{\mathfrak{m}_i}$ is projective and, therefore, so is I . It remains to see that the support of σ is $V(\mathfrak{p})$. We have the following cases.

- If \mathfrak{q} does not lie in $V(x)$, then $\sigma_{\mathfrak{q}}$ is an isomorphism because x is invertible in $A_{\mathfrak{q}}$;

- If \mathfrak{q} lies in $V(x) \setminus V(\mathfrak{p})$, i.e. $\mathfrak{q} = \mathfrak{m}_i$ for some i , then $I = xA_{\mathfrak{m}_i}$ and, since as seen above x is a regular element in $A_{\mathfrak{m}_i}$, it follows that $\sigma_{\mathfrak{m}_i}$ is an isomorphism;
- If \mathfrak{q} lies in $V(\mathfrak{p})$, then $I_{\mathfrak{q}} = A_{\mathfrak{q}}$ and $xA_{\mathfrak{q}}$ lies in $\mathfrak{p}A_{\mathfrak{q}} \subseteq \mathfrak{q}A_{\mathfrak{q}}$, thus showing that $\sigma_{\mathfrak{q}}$ is not surjective.

□

Combining Theorem 5.7 with the results of Section 4, we get the following immediate corollary.

Corollary 5.8. *If A is a commutative noetherian ring of Krull dimension at most one, then there are bijections between*

- (i) *epiclasses of universal localisations of A ;*
- (ii) *specialisation closed subsets of $\text{Spec}(A)$;*
- (iii) *smashing subcategories of $\mathcal{D}(A)$.*

Remark 5.9. Note that the theorem above also provides a refinement of the telescope conjecture for rings of Krull dimension one. Indeed, the telescope conjecture is known to hold for commutative noetherian rings: all smashing subcategories are compactly generated by [24]. Now we see that in Krull dimension one the set of compact generators can be chosen to consist of 2-term complexes.

5.3. Normal rings. We turn to a class of rings where we have additional information about the divisors and their groups. Recall that a commutative noetherian ring A is said to be **normal** if every localisation of A at a prime is an integrally closed domain.

Remark 5.10.

- (1) Regular rings are locally factorial (the localisations at primes are unique factorisation domains by Auslander-Buchsbaum's theorem). Moreover, locally factorial rings are normal since every unique factorisation domain is integrally closed. None of the implications is reversible.
- (2) A finite product of normal rings is normal and every normal ring is a finite product of normal domains. If a ring A is normal, then so are $A[X]$ and $A[[X]]$.
- (3) A ring of Krull dimension one is normal if and only if it is regular. More generally, if A is normal, then for every prime \mathfrak{p} of height one, the ring $A_{\mathfrak{p}}$ is regular (so a discrete valuation domain).
- (4) If a ring A is Cohen-Macaulay, then A is normal if and only if $A_{\mathfrak{p}}$ is regular for every prime \mathfrak{p} of height one, see [6, Theorem 11.5] and the discussion before it. In other words, a Cohen-Macaulay ring is normal if and only if it is regular in height one (the singular locus lives in height at least 2). Recall also that complete intersection rings and, more generally, Gorenstein rings are Cohen-Macaulay.

For commutative noetherian normal rings, we know more about the maps in (3.2).

Theorem 5.11. [6, Theorem 11.8b and Proposition 11.11] *If A is normal, then div and $\overline{\text{div}}$ are injective. If, moreover, A is locally factorial, then both div and $\overline{\text{div}}$ are in fact isomorphisms.*

Since every normal ring A decomposes into a finite product $A_1 \times A_2 \times \cdots \times A_n$ of normal domains, $\text{Mod-}A$ is consequently equivalent to $\text{Mod-}A_1 \times \text{Mod-}A_2 \times \cdots \times \text{Mod-}A_n$ and $\text{Spec}(A)$ is homeomorphic to the disjoint union $\text{Spec}(A_1) \amalg \text{Spec}(A_2) \amalg \cdots \amalg \text{Spec}(A_n)$. A specialisation closed subset $V \subseteq \text{Spec}(A)$ can thus be identified with an n -tuple (V_1, V_2, \dots, V_n) , where $V_i = V \cap \text{Spec}(A_i)$ is specialisation closed in $\text{Spec}(A_i)$. Correspondingly, we have an isomorphism $\text{Pic}(A) \cong \text{Pic}(A_1) \times \text{Pic}(A_2) \times \cdots \times \text{Pic}(A_n)$ as abelian groups, and similar isomorphisms for $\text{C}(A)$, $\text{Div}(A)$ and $\text{Cl}(A)$. Moreover, each ring epimorphism $f: A \rightarrow B$ decomposes into a product of ring epimorphisms $f_i: A_i \rightarrow B_i$. Clearly f is flat if and only if so are all the f_i , and the same is true for the property of being a classical or a universal localisation. As an upshot, whenever we ask for some general criteria to compare flat ring epimorphisms with universal or even classical localisations of A , we can restrict ourselves to the case when A is a domain.

For a (not necessarily normal) domain A there is a rather easy description of all flat ring epimorphisms. These are certain intermediate ring epimorphisms between A and its quotient field K . The following result essentially goes back to Richman [28].

Proposition 5.12. *Let A be a commutative noetherian domain and $f: A \rightarrow B$ be a non-zero flat ring epimorphism corresponding to a specialisation closed subset V of $\text{Spec}(A)$. Then f is equivalent to the inclusion*

$$A \longrightarrow \bigcap_{\mathfrak{p} \in \text{Spec}(A) \setminus V} A_{\mathfrak{p}} \quad (\subseteq K).$$

If, moreover, A is normal, then also B is normal and f is equivalent to

$$A \longrightarrow \bigcap_{\substack{\mathfrak{p} \in \text{Spec}(A) \setminus V \\ \text{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}} \quad (\subseteq K).$$

Proof. It follows from [31, Remark on p. 47] that f is equivalent to the inclusion map from A to a subring B' of K which is isomorphic to B . The formula $B' = \bigcap_{\mathfrak{p} \in \text{Spec}(A) \setminus V} A_{\mathfrak{p}}$ follows from [28, §2]. If A is normal, so is B' by [28, §4]. Then $B' = \bigcap_{\mathfrak{q} \in \text{Spec}(B'), \text{ht}(\mathfrak{q})=1} B'_{\mathfrak{q}}$ by [6, Corollary 11.4]. However, we have that $B'_{\mathfrak{q}}$ coincides with $A_{\mathfrak{p}}$ (for $\mathfrak{p} = A \cap \mathfrak{q}$) inside K . Moreover, the prime $\mathfrak{p} = A \cap \mathfrak{q}$ is of height one since the map $\mathfrak{q} \mapsto \mathfrak{q} \cap A$ identifies by [28, Theorem 3] (or Remark 3.4) the spectrum $\text{Spec}(B')$ with $\text{Spec}(A) \setminus V$, which is closed under generalisation. \square

Recall from Theorem 4.9 that the minimal primes in the specialisation closed subset associated with a flat ring epimorphism have height at most one. Let us first discuss the case of a locally factorial ring A . Note that in this case, prime ideals of height one are projective. Indeed, given \mathfrak{p} of height one and a prime ideal \mathfrak{q} containing \mathfrak{p} , we have that $\mathfrak{p}_{\mathfrak{q}}$ is a height one prime in $A_{\mathfrak{q}}$. Since $A_{\mathfrak{q}}$ is a unique factorisation domain, $\mathfrak{p}_{\mathfrak{q}}$ is a principal ideal and, thus, free as an $A_{\mathfrak{q}}$ -module. On the other hand, if \mathfrak{q} is a prime ideal not containing \mathfrak{p} , it is also clear that $\mathfrak{p}_{\mathfrak{q}} \cong A_{\mathfrak{q}}$ is a free $A_{\mathfrak{q}}$ -module. Hence \mathfrak{p} is projective as an A -module.

Theorem 5.13. *Let A be a locally factorial commutative noetherian ring. Suppose that V is a specialisation closed subset of $\text{Spec}(A)$ such that all minimal primes of V have height at most one. Then there is a set of maps Σ between finitely generated projective A -modules such that $\text{supp}(\Sigma) = V$. More precisely, we can take $\Sigma = \{\mathfrak{p} \hookrightarrow A : \mathfrak{p} \in \min V\}$.*

In particular, every flat ring epimorphism is a universal localisation. Moreover, every universal localisation is a classical ring of fractions if and only if $\text{Pic}(A)$ is torsion.

Proof. We assume without loss of generality that A is a domain. Excluding the trivial case where $V = \text{Spec}(A)$, we will assume that the minimal primes of V have height precisely one. Since A is locally factorial, it follows that every such prime ideal is projective as an A -module. Thus, the set of inclusion maps $\Sigma = \{\iota_{\mathfrak{p}} : \mathfrak{p} \hookrightarrow A : \mathfrak{p} \in \min V\}$ is indeed a set of maps between finitely generated projective A -modules and $\text{supp} \Sigma = V$ since $\text{supp} \iota_{\mathfrak{p}} = V(\mathfrak{p})$.

It follows from Lemma 5.1 that every flat ring epimorphism is a universal localisation. Since the if-part in the last statement is Theorem 5.4, it only remains to show that $\text{Pic}(A)$ is torsion if every universal localisation is a classical ring of fractions. Recall from Theorem 5.11 that $\text{Pic}(A) \cong \text{Cl}(A)$. Let \mathfrak{p} be a prime of height one. Since A is locally factorial, \mathfrak{p} is projective, and we may again consider the universal localisation at the inclusion map $\iota_{\mathfrak{p}} : \mathfrak{p} \hookrightarrow A$. By assumption, $A_{\iota_{\mathfrak{p}}}$ is the localisation at a multiplicative set S , hence $\bigcup_{s \in S} V(s) = \text{supp} \iota_{\mathfrak{p}} = V(\mathfrak{p})$ by Remark 5.2. As $V(\mathfrak{p})$ is irreducible, $V(\mathfrak{p}) = V(s)$ for some non-zero (hence regular) element s in S . That is, \mathfrak{p} is the unique prime ideal of height one containing s . It follows that $\text{div}(s) = n \cdot \mathfrak{p}$ for some $n > 0$ and we have $n \cdot \mathfrak{p} = 0$ in $\text{Cl}(A)$. \square

Note that for an arbitrary normal domain A , we cannot argue like in the theorem above, since a prime \mathfrak{p} of height one in $\text{Spec}(A)$ will not always be projective. We can, however, view \mathfrak{p} as an element in $\text{Div}(A)$ and we can ask if \mathfrak{p} (or finite sums of copies of \mathfrak{p}) can be written as $\text{div}(I)$ for an invertible ideal I of A . This amounts to asking whether \mathfrak{p} is a torsion element in $\text{Cl}(A)/\text{Pic}(A)$ rather than in $\text{Pic}(A)$.

Let us return to the pullback diagram of abelian groups from Section 3

$$(5.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \{sA : s \in K^{\times}/A^{\times}\} & \longrightarrow & \text{C}(A) & \longrightarrow & \text{Pic}(A) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \text{div} & & \downarrow \overline{\text{div}} \\ 0 & \longrightarrow & \text{PDiv}(A) & \longrightarrow & \text{Div}(A) & \longrightarrow & \text{Cl}(A) \longrightarrow 0 \end{array}$$

where the maps div and $\overline{\text{div}}$ are injective. The injectivity of div allows to reconstruct an invertible fractional ideal from its Weil divisor. The following lemma shows an explicit formula for such a reconstruction. We will use that if \mathfrak{p} is a prime ideal of A of height one, then the normality of A implies that $A_{\mathfrak{p}}$ is a discrete valuation domain (Remark 5.10(3)) and, thus, $\mathfrak{p}A_{\mathfrak{p}}$ is an invertible ideal of $A_{\mathfrak{p}}$.

Lemma 5.14. *Let A be normal and I be an invertible fractional ideal with $\text{div}(I) = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \cdot \mathfrak{p}$. Then $I = \bigcap_{\mathfrak{p}} (\mathfrak{p}A_{\mathfrak{p}})^{n_{\mathfrak{p}}}$, where \mathfrak{p} runs over all prime ideals of height one. If, moreover, $\text{div}(I)$ is effective, then I is an invertible ideal of A .*

Proof. As explained above, we can assume without loss of generality that A is a domain. We start with the case where $\text{div}(I)$ is effective. By assumption, there is a regular element g in A such that gI is an invertible ideal of A and we have $\text{div}(I) = \text{div}(gI) - \text{div}(gA)$. For any prime \mathfrak{p} in $\text{Spec}(A)$ of height one, it follows that $\ell(A_{\mathfrak{p}}/gI_{\mathfrak{p}}) \geq \ell(A_{\mathfrak{p}}/gA_{\mathfrak{p}})$. Since, by assumption, $A_{\mathfrak{p}}$ is a discrete valuation ring, this further implies that $gI_{\mathfrak{p}} \subseteq gA_{\mathfrak{p}}$ and, hence, $I_{\mathfrak{p}}$ is contained in $A_{\mathfrak{p}}$. As a consequence, I is contained in $\bigcap_{\mathfrak{p} \in \text{Spec}(A), \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ which coincides with A by [6, Corollary 11.4].

Now we write $\text{div}(I) = n_1 \cdot \mathfrak{p}_1 + n_2 \cdot \mathfrak{p}_2 + \cdots + n_s \cdot \mathfrak{p}_s$ with $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ pairwise distinct and $n_i \geq 0$ for all $i = 1, \dots, s$. We claim that the prime ideals associated to A/I are all among $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Indeed, if \mathfrak{p} is associated to A/I , then $\mathfrak{p}A_{\mathfrak{p}}$ is associated to $A_{\mathfrak{p}}/I_{\mathfrak{p}}$. However, since I is invertible, $I_{\mathfrak{p}}$ is principal and generated by a regular element and, consequently, \mathfrak{p} is of height one by [6, Theorem 11.5(i)]. Thus $\mathfrak{p} = \mathfrak{p}_i$ for some $i = 1, \dots, s$ by the very definition of the map div . This proves the claim.

Since $I \subseteq I_{\mathfrak{p}_i} = (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i}$, we clearly have $I \subseteq J$, where we put $J = A \cap \bigcap_{i=1}^s (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i}$. Moreover, since localisation commutes with finite intersections by left exactness, we have $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for each \mathfrak{p} of height one. In particular, $\text{supp } J/I$ does not intersect the associated primes of A/I , which implies that $J/I = 0$ by [6, Corollary 3.5(b)]. Using the equality $A = \bigcap_{\mathfrak{p} \in \text{Spec}(A), \text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ again, we obtain the desired expression $I = \bigcap_{\mathfrak{p} \in \text{Spec}(A), \text{ht}(\mathfrak{p})=1} (\mathfrak{p}A_{\mathfrak{p}})^{n_{\mathfrak{p}}}$.

Suppose finally that I is an arbitrary invertible fractional ideal and g in A is such that $gI \subseteq A$. If we write $\text{div}(gA) = \sum_{\mathfrak{p}} m_{\mathfrak{p}} \cdot \mathfrak{p}$, then $gI = \bigcap_{\mathfrak{p}} (\mathfrak{p}A_{\mathfrak{p}})^{m_{\mathfrak{p}} + n_{\mathfrak{p}}}$. Now it suffices to notice that the multiplication by g^{-1} induces an A -module automorphism of the total ring of fractions of A which maps gI to I and $(\mathfrak{p}A_{\mathfrak{p}})^{m_{\mathfrak{p}} + n_{\mathfrak{p}}}$ to $(\mathfrak{p}A_{\mathfrak{p}})^{n_{\mathfrak{p}}}$. \square

Remark 5.15. If A is normal, $I \subseteq A$ is an invertible ideal and $\text{div}(I) = n_1 \cdot \mathfrak{p}_1 + n_2 \cdot \mathfrak{p}_2 + \cdots + n_s \cdot \mathfrak{p}_s$, then $I = \bigcap_{i=1}^s \mathfrak{p}_i^{(n_i)}$, where $\mathfrak{p}_i^{(n_i)}$ is the n_i -**th symbolic power** of \mathfrak{p}_i , [6, §3.9]. Indeed, recall that $\mathfrak{p}_i^{(n_i)}$ is defined as the \mathfrak{p}_i -primary component in the primary decomposition of \mathfrak{p}^{n_i} and, by [6, §3.3], we have $\mathfrak{p}_i^{(n_i)} = A \cap (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i}$. We have seen in the proof of Lemma 5.14 that $I = A \cap \bigcap_{i=1}^s (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i} = \bigcap_{i=1}^s (A \cap (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i})$, which gives the desired equality.

The next example shows that, without the assumption of normality, even the last statement in the lemma above may fail.

Example 5.16. Consider the ring $A = \mathbb{C}[X, Y]/(Y^2 - X^3)$ and its invertible fractional ideal I generated by Y/X . We claim that $\text{div}(I)$ is effective. Since $\text{div}(I) = \text{div}(YA) - \text{div}(XA)$, it is enough to check that $\ell(A_{\mathfrak{p}}/YA_{\mathfrak{p}}) \geq \ell(A_{\mathfrak{p}}/XA_{\mathfrak{p}})$ for all non-zero prime ideals \mathfrak{p} in $\text{Spec}(A)$. But $\ell(A_{\mathfrak{p}}/XA_{\mathfrak{p}}) \neq 0$ is only possible if X lies in \mathfrak{p} or, equivalently, if \mathfrak{p} is the prime ideal of A generated by X and Y . But in this case, we have $3 = \ell(A_{\mathfrak{p}}/YA_{\mathfrak{p}}) > \ell(A_{\mathfrak{p}}/XA_{\mathfrak{p}}) = 2$. Therefore, $\text{div}(I)$ is effective even though I is not an ideal of A .

We are now ready to prove the main result of this subsection.

Theorem 5.17. *Let A be a normal commutative noetherian ring and V be a specialisation closed subset of $\text{Spec}(A)$ such that all minimal primes of V have height at most one. Then the following holds.*

- *If all minimal primes \mathfrak{p} in V which are of height exactly one are torsion in $\text{Cl}(A)/\text{Pic}(A)$, then there exists a universal localisation $f: A \rightarrow B$ with associated specialisation closed subset V . Conversely, if \mathfrak{p} is a prime ideal of height one and $f: A \rightarrow B$ is a universal localisation with associated specialisation closed subset $V = V(\mathfrak{p})$, then \mathfrak{p} is torsion in $\text{Cl}(A)/\text{Pic}(A)$.*
- *If all minimal primes \mathfrak{p} in V which are of height one are torsion in $\text{Cl}(A)$, then f as above is a classical localisation. Conversely, if $V = V(\mathfrak{p})$ and f is a classical localisation, \mathfrak{p} is torsion in $\text{Cl}(A)$.*

Proof. We assume without loss of generality that A is a domain and 0 does not lie in V . In order to show that there exists a universal localisation $f: A \rightarrow B$ corresponding to V it suffices to find for each minimal prime \mathfrak{p} in V a map σ between finitely generated projective A -modules such that $\text{supp } \sigma = V(\mathfrak{p})$. Let \mathfrak{p} be such a prime and suppose that \mathfrak{p} is torsion when seen as an element in $\text{Cl}(A)/\text{Pic}(A)$. Then there is an invertible fractional ideal I of A whose isoclass is mapped via $\overline{\text{div}}$ to $n \cdot \mathfrak{p}$ in $\text{Cl}(A)$ for some $n > 0$. Using the pullback diagram (5.1), it follows that we can choose I in a way such that also $\text{div}(I)$ equals $n \cdot \mathfrak{p}$ in $\text{Div}(A)$. By Lemma 5.14, this implies that I is an ideal of A . Define σ to be the inclusion $I \rightarrow A$. It follows from Remark 3.9 that $\text{supp } \sigma = V(\mathfrak{p})$. Note that if \mathfrak{p} is even torsion when seen as an element in $\text{Cl}(A)$, then the isoclass of I is torsion in $\text{Pic}(A)$, that is, there is some $m > 0$ such that $I^m = sA$ is a principal ideal of A . Instead of localising at σ we can now equivalently localise at the map $A \xrightarrow{s} A$ turning f into a classical localisation.

Conversely, suppose that $V = V(\mathfrak{p})$ and that f is the universal localisation of A at a set Σ of maps between finitely generated projective A -modules whose associated specialisation closed subset is V . By Lemma 5.1, we have $V(\mathfrak{p}) = \text{supp } \Sigma$. Since $V(\mathfrak{p})$ is irreducible, there is some map σ in Σ whose support is precisely $V(\mathfrak{p})$. By Proposition 5.3, we can assume that σ is just the inclusion map of an invertible ideal I into A . It then follows that $\text{div}(I)$ is of the form $n \cdot \mathfrak{p}$ for some $n \in \mathbb{N}$, which clearly implies that \mathfrak{p} is torsion when seen as an element in $\text{Cl}(A)/\text{Pic}(A)$. Note that in case f is even a classical localisation, I is a principal ideal of A and $\text{div}(I) = n \cdot \mathfrak{p}$ becomes zero in $\text{Cl}(A)$ or, in other words, \mathfrak{p} is torsion when seen as an element in $\text{Cl}(A)$. \square

Remark 5.18. When A is locally factorial, the quotient $\text{Cl}(A)/\text{Pic}(A)$ will always be zero, and we recover Theorem 5.13.

Given a prime \mathfrak{p} of height one over a normal domain A , we will see in the forthcoming section that the specialisation closed subset $V(\mathfrak{p})$ does not necessarily arise from a flat ring epimorphism $A \rightarrow B$ or, equivalently, its complement in $\text{Spec}(A)$ is not necessarily coherent. So far we only know from Theorem 5.17 that whenever \mathfrak{p} is torsion when seen as an element in $\text{Cl}(A)/\text{Pic}(A)$, then $V(\mathfrak{p})$ must be of the form $\text{supp } \sigma$ for a map σ between finitely generated projective A -modules. In particular, $V(\mathfrak{p})$ is then associated to a universal localisation of A . In order to discuss examples of flat ring epimorphisms that are not universal localisations, we need to find some different criteria to guarantee that a subset of the form $V(\mathfrak{p})$ admits coherent complement in $\text{Spec}(A)$. We will also deal with this in the next section.

6. EXAMPLES

In this section we discuss some examples of noetherian normal domains A and

- (1) primes \mathfrak{p} of height one such that the complement of $V(\mathfrak{p})$ is not coherent (i.e. there is no flat ring epimorphism $A \rightarrow B$ associated with $V(\mathfrak{p})$), and
- (2) primes \mathfrak{p} of height one such that there is a flat ring epimorphism $A \rightarrow B$ associated with $V(\mathfrak{p})$, but it is not a universal localisation.

By Theorems 5.7 and 5.13, such examples must be at least 2-dimensional and cannot be locally factorial. In the case where A has Krull dimension exactly two, the singular locus must consist of maximal ideals of height two by normality.

Along the way, we obtain a useful criterion for the existence of flat ring epimorphisms in the case of normal rings of Krull dimension two (Proposition 6.4).

6.1. Closed subsets with non-coherent complement. We start out with the first type of examples, i.e. primes \mathfrak{p} of height one such that the complement of $V(\mathfrak{p})$ is not coherent. We have, in fact, already seen this phenomenon in Example 4.13, although in that case A was not a domain. We provide a first easy example in a domain of Krull dimension three (a more comprehensive analysis of flat epimorphisms originating from a non-complete version of this ring will be given in Proposition 6.13).

Example 6.1. [14, Exercise 33] Let $A = k[[X, Y, U, V]]/(XU - YV)$ and $\mathfrak{p} = (X, Y)$. Clearly, A is a three-dimensional domain and, as $A/\mathfrak{p} \cong k[[U, V]]$, \mathfrak{p} is a prime ideal of height one. Moreover, A is normal by Remark 5.10(4) since it is a complete intersection ring and the singular locus of A consists only of the maximal ideal $\mathfrak{m} = (X, Y, U, V)$. We claim that $H_{\mathfrak{p}}^2(A) \neq 0$ and, hence, $V(\mathfrak{p})$ does not have coherent complement by Theorem 4.9. Indeed, notice first that $H_{\mathfrak{p}}^3 \cong 0$ since \mathfrak{p} has two generators, [4, Theorem 3.3.1]. In particular, $H_{\mathfrak{p}}^2$ is right exact and it suffices to show that $H_{\mathfrak{p}}^2(R) \neq 0$, where $R = A/(V) \cong k[[X, Y, U]]/(XU)$. However, $H_{\mathfrak{p}}^2(R) \cong H_{\mathfrak{p}_R}^2(R)$ by the independence of the base (see [4, §4.2]), and the latter local cohomology module is non-zero by Example 4.13.

There also exist examples of Krull dimension two, but they are more demanding as it turns out that they cannot be localisations of finitely generated algebras over a field. We will explain the reason here as this discussion is also necessary for §6.3.

Recall from [22, §32] that a commutative noetherian ring A is a **G-ring**, if the completion morphism $A_{\mathfrak{p}} \rightarrow \widehat{A}_{\mathfrak{p}}$ is a regular map for each $\mathfrak{p} \in \text{Spec}(A)$. We will not discuss the definition more in detail here, but we rather point out the following classes of G-rings.

Lemma 6.2. *Finitely generated commutative algebras over a field as well as complete local commutative noetherian rings are G-rings. Furthermore, the class of G-rings is closed under flat epimorphic images (in particular, under classical localisations).*

Proof. The first sentence follows from [22, Theorem 32.3] and Corollary to [22, Theorem 32.6]. If $A \rightarrow B$ is a flat epimorphism, $\mathfrak{q} \in \text{Spec}(B)$ and \mathfrak{p} the preimage of \mathfrak{q} , then $A_{\mathfrak{p}} \cong B_{\mathfrak{q}}$ by [19, Proposition 2.4(iii)]. Since G-rings are defined in terms of stalks, it follows that B is a G-ring provided that A is such. \square

Recall further that a commutative noetherian local ring A is called **analytically irreducible** if its completion \widehat{A} is a domain. If A is regular, it is analytically irreducible since the completion \widehat{A} is regular local and so, in particular, a (unique factorisation) domain. Normality of A , in general, does not ensure analytic irreducibility, but it does for G-rings.

Lemma 6.3. *Let A be a local normal G-ring. Then A is analytically irreducible.*

Proof. This goes back to [34], and in this generality it follows from [22, Theorems 32.2(1)]. \square

To see the connection to flat epimorphisms, notice that the analytic irreducibility of a commutative noetherian local ring A of Krull dimension d ensures that the only prime ideal $\mathfrak{q} \in \text{Spec}(\widehat{A})$ for which $\dim \widehat{A}/\mathfrak{q} = d$ is $\mathfrak{q} = 0$. Hence, by Theorem 3.7, the local cohomology $H_I^d(A)$ vanishes for each proper ideal $I \subseteq A$ which is not \mathfrak{m} -primary. This allows us to deduce the following criterion for the existence of flat epimorphisms with given support originating at rings of Krull dimension two.

Proposition 6.4. *Let A be a commutative noetherian ring of Krull dimension two.*

- (1) *If \mathfrak{p} is a prime ideal in $\text{Spec}(A)$ of height at most one such that for all $\mathfrak{m} \in V(\mathfrak{p})$ of height two, the localisation $A_{\mathfrak{m}}$ is analytically irreducible, then $V(\mathfrak{p})$ has coherent complement in $\text{Spec}(A)$.*
- (2) *If A is a normal G-ring and $V \subseteq \text{Spec}(A)$ is any specialisation closed set such that the minimal primes in V are of height at most one, then V has coherent complement.*

Proof. (1) By Theorem 4.9, it is enough to show that $H_{\mathfrak{p}}^2(A) = 0$ or, equivalently, that $H_{\mathfrak{p}}^2(A)_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subseteq A$. Recall also that we have $H_{\mathfrak{p}}^2(A)_{\mathfrak{m}} \cong H_{\mathfrak{p}A_{\mathfrak{m}}}^2(A_{\mathfrak{m}})$ by the flat base change (see [4, §4.3]). Now, we certainly have $H_{\mathfrak{p}A_{\mathfrak{m}}}^2(A_{\mathfrak{m}}) = 0$ if \mathfrak{p} is not contained in \mathfrak{m} (since then $\mathfrak{p}A_{\mathfrak{m}} = A_{\mathfrak{m}}$) or if the height of \mathfrak{m} is at most one (since in this case the Krull dimension of $A_{\mathfrak{m}}$ is at most one, see Grothendieck's vanishing theorem [4, Theorem 6.1.2]). If \mathfrak{m} is of height two and $\mathfrak{p} \subseteq \mathfrak{m}$, then $H_{\mathfrak{p}A_{\mathfrak{m}}}^2(A_{\mathfrak{m}}) = 0$ directly by the discussion before the proposition.

(2) This follows immediately from [16, Proposition 4.1(1)], the first part and Lemma 6.3. \square

Thus, in order to exhibit an example of a local normal domain A of dimension two and $\mathfrak{p} \in \text{Spec}(A)$ of height one such that the complement of $V(\mathfrak{p})$ is not coherent, we need that A is not analytically irreducible, so in particular not a G-ring. An example of such a ring was given by Nagata [23].

Example 6.5. Let k be a field of characteristic different from two, and let X, Y and Z be algebraically independent elements over k . Let $w = \sum_{i>0} a_i X^i \in k[[X]]$ be a transcendental element over $k(X)$ and set $Z_1 = Z$ and $Z_{i+1} = X^{-i}(Z - (Y + \sum_{j<i} a_j X^j)^2) \in k(X)[Y, Z]$ for all $i \geq 1$. Consider $R = k[X, Y, Z_1, Z_2, \dots]_{\mathfrak{m}}$, where $\mathfrak{m} = (X, Y, Z_1, Z_2, \dots)$, and $A = R[W]/(W^2 - Z)$, where W is a new variable. By [23], A is a noetherian normal local domain of Krull dimension two, and $\widehat{A} = k[[X, Y]][W]/((W - (Y + w))(W + (Y + w)))$.

Put $\mathfrak{p} = (X, W + Y) \in \text{Spec}(A)$ and $\mathfrak{q} = (W - (Y + w)) \in \text{Spec}(\widehat{A})$. We claim that \mathfrak{p} is of height one. To see that, note first that by the arguments in [23, part (2)], $R/(X) \cong k[[Y]]$. The relation $Z_2 X = Z - Y^2$ in R implies that the coset of Z is sent to Y^2 under this isomorphism. Hence $A/(X) \cong k[[Y, W]]/(W^2 - Y^2) = k[[Y, W]]/((W - Y)(W + Y))$ and $A/\mathfrak{p} \cong k[[Y, W]]/(W + Y) \cong k[[Y]]$ has dimension one, proving the claim.

Now $\dim \widehat{A}/\mathfrak{q} = \dim \widehat{A} = 2$ and $\mathfrak{p}\widehat{A} + \mathfrak{q} = (X, W + Y, W - Y + w) = (X, Y, W)$ since $w \in (X)$ in \widehat{A} and $\text{char } k \neq 2$. Hence $\dim \widehat{A}/(\mathfrak{p}\widehat{A} + \mathfrak{q}) = 0$ and Theorem 3.7 implies $H_{\mathfrak{p}}^2(A) \neq 0$. Finally, by Theorem 4.9, the complement of $V(\mathfrak{p})$ in $\text{Spec}(A)$ is not coherent.

6.2. Groups of divisors: the graded and the projective case. In order to give examples of flat epimorphisms which are not universal localisations, we need to compute the Picard and the divisor class groups for certain rings of Krull dimension two and three. We will focus on homogeneous coordinate rings of one and two-dimensional projective varieties where the projective version of the divisor class groups is well-known and extract the groups we need from that piece of information. This subsection is devoted to describing the necessary relations between the various types of groups of divisors.

Suppose that $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a \mathbb{Z} -graded commutative noetherian normal ring and denote by $L = A_S$ the localisation at the multiplicative set $S \subseteq A$ consisting of all homogeneous non-zero-divisors. The

finitely generated submodules $I \subseteq L$ will be called **homogeneous fractional ideals**. We will denote by $\text{HC}(A)$ the subgroup of $C(A)$ consisting of the homogeneous invertible fractional ideals, and by $\text{HPic}(A)$ the **homogeneous Picard group**, i.e. the quotient of $\text{HC}(A)$ by the subgroup of principal homogeneous invertible fractional ideals. In the same vein, we will denote by $\text{HDiv}(A) \subseteq \text{Div}(A)$ the free subgroup generated by the homogeneous ideals. We also define the subgroup $\text{HPDiv}(A) \subseteq \text{PDiv}(A)$ consisting of the divisors of the homogeneous elements of L . We will call the quotient $\text{HCl}(A) = \text{HDiv}(A)/\text{HPDiv}(A)$ the **homogeneous divisor class group** of A .

Since associated prime ideals of graded A -modules are homogeneous by [6, §3.5], the map $\text{div}: C(A) \rightarrow \text{Div}(A)$ restricts to $\text{div}: \text{HC}(A) \rightarrow \text{HDiv}(A)$ and we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \{sA : s \in L^\times \text{ homogeneous}\} & \longrightarrow & \text{HC}(A) & \longrightarrow & \text{HPic}(A) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \text{div} & & \downarrow \overline{\text{div}} \\ 0 & \longrightarrow & \text{HPDiv}(A) & \longrightarrow & \text{HDiv}(A) & \longrightarrow & \text{HCl}(A) \longrightarrow 0, \end{array}$$

where the maps div and $\overline{\text{div}}$ are again injective. The following lemma is the first step of relating this diagram to the similar diagram (3.2) for not necessarily homogeneous divisors.

Lemma 6.6. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a \mathbb{Z} -graded commutative noetherian normal ring, and let I be an invertible fractional ideal of A . If $\text{div}(I)$ lies in $\text{HDiv}(A)$, then I lies in $\text{HC}(A)$. In particular, we have $\text{HPDiv}(A) = \text{HDiv}(A) \cap \text{PDiv}(A)$.*

Proof. If $I \subseteq A$, this is an immediate consequence of Remark 5.15 and the fact that $\mathfrak{p}^{(n)}$ is homogeneous whenever \mathfrak{p} is homogeneous and $n \geq 0$ by [6, §3.5]. If I is any invertible fractional ideal of A , this follows from the fact that there exists a homogeneous element $g \in A$ such that $\text{div}(I) + \text{div}(gA)$ is effective and, hence, $gI \subseteq A$ by Lemma 5.14. \square

The main result about homogeneous class groups now says that they are canonically isomorphic to the non-homogeneous ones for normal rings.

Proposition 6.7. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a \mathbb{Z} -graded commutative noetherian normal ring. The inclusions $\text{HC}(A) \rightarrow C(A)$ and $\text{HDiv}(A) \rightarrow \text{Div}(A)$ induce isomorphisms $\text{HPic}(A) \cong \text{Pic}(A)$ and $\text{HCl}(A) \cong \text{Cl}(A)$.*

Proof. This was proved in [29, Proposition 7.1], but we recall the proof for the reader's convenience. We suppose without loss of generality that A is a domain and, by excluding a trivial case, also that $A \neq A_0$.

If we denote by $S \subseteq A$ the set of non-zero homogeneous elements and by $L = A_S$ the localisation at S , then $L \cong K[T^{\pm 1}]$, where $K = L_0$ is a field and T is any fixed homogeneous element of the smallest possible positive degree. Let \mathfrak{p} be a prime ideal of A of height one, but not necessarily homogeneous. Since L is a principal ideal domain, there exists an element f in \mathfrak{p} such that $\mathfrak{p}L = fL$. This implies that there exists g in S such that already $\mathfrak{p}A_g = fA_g$. Hence the divisor $D := \mathfrak{p} - \text{div}(f)$ is supported in $V(g)$ and, consequently, D lies in $\text{HDiv}(A)$ and $[D] = [\mathfrak{p}]$ in $\text{Cl}(A)$. This implies that the induced map $\text{HCl}(A) \rightarrow \text{Cl}(A)$ is surjective and the injectivity follows from Lemma 6.6. We thus have a commutative square with monomorphisms in the columns

$$\begin{array}{ccc} \text{HPic}(A) & \longrightarrow & \text{Pic}(A) \\ \overline{\text{div}} \downarrow & & \downarrow \overline{\text{div}} \\ \text{HCl}(A) & \xrightarrow{\cong} & \text{Cl}(A) \end{array}$$

which is a pull-back by Lemma 6.6, showing that also $\text{HPic}(A) \rightarrow \text{Pic}(A)$ is an isomorphism. \square

If k is a field and $A = \bigoplus_{i \geq 0} A_i$ is non-negatively graded with $A_0 = k$, we can form the projective scheme $\text{Proj}(A)$. As a set, $X = \text{Proj}(A)$ is the set of homogeneous prime ideals of A different from the irrelevant ideal $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$. To X one can again attach its group of divisors $\text{Div}(X)$ and its class group $\text{Cl}(X)$. Divisors are again formal \mathbb{Z} -linear combinations of height one subvarieties of X , i.e. $\text{Div}(X) = \text{HDiv}(A)$. In the class group one factors out the subgroup of divisors of degree zero elements of $L = A_S$, where S is the set of homogeneous regular elements of A . This allows us to quickly deduce the relation between $\text{Cl}(A)$ and $\text{Cl}(X)$.

Proposition 6.8. *Let k be a field, $A = \bigoplus_{i \geq 0} A_i$ a graded normal domain of Krull dimension at least one with $A_0 = k$ and $X = \text{Proj}(A)$ its associated projective scheme. Let also L be the ring of fractions*

obtained by localising A at the set of non-zero homogeneous elements. For any homogeneous element T in L of smallest possible positive degree, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \text{Cl}(X) \longrightarrow \text{Cl}(A) \longrightarrow 0.$$

where $\varphi(1)$ is the class of $\text{div}(T)$.

Proof. This is in fact a special case of [33, Theorem 1.6]. As in the proof of Proposition 6.7, we have $L \cong K[T^{\pm 1}]$, where $K = L_0$ is a field. Then we have a short exact sequence

$$0 \longrightarrow \{\text{div}(gA) : g \in K\} \longrightarrow \text{HPDiv}(A) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the quotient \mathbb{Z} is generated by the coset of T . Now we have $\text{Cl}(A) \cong \text{HDiv}(A)/\text{HPDiv}(A)$ by Proposition 6.7, whereas $\text{Cl}(X) = \text{HDiv}(A)/\{\text{div}(gA) : g \in K\}$. \square

Remark 6.9. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded commutative noetherian normal ring with $A_0 = k$ and let $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$ be the unique maximal graded ideal. Kurano established in [18, Lemma (4.1)] an isomorphism $\text{Cl}(A) \cong \text{Cl}(A_{\mathfrak{m}})$. Since every projective graded A -module is free, we have $\text{Pic}(A) = 0$ by Proposition 6.7 and, hence, we also have an isomorphism $\text{Pic}(A) \cong \text{Pic}(A_{\mathfrak{m}}) (= 0)$.

6.3. Flat epimorphisms which are not localisations. If $A = \bigoplus_{i \geq 0} A_i$ is the homogeneous coordinate ring of a $(d-1)$ -dimensional smooth projective variety $X = \text{Proj } A$ over a field k , then the singular locus of $\text{Spec}(A)$ (which is geometrically the affine cone of X) contains at most the irrelevant ideal $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$. If $d = 2$ (i.e. if X is a smooth projective curve) and A is normal (which is automatic if A is Cohen-Macaulay by Remark 5.10(4)), then any specialisation closed set $V \subseteq \text{Spec}(A)$ with minimal primes of height at most one has coherent complement by Proposition 6.4 (together with Lemma 6.2). In view of Theorem 5.17, all we need to do to give two-dimensional examples of flat epimorphisms which are not universal localisations is to find such A with non-torsion $\text{Cl}(A)/\text{Pic}(A)$. We moreover know that $\text{Pic}(A) = 0$ by Remark 6.9.

Now we can give particular examples. Recall that an **elliptic curve** E (embedded into the projective plane $\mathbb{P}_k^2 = \text{Proj}(k[X, Y, Z])$ over a field k) is defined by $E = \text{Proj}(A)$, where $A = k[X, Y, Z]/(f)$ is a standard graded ring with f a homogeneous polynomial of degree 3 such that $A_{\mathfrak{p}}$ is a regular local ring for all homogeneous prime ideals \mathfrak{p} not containing (or, equivalently, not equal to) the irrelevant ideal (X, Y, Z) . In particular, E is a smooth projective curve.

We will also assume for simplicity that the set $E(k)$ of k -rational points of E contains an inflection point \mathcal{O} . Then $E(k)$ has a well-known group structure with the neutral element \mathcal{O} and the group operation given by the rule that if a line in \mathbb{P}_k^2 intersects $E(k)$ in three points (when counting the intersection with multiplicities), then the sum of the three points is \mathcal{O} . The divisor class group $\text{Cl}(E)$ is well-known to be isomorphic to $E(k) \times \mathbb{Z}$ in this case. More specifically, there is a well-defined group homomorphism $\text{deg} : \text{Cl}(E) \longrightarrow \mathbb{Z}$ which sends the class of a closed point \mathfrak{p} of E to the degree $\text{deg}([\mathfrak{p}]) = \dim_k k(\mathfrak{p})$. The kernel of deg is then identified with $E(k)$ via a group isomorphism which sends \mathfrak{p} in $E(k)$ to $[\mathfrak{p}] - [\mathcal{O}] \in \text{Cl}(E)$. Now we can use Propositions 6.7 and 6.8 to compute $\text{Cl}(A)$. If $T = aX + bY + cZ$ is an equation for the tangent of E at \mathcal{O} , then $\text{div}(T) = 3 \cdot \mathcal{O}$ and, hence

$$(6.1) \quad \text{Cl}(A) \cong E(k) \times \mathbb{Z}/(3).$$

We summarise the results about homogeneous coordinate rings of elliptic curves in the following theorem.

Theorem 6.10. *Let A be the homogeneous coordinate ring of an elliptic curve E over k with a k -rational inflection point \mathcal{O} . If \mathfrak{p} lies in $E(k)$ (in particular, \mathfrak{p} is a homogeneous prime ideal of A of height one) and \mathfrak{p} is a non-torsion point of the group $E(k)$, then there is a flat ring epimorphism $f : A \longrightarrow B$ associated with $V(\mathfrak{p})$ and f is not a universal localisation.*

Proof. The flat ring epimorphism exists by Proposition 6.4 and Lemma 6.2. It is not a universal localisation by Theorem 5.17 and the isomorphism (6.1). \square

Example 6.11. Let $A = \mathbb{Q}[X, Y, Z]/(X^3 - Y^2Z - 4Z^3)$ be a graded ring, with the grading defined by $\text{deg}(X) = \text{deg}(Y) = \text{deg}(Z) = 1$, and let \mathfrak{p} denote the prime ideal $(X - 2Z, Y - 2Z)$. Then the assumption of Theorem 6.10 is satisfied; see [15, Example 1.10]. Hence we have flat ring epimorphisms originating from A which are not universal localisations.

If we, on the other hand, let $A = \mathbb{Q}[X, Y, Z]/(X^3 - XZ^2 - Y^2Z)$ or $A = \mathbb{Q}[X, Y, Z]/(X^3 - Y^2Z + Z^3)$, then $\text{Cl}(A)$ is torsion ([15, Examples 1.8 and 1.9]). In particular, every flat ring epimorphism originating from A is a universal localisation.

We conclude the paper with an analysis of flat epimorphisms and universal localisations originating from the algebra $A = k[X, Y, U, V]/(XU - YV)$ for an algebraically closed field k . This is a three-dimensional normal domain (compare with Example 6.1). It turns out that there are both primes $\mathfrak{p} \in \text{Spec}(A)$ of height one such that $V(\mathfrak{p})$ does not have coherent complement and primes \mathfrak{p} where $V(\mathfrak{p})$ has coherent complement, but the corresponding flat epimorphism is not a universal localisation.

In order to compute the divisor class and the Picard groups, we equip A with the standard grading with all X, Y, U, V in degree 1 and put $\mathfrak{m} = (X, Y, U, V) = \bigoplus_{i \geq 1} A_i$. Then $\text{Pic}(A) = 0$ by Remark 6.9 and to compute $\text{Cl}(A)$, we will use Propositions 6.7 and 6.8.

To this end, it is well-known that $X := \text{Proj}(A) \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ via the Segre embedding and that $\text{Cl}(X) \cong \text{Cl}(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \cong \mathbb{Z} \times \mathbb{Z}$, [11, Example II.6.6.1]. In fact, we can be more specific about this isomorphism. Consider the embedding $A \rightarrow k[S_0, S_1, T_0, T_1]$ of k -algebras given by

$$X \mapsto S_0 T_0, \quad Y \mapsto S_1 T_0, \quad U \mapsto S_1 T_1, \quad V \mapsto S_0 T_1.$$

This is essentially the Segre embedding at the level of homogeneous coordinate rings, where we equip $k[S_0, S_1, T_0, T_1]$ with the $\mathbb{Z} \times \mathbb{Z}$ -grading with S_0, S_1 in degree $(1, 0)$ and T_0, T_1 in degree $(0, 1)$. The isomorphism of schemes $X \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ in particular induces a bijection between homogeneous prime ideals $\mathfrak{p} \subseteq A$ of height one and homogeneous prime ideals $\mathfrak{p}' \subseteq k[S_0, S_1, T_0, T_1]$ of height one, which is given by $\mathfrak{p} = \mathfrak{p}' \cap A$. Since every height one prime ideal of a polynomial ring is principal, the following fact immediately follows from the discussion.

Lemma 6.12. *For any homogeneous prime ideal $\mathfrak{p} \subseteq A = k[X, Y, U, V]/(XU - YV)$, there is a homogeneous (with respect to the $\mathbb{Z} \times \mathbb{Z}$ -grading) irreducible polynomial $f_{\mathfrak{p}} \in k[S_0, S_1, T_0, T_1]$ such that $\mathfrak{p} = (f_{\mathfrak{p}} \cdot k[S_0, S_1, T_0, T_1]) \cap A$. Moreover, such $f_{\mathfrak{p}}$ is unique up to a non-zero scalar multiple.*

Now, the bijection $\psi: \text{Cl}(X) \rightarrow \mathbb{Z} \times \mathbb{Z}$ is explicitly given as follows. If $[\mathfrak{p}] \in \text{Cl}(X)$ is a divisor class represented by a homogeneous prime ideal $\mathfrak{p} \subseteq A$, $f_{\mathfrak{p}}$ is the homogeneous polynomial as in the above lemma, $d_{\mathfrak{p}}$ is the degree of $f_{\mathfrak{p}}$ in the variables S_0, S_1 and $e_{\mathfrak{p}}$ is the degree of $f_{\mathfrak{p}}$ in T_0, T_1 , then we have

$$\begin{aligned} \psi: \text{Cl}(\text{Proj}(A)) &\xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}, \\ [\mathfrak{p}] &\longmapsto (d_{\mathfrak{p}}, e_{\mathfrak{p}}). \end{aligned}$$

In particular, one readily checks that

$$\psi([(X, V)]) = \psi([(Y, U)]) = (1, 0) \quad \text{and} \quad \psi([(X, Y)]) = \psi([(U, V)]) = (0, 1),$$

since $f_{(X, V)} = S_0$, $f_{(Y, U)} = S_1$, $f_{(X, Y)} = T_0$ and $f_{(U, V)} = T_1$. Since $(X, V) \cdot (X, Y) = X \cdot \mathfrak{m}$ as ideals in A , we have that $\text{div}(X) = (X, V) + (X, Y)$ in $\text{HDiv}(A)$, which corresponds to $(1, 1) \in \mathbb{Z} \times \mathbb{Z}$ under ψ (see also [11, Example II.6.6.2]). Now we can use Proposition 6.8 with $T = X$ to conclude that $\text{Cl}(A) \cong \mathbb{Z} \times \mathbb{Z}/\langle (1, 1) \rangle \cong \mathbb{Z}$. With the notation above, if \mathfrak{p} is a homogeneous prime ideal of A of height one and $d_{\mathfrak{p}}, e_{\mathfrak{p}}$ are the degrees of $f_{\mathfrak{p}}$, we explicitly have

$$\begin{aligned} \rho: \text{Cl}(A) &\xrightarrow{\cong} \mathbb{Z}, \\ [\mathfrak{p}] &\longmapsto e_{\mathfrak{p}} - d_{\mathfrak{p}}. \end{aligned}$$

The existence of flat epimorphisms and universal localisations starting in A , depending on the class groups, is summarised in the following proposition. For a particular instance of case (2), we refer to [21, Example 6.14] and [13, Remarks 2.1].

Proposition 6.13. *Let k be an algebraically closed field and $A = k[X, Y, U, V]/(XU - YV)$ with the standard \mathbb{Z} -grading. If $\mathfrak{p} \subseteq A$ is a homogeneous prime ideal of height one and $\psi([\mathfrak{p}]) = (d_{\mathfrak{p}}, e_{\mathfrak{p}})$ with the notation above, exactly one of the following three cases occurs.*

- (1) *If $d_{\mathfrak{p}} = 0$ or $e_{\mathfrak{p}} = 0$, then $V(\mathfrak{p})$ does not have coherent complement. This case occurs precisely if \mathfrak{p} is of the form $(g(X, Y), g(V, U))$ or $(g(X, V), g(Y, U))$, where g is a linear form.*
- (2) *If $d_{\mathfrak{p}} \neq 0 \neq e_{\mathfrak{p}}$ and $d_{\mathfrak{p}} \neq e_{\mathfrak{p}}$, then $V(\mathfrak{p})$ has coherent complement, but the corresponding flat epimorphism $A \rightarrow B$ is not a universal localisation.*
- (3) *If $d_{\mathfrak{p}} = e_{\mathfrak{p}}$, then $V(\mathfrak{p})$ corresponds to a classical localisation.*

Proof. Since clearly $(d_{\mathfrak{p}}, e_{\mathfrak{p}}) \neq (0, 0)$, exactly one of the three cases occurs.

In case (1), assume without loss of generality that $e_{\mathfrak{p}} = 0$. Then $f_{\mathfrak{p}}$ (in the notation above) is an irreducible homogeneous polynomial in $k[S_0, S_1]$ with the standard grading and $\mathfrak{p} = (f_{\mathfrak{p}} \cdot k[S_0, S_1, T_0, T_1]) \cap A$. In particular, $f_{\mathfrak{p}}(X, Y)$ and $f_{\mathfrak{p}}(V, U)$ belong to \mathfrak{p} . Since k is algebraically closed, $f_{\mathfrak{p}}$ must be linear and one readily checks that $\mathfrak{p} = (f_{\mathfrak{p}}(X, Y), f_{\mathfrak{p}}(V, U))$.

Now, we can without loss of generality assume that S_1 does not divide $f_{\mathfrak{p}}$, i.e. $f_{\mathfrak{p}}(S_0, 0) \neq 0$. As in Example 6.1, we have $H_{\mathfrak{p}}^i(A) = 0$ for $i \geq 3$ by [4, Theorem 3.3.1] since \mathfrak{p} has two generators. To prove that $V(\mathfrak{p})$ does not have coherent complement, we need to show that $H_{\mathfrak{p}}^2(A) \neq 0$ (Theorem 4.9). Since $H_{\mathfrak{p}}^2$ is right exact, it again suffices to prove by the independence of the base (Proposition ??) that $H_{\mathfrak{p}C}^2(C) \neq 0$ for $C = A/(Y) \cong k[X, U, V]/(XU)$. Note that $\mathfrak{q} = (U) \subseteq \widehat{C}$ satisfies $\dim \widehat{C} = \dim \widehat{C}/\mathfrak{q} = 2$ and, since $\widehat{C}/(\mathfrak{p}\widehat{C} + \mathfrak{q}) \cong K[X, V]/I$ for an ideal I containing $f_{\mathfrak{p}}(X, 0)$ and $f_{\mathfrak{p}}(V, 0)$, also $\dim \widehat{C}/(\mathfrak{p}\widehat{C} + \mathfrak{q}) = 0$. The non-vanishing $H_{\mathfrak{p}C}^2(C)$ then follows from Theorem 3.7.

In case (2), assume without loss of generality that $d_{\mathfrak{p}} < e_{\mathfrak{p}}$ and put $\delta_{\mathfrak{p}} := e_{\mathfrak{p}} - d_{\mathfrak{p}} = \rho([\mathfrak{p}])$, where $\rho: \text{Cl}(A) \xrightarrow{\cong} \mathbb{Z}$ is as above. We can equip $k[S_0, S_1, T_0, T_1]$ with a \mathbb{Z} -grading such that S_0, S_1 are in degree -1 and T_0, T_1 are in degree 1 . This means that A identifies with $k[S_0, S_1, T_0, T_1]_0$ and $f_{\mathfrak{p}} \in k[S_0, S_1, T_0, T_1]_{\delta_{\mathfrak{p}}}$. Since then $f_{\mathfrak{p}}S_0^{\delta_{\mathfrak{p}}}, f_{\mathfrak{p}}S_1^{\delta_{\mathfrak{p}}} \in \mathfrak{p}$ and $f_{\mathfrak{p}}^{2\delta_{\mathfrak{p}}-1} \in (S_0^{\delta_{\mathfrak{p}}}, S_1^{\delta_{\mathfrak{p}}})$, we have $f_{\mathfrak{p}}^{2\delta_{\mathfrak{p}}} \in \mathfrak{p} \cdot k[S_0, S_1, T_0, T_1]$. It follows from (the proof of) [21, Proposition 6.11] that there exists a flat epimorphism $A \rightarrow B$ associated with $V(\mathfrak{p})$. On the other hand $[\mathfrak{p}] \in \text{Cl}(A) \cong \mathbb{Z}$ corresponds to the torsion-free element $0 \neq \delta_{\mathfrak{p}}$, so it cannot be a universal localisation by Theorem 5.17.

Finally, in case (3), $f_{\mathfrak{p}}$ lies in the image of the embedding $A \rightarrow k[S_0, S_1, T_0, T_1]$ and, hence $\mathfrak{p} = f_{\mathfrak{p}}A$ is principal. The classical localisation $A \rightarrow A_{f_{\mathfrak{p}}}$ is associated with $V(\mathfrak{p})$ in this case. \square

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